Journal of Geometry and Physics 165 (2021) 104191

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

N-soliton solution of a combined pKP–BKP equation

Wen-Xiu Ma *

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
School of Mathematics, South China University of Technology, Guangzhou 510640, China
School of Mathematical and Statistical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

A R T I C L E I N F O

Article history:
Received 24 July 2020
Accepted 22 February 2021
Available online 6 March 2021

MSC:
35Q51
35Q53
37K40

Keywords:
N-soliton solution
Hirota N-soliton condition
Integrable equations in (2+1)-dimensions

A B S T R A C T

We consider a linear combination of the potential KP equation and the BKP equation, and call it a combined pKP–BKP equation. We prove that the combined pKP–BKP equation satisfies the Hirota N-soliton condition and thus it possesses an N-soliton solution. The proof is an application of an algorithm to compare degrees of homogeneous polynomials associated with the Hirota function in N wave vectors.

1. Introduction

When studying differential equations, any one solution is called a particular solution, while a general formula for all possible solutions is called a general solution. General solutions can often be worked out for linear differential equations, but it is generally impossible to obtain general solutions for nonlinear differential equations, particularly nonlinear partial differential equations (PDEs). Nevertheless, there is a class of nonlinear PDEs generated from Lax pairs, for which one can find N-soliton solutions, a kind of general solutions in the case of nonlinear PDEs. Such PDEs are often called soliton equations [1,21], and can possess lump solutions as well [19].

The Hirota bilinear method [10] was developed for constructing soliton solutions, including lump solutions. The key elements of the method are Hirota bilinear derivatives [6]:

$$D_x^m f\cdot g=(\partial_x-\partial_{x'})^m f(x)g(x')|_{x'=x}=\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(\partial_x^i f)(\partial_x^{m-i}g),\ \ m\geq 1, \tag{1.1}$$

and more generally, bilinear partial derivatives:

$$(D_x^m D_t^n f\cdot g)(x,t)=(\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t},\ \ m,n\geq 1. \tag{1.2}$$

* Correspondence to: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.
E-mail address: mawx@cas.usf.edu.

https://doi.org/10.1016/j.geomphys.2021.104191
0393-0440/

When $f = g$, we obtain Hirota bilinear expressions:

$$D_x^m f \cdot f = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\partial_x^i f)(\partial_x^{m-i} f), \ m \geq 1, \tag{1.3}$$

and similarly, bilinear partial derivative expressions:

$$D_x^m D_t^n f \cdot f = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (\partial_x^i \partial_t^j f)(\partial_x^{m-i} \partial_t^{n-j} f), \ m, n \geq 1. \tag{1.4}$$

By means of Hirota bilinear expressions, we can define Hirota bilinear equations. Noting that Hirota bilinear expressions of odd orders are all zero, we take an even polynomial $P(x, y, t)$ in the space variables x, y and the time variable t, and assume that $P(0, 0, 0) = 0$, i.e., no constant term is involved. Then formulate an associated Hirota bilinear equation

$$P(D_x, D_y, D_t) f \cdot f = 0, \tag{1.5}$$

every term of which is a Hirota bilinear expression. If a nonlinear PDE can be transformed into a Hirota bilinear equation, we say that it possesses a Hirota bilinear form.

One well-known example of integrable equations in (2+1)-dimensions is the KP equation [11]:

$$N_1(u) := (u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{1.6}$$

whose Hirota bilinear form is

$$\begin{aligned} B_1(f) &:= (D_x^4 + D_x D_t - D_y^2) f \cdot f \\ &= 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_{xt} f - f_x f_t - f_{yy} f + f_y^2) = 0. \end{aligned} \tag{1.7}$$

Under the logarithmic derivative transformation $u = 2(\ln f)_{xx}$, they are linked together, and the link is $N_1(u) = (B_1(f)/f^2)_{xx}$. The KP equation is associated with the A-type infinite dimensional Lie algebra gl(∞) [12] and has N-soliton solutions [22]. The potential KP equation

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} - u_{yy} = 0 \tag{1.8}$$

is generated from the KP equation by replacing u with u_x and integrating with respect to x once.

Another well-studied integrable equation in (2+1)-dimensions is the BKP equation associated with the B-type infinite dimensional Lie algebra o(∞) [2]:

$$N_2(u) := (15u_x^3 + 15u_x u_{3x} + u_{5x})_x + 5[u_{3x,y} + 3(u_x u_y)_x] + u_{xt} - 5u_{yy} = 0, \tag{1.9}$$

and its Hirota bilinear form is

$$\begin{aligned} B_2(f) &:= (D_x^6 + 5D_x^3 D_y + D_x D_t - 5D_y^2) f \cdot f \\ &= 2[f_{6x} f - 6 f_{5x} f_x + 15 f_{4x} f_{xx} - 10 f_{3x}^2 \\ &\quad + 5(f_{3x,y} f - 3 f_{xxy} f_x + 3 f_{xy} f_{xx} - f_y f_{3x}) \\ &\quad + f_{xt} f - f_x f_t - 5(f_{yy} f - f_y^2)] = 0. \end{aligned} \tag{1.10}$$

This is equivalent to the BKP equation, under the logarithmic derivative transformation $u = 2(\ln f)_x$, and the link is $N_2(u) = (B_2(f)/f^2)_x$. Based on the above Hirota bilinear form, soliton solutions were analyzed for the BKP equation via the τ-function [2] and the Pfaffian technique [8].

In this paper, we would like to consider a linear combination of the potential KP equation and the BKP equation, and call it a combined pKP–BKP equation. We will present soliton solutions for the combined pKP–BKP equation by verifying the corresponding Hirota N-soliton condition. An algorithm will be used for comparing degrees of homogeneous polynomials generated from the Hirota function in N wave vectors, while verifying the Hirota N-soliton condition. Our result shows that the combined pKP–BKP equation is another example of nonlinear PDEs, which possess N-soliton solutions, in (2+1)-dimensions.

2. Soliton solutions

Let N be an arbitrary natural number. An N-soliton solution to a Hirota bilinear equation (1.5) is given by [5]:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} a_{ij} \mu_i \mu_j\right), \tag{2.1}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, $\mu = 0, 1$ means that each μ_i takes 0 or 1, and

$$\eta_i = k_i x + l_i y - \omega_i t + \eta_{i,0}, \ 1 \leq i \leq N, \tag{2.2}$$

$$e^{a_{ij}} = A_{ij} := -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)},\ 1 \le i < j \le N, \tag{2.3}$$

$\eta_{i,0}$'s being arbitrary phase shifts, when the following dispersion relations hold:

$$P(\mathbf{k}_i) = 0,\ \mathbf{k}_i = (k_i, l_i, -\omega_i),\ 1 \le i \le N. \tag{2.4}$$

The fundamental question in soliton theory is what condition can guarantee the existence of N-soliton solitons. To this end, let us define

$$H(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \sum_{\sigma=\pm 1} P(\sum_{i=1}^{n} \sigma_i \mathbf{k}_i) \prod_{1\le i<j\le n} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j)\sigma_i\sigma_j,\ 1 \le n \le N, \tag{2.5}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, and $\sigma = \pm 1$ means that each σ_i takes 1 or -1. We call these functions the Hirota functions. In particular, we have $H(\mathbf{k}_1) = P(\mathbf{k}_1) + P(-\mathbf{k}_1) = 2P(\mathbf{k}_1)$.

Using the basic properties

$$P(D_x, D_y, D_t)e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j)e^{\eta_i+\eta_j}, \tag{2.6}$$

and

$$P(D_x, D_y, D_t)e^{\eta_n} f \cdot e^{\eta_n} g = e^{2\eta_n} P(D_x, D_y, D_t) f \cdot g, \tag{2.7}$$

where η_i, η_j and η_n are arbitrary linear functions of x, y and t, defined as in (2.2), we can work out the following expression [16].

Theorem 2.1. *Let f be defined by (2.1), and $\hat{\xi}$ mean that no ξ is involved. Then we have*

$$\begin{aligned} &P(D_x, D_y, D_t) f \cdot f \\ &= (-1)^{\frac{1}{2}N(N-1)} \frac{H(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)}{\prod_{1\le i<j\le N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1+\eta_2+\cdots+\eta_N} \\ &+ \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1\le i_1<\cdots<i_n\le N} \frac{H(\mathbf{k}_1, \ldots, \hat{\mathbf{k}}_{i_1}, \ldots, \hat{\mathbf{k}}_{i_n}, \ldots, \mathbf{k}_N)}{\prod_{\substack{1\le i<j\le N \\ i,j\notin\{i_1,\ldots,i_n\}}} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1+\cdots+\hat{\eta}_{i_1}+\cdots+\hat{\eta}_{i_n}+\cdots+\eta_N} \\ &+ \sum_{n=1}^{N-1} \sum_{1\le i_1<\cdots<i_n\le N} e^{2(\eta_{i_1}+\cdots+\eta_{i_n}+\sum_{1\le r<s\le n} a_{i_r i_s})} P(D_{x_1}, \ldots, D_{x_M}) \tilde{f}_{i_1\cdots i_n} \cdot \tilde{f}_{i_1\cdots i_n} \end{aligned} \tag{2.8}$$

with

$$\tilde{f}_{i_1\cdots i_n} = \sum_{\tilde{\mu}_{i_1\cdots i_n}=0,1} exp(\sum_{\substack{1\le i\le N \\ i\notin\{i_1,\ldots,i_n\}}} \mu_i \tilde{\eta}_i + \sum_{\substack{1\le i<j\le N \\ i,j\notin\{i_1,\ldots,i_n\}}} a_{ij}\mu_i\mu_j),\ \ \tilde{\eta}_i = \eta_i + \sum_{r=1}^{n} a_{i i_r},$$

where $\tilde{\mu}_{i_1\cdots i_n} = (\mu_1, \ldots, \hat{\mu}_{i_1} \cdots, \hat{\mu}_{i_n}, \ldots, \mu_N)$ and $\tilde{\mu}_{i_1\cdots i_n} = 0, 1$ means that each μ_i in $\tilde{\mu}_{i_1\cdots i_n}$ takes 0 or 1.

Based on this theorem, we can see, using a recursive idea, that a Hirota bilinear equation (1.5) has an N-soliton solution (2.1) if and only if the condition

$$H(\mathbf{k}_1, \ldots, \mathbf{k}_n) = 0,\ 1 \le n \le N, \tag{2.9}$$

is satisfied. This is called the Hirota condition for an N-soliton solution, or simply, the N-soliton condition [7,20]. There are very few studies on this Hirota N-soliton condition, due to its high complexity [7].

The one-soliton condition is exactly the dispersion relation: $P(\mathbf{k}_1) = 0$, which means that $f = 1 + e^{\eta_1}$ is a solution. In addition to the dispersion relations, the two-soliton condition reads

$$2(P(\mathbf{k}_1 + \mathbf{k}_2)P(\mathbf{k}_1 - \mathbf{k}_2) - P(\mathbf{k}_1 - \mathbf{k}_2)P(\mathbf{k}_1 + \mathbf{k}_2)) = 0,$$

which is an identity. Therefore, there always exists a two-soliton solution:

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}, \tag{2.10}$$

to a Hirota bilinear equation. When $N = 3$, we obtain the three-soliton condition [3,4]:

$$\begin{aligned} \sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} & P(\sigma_1\mathbf{k}_1 + \sigma_2\mathbf{k}_2 + \sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1 - \sigma_2\mathbf{k}_2) \\ & \times P(\sigma_2\mathbf{k}_2 - \sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1 - \sigma_3\mathbf{k}_3) = 0, \end{aligned}$$

which equivalently reads

$$\sum_{(\sigma_1,\sigma_2,\sigma_3)\in S} P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0, \tag{2.11}$$

where $S=\{(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\}$. The three-soliton solution is given by

$$f=1+e^{\eta_1}+e^{\eta_2}+e^{\eta_3}+A_{12}e^{\eta_1+\eta_2}+A_{13}e^{\eta_1+\eta_3} +A_{23}e^{\eta_2+\eta_3}+A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}. \tag{2.12}$$

It is broadly recognized that the three-soliton condition implies the N-soliton condition, without objective evidence.

If we take a sufficient Hirota N-soliton condition [17]:

$$P(\mathbf{k}_i-\mathbf{k}_j)=0,\ 1\le i<j\le N, \tag{2.13}$$

we can get the resonant N-soliton solution:

$$f=1+c_1e^{\eta_1}+c_2e^{\eta_2}+\cdots+c_Ne^{\eta_N}, \tag{2.14}$$

where the coefficients c_i's are arbitrary constants. All wave vectors $\mathbf{k}_i$'s leading to resonant solutions form an affine space in $\mathbb{R}^3$ [18].

To verify the Hirota N-soliton condition, we need to factor out as many common factors out of the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ as possible. The following result will be helpful in this regard, which is a direct consequence of the definition of the Hirota functions.

Theorem 2.2. *The Hirota functions defined by (2.5) are all symmetric and even functions in the wave vectors.*

Taking $\mathbf{k}_2=\pm\mathbf{k}_1$, we have

$$P(\sigma_i\mathbf{k}_i-\mathbf{k}_2)P(\sigma_i\mathbf{k}_i\pm\mathbf{k}_1)=P(\mathbf{k}_i-\mathbf{k}_1)P(\mathbf{k}_i+\mathbf{k}_1) \tag{2.15}$$

in each case of $\sigma_i=\pm1$, by cause of the even property of the polynomial P. An application of this property can yield the following consequence [16].

Theorem 2.3. *If $\mathbf{k}_2=\pm\mathbf{k}_1$, then we have*

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=2H(\mathbf{k}_3,\ldots,\mathbf{k}_N)P(2\mathbf{k}_1)\prod_{i=3}^{N}P(\mathbf{k}_i-\mathbf{k}_1)P(\mathbf{k}_i+\mathbf{k}_1). \tag{2.16}$$

This theorem will be used to factor out the required common factors out of the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$, when we prove the Hirota N-soliton condition.

3. Verification of the Hirota N-soliton condition

Let us recall that the wave vectors read

$$\mathbf{k}_i=(k_i,l_i,-\omega_i),\ 1\le i\le N,$$

and we assume that the dispersion relations (2.4) determine all frequencies $\omega_i=\omega(k_i,l_i)$, $1\le i\le N$. Therefore, $P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)$ will be functions of k_i,l_i and k_j,l_j only,

First, let us make an assumption that under the substitution

$$l_i=l_ik_i^w,\ 1\le i\le N, \tag{3.1}$$

for some integer weight w, $P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)$ and $P(\sigma_1\mathbf{k}_1+\cdots+\sigma_N\mathbf{k}_N)$ can be simplified into the following rational functions:

$$P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)=\frac{\sigma_i\sigma_jk_ik_jQ_1(k_i,l_i,k_j,l_j,\sigma_i,\sigma_j)}{Q_2(k_i,l_i,k_j,l_j)}, \tag{3.2}$$

where Q_1 and Q_2 are polynomial functions, and

$$P(\sigma_1\mathbf{k}_1+\cdots+\sigma_N\mathbf{k}_N)=\frac{Q_3(k_1,l_1,\ldots,k_N,l_N,\sigma_1,\ldots,\sigma_N)}{Q_4(k_1,l_1,\ldots,k_N,l_N)}, \tag{3.3}$$

where Q_3 and Q_4 are polynomial functions. It is crucial for guaranteeing the existence of N-soliton solutions to identify a factor of k_ik_j in $P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)$.

Second, by Theorem 2.3, the induction assumption implies that the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ will be zero, when two of the wave vectors are equal, i.e., $\mathbf{k}_i=\mathbf{k}_j$ for a pair (i,j) with $1\le i\ne j\le N$. Based on the symmetric property in Theorem 2.2, we see that under the substitution in (3.1), the Hirota polynomial function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is still even with respect to k_i, l_i $1\le i\le N$, when w is an even integer, and it is even only with respect to $k_i,\ 1\le i\le N$, when w is an odd integer. However, in both cases, we can have the simplified form for the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$:

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=(k_i^2-k_j^2)^2g_{ij}+(l_i-l_j)^2h_{ij},\ \text{for } 1\le i<j\le N,$$

where g_{ij} and h_{ij} are rational functions of $k_n, l_n,\ 1\le n\le N$.

Finally, it follows from (3.2) and (3.3) that the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ can be written as

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=\frac{\prod_{1\le i<j\le N}k_i^2k_j^2[\prod_{1\le i<j\le N}(k_i^2-k_j^2)^2g+\prod_{1\le i<j\le N}(l_i-l_j)^2h]}{Q_4(k_1,l_1,\ldots,k_N,l_N)\prod_{1\le i<j\le N}Q_2(k_i,l_i,k_j,l_j)} \tag{3.4}$$

under the substitution (3.1), where g and h are homogeneous polynomials of $k_n, l_n,\ 1\le n\le N$. If $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\ne 0$, we can have a nonzero polynomial function g at least. Introduce a new homogeneous polynomial function

$$\tilde H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=H(\mathbf{k}_1,\ldots,\mathbf{k}_N)Q_4(k_1,l_1,\ldots,k_N,l_N)\prod_{1\le i<j\le N}Q_2(k_i,l_i,k_j,l_j). \tag{3.5}$$

Then, from the relation

$$\tilde H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=\prod_{1\le i<j\le N}k_i^2k_j^2[\prod_{1\le i<j\le N}(k_i^2-k_j^2)^2g+\prod_{1\le i<j\le N}(l_i-l_j)^2h], \tag{3.6}$$

we see that the degree of the homogeneous polynomial $\tilde H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is at least $2N(N-1)+2N(N-1)=4N(N-1)$, if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\ne 0$. In other words, if the degree of the polynomial $\tilde H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is less than $4N(N-1)$, then $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=0$. Now, we can say, based on

$$\begin{aligned}&\tilde H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\\&=\sum_{\sigma=\pm1}Q_3(k_1,l_1,\ldots,k_N,l_N,\sigma_1,\ldots,\sigma_N)\prod_{1\le i<j\le N}\sigma_i\sigma_jk_ik_jQ_1(k_i,l_i,k_j,l_j,\sigma_i,\sigma_j),\end{aligned} \tag{3.7}$$

that the final step to completing the proof is to work out Q_1 and Q_3 to determine if the degree of $\tilde H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is less than $4N(N-1)$. Otherwise, we will have $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=0$, which is what we need to prove.

Let us now check the situation in the combined pKP–BKP equation. The equation is associated with

$$P(x,y,t)=a_1x^6+a_2x^4+a_3x^3y+a_4x^2+a_5xt+a_6y^2, \tag{3.8}$$

where $a_5\ne 0$, and the combined bilinear pKP–BKP equation reads

$$\begin{aligned}B(f)&:=(a_1D_x^6+a_2D_x^4+a_3D_x^3D_y+a_4D_x^2+a_5D_xD_t+a_6D_y^2)f\cdot f\\&=2[a_1(f_{6x}f-6f_{5x}f_x+15f_{4x}f_{xx}-10f_{3x}^2)\\&\quad+a_2(f_{4x}f-4f_{3x}f_x+3f_{xx}^2)+a_3(f_{3x,y}f-3f_{xxy}f_x+3f_{xy}f_{xx}-f_yf_{3x})\\&\quad+a_4(f_{xx}f-f_x^2)+a_5(f_{xt}f-f_xf_t)+a_6(f_{yy}f-f_y^2)]=0.\end{aligned} \tag{3.9}$$

This is equivalent to the combined pKP–BKP equation:

$$\begin{aligned}N(u)&:=a_1(15u_x^3+15u_xu_{3x}+u_{5x})_x+a_2(6u_xu_{xx}+u_{4x})\\&\quad+a_3[u_{3x,y}+3(u_xu_y)_x]+a_4u_{xx}+a_5u_{xt}+a_6u_{yy}=0,\end{aligned} \tag{3.10}$$

under the logarithmic derivative transformation $u=2(\ln f)_x$. The link is $N(u)=(B(f)/f^2)_x$. To guarantee the nonlinearity of this equation, we impose that $a_1^2+a_2^2+a_3^2\ne 0$.

A direct computation can show the following result.

Theorem 3.1. *Let $a_5\ne 0$ and $a_1^2+a_2^2+a_3^2\ne 0$. The combined bilinear pKP–BKP equation (3.9) or the combined nonlinear pKP–BKP equation (3.10) possesses the three-soliton solution (2.12) under $P(\mathbf{k}_i)=0,\ 1\le i\le 3$, if and only if $a_3^2+5a_1a_6=0$.*

Due to $a_5\ne 0$, it is direct to work out that

$$\omega_i=\frac{a_1k_i^5+a_2k_i^3+a_3k_i^3l_i+a_4k_i+a_6k_il_i^2}{a_5},\ 1\le i\le N, \tag{3.11}$$

and

$$\begin{cases}Q_1=-5[a_1(k_i^4-3\sigma_i\sigma_jk_i^3k_j+4k_i^2k_j^2-3\sigma_i\sigma_jk_ik_j^3+k_j^4)\\\qquad+\frac{1}{5}a_3[-3\sigma_i\sigma_j(l_i+l_j)k_ik_j+(2l_i+l_j)k_i^2+(l_i+2l_j)k_j^2]\\\qquad+\frac{3}{5}a_2(\sigma_ik_i-\sigma_jk_j)^2-\frac{1}{5}a_6(l_i-l_j)^2],\\\deg Q_3\le 6,\ Q_2=1,\ Q_4=1,\end{cases} \tag{3.12}$$

under the substitution (3.1) with $w = 1$. Then, if $H(\mathbf{k}_1, \ldots, \mathbf{k}_N) \neq 0$, we know that based on (3.7), the degree of the polynomial $\tilde{H}(\mathbf{k}_1, \ldots, \mathbf{k}_N)\,(= H(\mathbf{k}_1, \ldots, \mathbf{k}_N))$ is less than $3N(N-1)+6$, which could not be greater than $4N(N-1)$ when $N \geq 4$. Therefore, it follows from Theorem 3.1 that we have $H(\mathbf{k}_1, \ldots, \mathbf{k}_N) = 0,\ N \geq 1$. The proof is then finished. We, therefore, arrive at the following conclusion.

Theorem 3.2. *Let $a_5 \neq 0$ and $a_1^2 + a_2^2 + a_3^2 \neq 0$. The combined bilinear pKP–BKP equation (3.9) or the combined nonlinear pKP–BKP equation (3.10) possesses the N-soliton solution (2.1) under the dispersion relations (2.4) if and only if $a_3^2+5a_1a_6 = 0$.*

Taking $a_1 = a_3 = a_4 = 0$, $a_2 = a_5 = 1$, and $a_6 = -1$, we obtain the potential KP equation (1.8). Its N-soliton solution has been presented [22]. Taking $a_1 = a_5 = 1,\ a_3 = 5,\ a_4 = 0$, $a_2 = a_4 = 0$, and $a_6 = -5$, we obtain the BKP equation (1.9) [2]. The N-soliton solution of the BKP equation has also been analyzed by using the τ-function [2], the Pfaffian technique [8] and the Gel'fand–Levitan–Marchenko integral equation [9]. The case that $a_6 = 0$ (and so $a_3 = 0$) is reduced to a generalized KdV equation [16].

4. Concluding remarks

We have introduced a combined pKP–BKP equation and verified its Hirota N-soliton condition. Therefore, soliton solutions under general dispersion relations have been presented for the combined pKP–BKP equation.

It would be interesting to see if there are any other bilinear equations in (2+1)-dimensions, to which there exist N-soliton solutions. Symbolic computations and theoretical proofs could be used together to look for new equations in the (2+1)-dimensional case or even higher-dimensional cases.

The Hirota bilinear derivatives have been generalized to work with bilinear differential equations involving odd-order derivatives. The generalized bilinear $D_{p,x}$-operators are defined by [13]:

$$D_{p,x}^m D_{p,t}^n f \cdot g = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_p^{i+j} (\partial_x^{m-i} \partial_t^{n-j} f)(\partial_x^i \partial_t^j g),\ \ m, n \geq 0,\ \ m+n \geq 1, \tag{4.1}$$

where p is an arbitrary natural number as well, and the powers of α_p are determined by

$$\alpha_p^i = (-1)^{r(i)},\ \ i = r(i) \bmod p,\ \ i \geq 0, \tag{4.2}$$

with $0 \leq r(i) < p$. The patterns of those powers α_p^i for $i = 1, 2, 3, \ldots$ are

$$\begin{aligned} &p = 3:\ \ -, +, +, -, +, +, \ldots; \\ &p = 5:\ \ -, +, -, +, +, -, +, -, +, +, \ldots; \\ &p = 7:\ \ -, +, -, +, -, +, +, -, +, -, +, -, +, +, \ldots. \end{aligned}$$

For example, when $p = 3$ and $p = 5$, we have the generalized bilinear derivatives $D_{3,x}$ and $D_{5,x}$, respectively. The cases of $p = 2k$, $k \in \mathbb{N}$, present exactly the same Hirota case. The corresponding generalized bilinear expressions exhibit new characteristics. For instance, we have

$$D_{3,x}^3 f \cdot f = 2f_{xxx}f,\ \ D_{3,x}^4 f \cdot f = 6f_{xx}^2, \tag{4.3}$$

which is completely different from the results in the Hirota case. Naturally, there are other generalized bilinear derivatives such as $D_{9,x}$ and $D_{15,x}$, and it is interesting to see if there exist any relations with $D_{3,x}$. Actually, we can define two larger classes of generalized bilinear derivatives:

$$D_{(p_1,p_2),x}^m f \cdot g = \sum_{i=0}^{m} \binom{m}{i} \alpha_{p_1}^i \alpha_{p_2}^i (\partial_x^{m-i} f)(\partial_x^i g),\ \ m \geq 1, \tag{4.4}$$

and

$$D_{(p_1;p_2),x}^m f \cdot g = \sum_{i=0}^{m} \binom{m}{i} \alpha_{p_1}^{m-i} \alpha_{p_2}^i (\partial_x^{m-i} f)(\partial_x^i g),\ \ m \geq 1, \tag{4.5}$$

by using two arbitrary natural numbers, p_1 and p_2.

We point out that resonant N-solitons have been presented for generalized bilinear equations [14] or trilinear equations [15]. A (2+1)-dimensional generalized bilinear equation

$$P(D_{p,x}, D_{p,y}, D_{p,t}) f \cdot f = 0 \tag{4.6}$$

possesses a resonant N-soliton solution [14]:

$$f = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + \cdots + c_N e^{\eta_N} \tag{4.7}$$

where c_i's are arbitrary constants and $\eta_i = k_i x + l_i y - \omega_i t + \eta_{i,0},\ 1 \le i \le N$, if and only if

$$P(\mathbf{k}_i + \alpha_p \mathbf{k}_j) + P(\mathbf{k}_j + \alpha_p \mathbf{k}_i) = 0,\ 1 \le i \le j \le N, \tag{4.8}$$

where $\mathbf{k}_i = (k_i, l_i, -\omega_i),\ 1 \le i \le N$. One interesting question is whether there is any generalized bilinear equation that has general N-soliton solutions, for instance,

$$P(D_{3,x}, D_{3,t})f \cdot f = 0 \text{ or } P(D_{3,x}, D_{3,y}, D_{3,t}) = 0,$$

with $p = 3$, in (1+1)- or (2+1)-dimensions. We need to determine first what a generalized N-soliton condition should be, i.e., an N-soliton condition for a generalized bilinear equation. Any result will amend the basic theory of bilinear equations possessing soliton solutions beautifully.

Acknowledgments

The work was supported in part by NSFC, China under the grants 11975145 and 11972291. The author would also like to thank Alle Adjiri, Ahmed Ahmed, Yushan Bai, Qingxian Chen, Liyuan Ding, Yehui Huang, Xing Lü, Yan Jiang, Wenting Li, Solomon Manukure, Morgan McAnally, Fudong Wang and Yong Zhang for their valuable discussions.

References

[1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, KP hierarchies of orthogonal and symplectic type - transformation groups for soliton equations VI, J. Phys. Soc. Japan 50 (1981) 3813–3818.
[3] J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations, J. Math. Phys. 28 (1987) 1732–1742.
[4] J. Hietarinta, Introduction to the Hirota bilinear method, in: Y. Kosmann-Schwarzbach, B. Grammaticos, K.M. Tamizhmani (Eds.), Integrability of Nonlinear Systems, in: Lecture Notes in Phys, vol. 495, Springer, Berlin, 1997, pp. 95–103.
[5] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192–1194.
[6] R. Hirota, A new form of Bäcklund transformations and its relation to the inverse scattering problem, Progr. Theoret. Phys. 52 (1974) 1498–1512.
[7] R. Hirota, Direct methods in soliton theory, in: R.K. Bullough, P. Caudrey (Eds.), Solitons, Springer-Verlag, Berlin, Heidelberg, 1980, pp. 157–176.
[8] R. Hirota, Soliton solutions to the BKP equations I - the Pfaffian technique, J. Phys. Soc. Japan 58 (1989) 2285–2296.
[9] R. Hirota, Soliton solutions to the BKP equations II - the integral equation, J. Phys. Soc. Japan 58 (1989) 2705–2712.
[10] R. Hirota, Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[11] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. 15 (1970) 539–541.
[12] M. Kashiwara, T. Miwa, The τ function of the Kadomtsev-Petviasgvili equation, Proc. Japan Acad. Ser. A 57 (1981) 342–347.
[13] W.X. Ma, Generalized bilinear differential equations, Stud. Nonlinear Sci. 2 (2011) 140–144.
[14] W.X. Ma, Bilinear equations and resonant solutions characterized by Bell polynomials, Rep. Math. Phys. 72 (2013) 41–56.
[15] W.X. Ma, Trilinear equations, Bell polynomials, and resonant solutions, Front. Math. China 8 (2013) 1139–1156.
[16] W.X. Ma, N-Soliton solutions and the Hirota conditions in (1+1)-dimensions, Int. J. Nonlinear Sci. Numer. Simul. 22 (2021) http://dx.doi.org/10.1515/ijnsns-2020-0214.
[17] W.X. Ma, E.G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011) 950–959.
[18] W.X. Ma, Y. Zhang, Y.N. Tang, J.Y. Tu, Hirota bilinear equations with linear subspaces of solutions, Appl. Math. Comput. 218 (2012) 7174–7183.
[19] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Differential Equations 264 (2018) 2633–2659.
[20] A.C. Newell, Y.B. Zeng, The Hirota conditions, J. Math. Phys. 27 (1986) 2016–2021.
[21] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Consultants Bureau, New York, 1984.
[22] J. Satsuma, N-Soliton solution of the two-dimensional Kortweg-de Vries equation, J. Phys. Soc. Japan 40 (1976) 286–290.