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Inverse scattering and soliton solutions of nonlocal complex reverse-spacetime mKdV equations

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ABSTRACT

The paper deals with the inverse scattering transforms for nonlocal complex reverse-spacetime multicomponent integrable modified Korteweg–de Vries (mKdV) equations. We establish associated Riemann–Hilbert problems and determine their solutions by the Sokhotski–Plemelj formula. The inverse scattering problems consist of Gelfand–Levitan–Marchenko type equations for the generalized matrix Jost solutions and the recovery formula for the potential. When reflection coefficients are zero, the corresponding Riemann–Hilbert problems yield soliton solutions to the nonlocal complex reverse-spacetime mKdV equations.

1. Introduction

Nonlocal integrable equations have been studied very recently, including nonlocal scalar nonlinear Schrödinger (NLS) equations [3,4] and nonlocal scalar modified Korteweg–de Vries (mKdV) equations [5,17]. Their inverse scattering transforms were established under zero or nonzero boundary conditions [2,4,17]. The N-soliton solutions were generated from the Riemann–Hilbert problems [22,34], via Darboux transformations [16,23], and through the Hirota bilinear method [14]. A few multicomponent generalizations [5,7,13,22,31] were also presented and analyzed.

We will apply the Riemann–Hilbert technique [30] to study the inverse scattering transforms and particularly generate soliton solutions. Various integrable equations, such as the multiple wave interaction equations [30], the general coupled nonlinear Schrödinger equations [32], the Harry Dym equation [33], and the generalized Sasa–Satsuma equation [10], have been studied by analyzing associated Riemann–Hilbert problems. In this paper, we would like to propose a kind of multicomponent nonlocal complex reverse-spacetime mKdV equations, and construct their inverse scattering transforms and soliton solutions through establishing associated Riemann–Hilbert problems.

The rest of the letter is structured as follows. In Section 2, we make a kind of nonlocal group reductions to generate nonlocal complex reverse-spacetime mKdV equations. In Section 3, we establish associated Riemann–Hilbert problems and

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determine their solutions by the Sokhotski–Plemelj formula to present the inverse scattering transforms. In Section 4, we construct soliton solutions from the reflectionless transforms, whose Riemann–Hilbert problems have the identity jump matrix. In the last section, we give a conclusion and some concluding remarks.

2. Nonlocal complex reverse-spacetime mKdV equations

Let n be an arbitrary natural number. Assume that λ stands for a spectral parameter, and u, a $2n$-dimensional potential

$$u = u(x,t) = (p, q^T)^T,\ p = (p_1, p_2, \ldots, p_n),\ q = (q_1, q_2, \ldots, q_n)^T. \tag{2.1}$$

Let us consider the multicomponent AKNS matrix spectral problems (see, e.g., [26]):

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{2.2}$$

with the Lax pair

$$U = \lambda\Lambda + P,\ V = \lambda^3\Omega + Q. \tag{2.3}$$

The involved four matrices are defined by $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2 I_n)$,

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.4}$$

and

$$Q = \frac{\beta}{\alpha}\lambda^2 \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\lambda \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} - \frac{\beta}{\alpha^3} \begin{bmatrix} i(pq_x - p_x q) & p_{xx} + 2pqp \\ q_{xx} + 2qpq & i(qp_x - q_x p) \end{bmatrix}, \tag{2.5}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary real constants, and $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$. It is clear that if $p_j = q_j = 0,\ 2 \le j \le n$, the matrix spectral problem (2.2) reduces to the original AKNS one [1]. The compatibility condition of the spectral problems in (2.2), i.e., the zero curvature equation

$$U_t - V_x + i[U, V] = 0, \tag{2.6}$$

presents the multicomponent standard mKdV equations

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp),\ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x). \tag{2.7}$$

We take a specific kind of nonlocal group reductions for the spectral matrix

$$U^\dagger(-x,-t,-\lambda^*) = -CU(x,t,\lambda)C^{-1},\ C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix},\ \Sigma^\dagger = \Sigma, \tag{2.8}$$

where $\dagger$ denotes the Hermitian transpose and $*$, the complex conjugate. This requires

$$P^\dagger(-x,-t) = -CP(x,t)C^{-1}, \tag{2.9}$$

where Σ is a constant invertible Hermitian matrix. This Eq. (2.9) is equivalent to

$$q(x,t) = -\Sigma^{-1}p^\dagger(-x,-t). \tag{2.10}$$

Under the reduction (2.10), one has

$$V^\dagger(-x,-t,-\lambda^*) = -CV(x,t,\lambda)C^{-1},\ Q^\dagger(-x,-t,-\lambda^*) = -CQ(x,t,\lambda)C^{-1}.$$

These imply that we can apply the group reductions in (2.9) to the zero curvature equation (2.6) to get reduced equations. It is now clear that the multicomponent standard mKdV equations (2.7) become the following multicomponent nonlocal complex reverse-spacetime mKdV equations

$$p_t(x,t) = -\frac{\beta}{\alpha^3}[p_{xxx}(x,t) - 3p(x,t)\Sigma^{-1}p^\dagger(-x,-t)p_x(x,t) - 3p_x(x,t)\Sigma^{-1}p^\dagger(-x,-t)p(x,t)]. \tag{2.11}$$

When $n = 1$, we can obtain a pair of scalar examples [5,17]:

$$p_t(x,t) = p_{xxx}(x,t) - 6\sigma p(x,t)p^*(-x,-t)p_x(x,t),\ \sigma = \pm 1. \tag{2.12}$$

3. Inverse scattering transforms

Let q be determined by (2.10). In what follows, we discuss the scattering and inverse scattering for the nonlocal complex reverse-spacetime mKdV equations (2.11) by the Riemann–Hilbert approach [30] (see also [6,11] for the local case). The results will be the basis for generating soliton solutions later.

3.1. Distribution of eigenvalues

Suppose that all the potentials rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$. Upon setting $\check{P} = iP$ and $\check{Q} = iQ$, we obtain an equivalent pair of matrix spectral problems to (2.2):

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.1}$$

$$\psi_t = i\lambda^3[\Omega, \psi] + \check{Q}\psi, \tag{3.2}$$

Through a generalized Liouville's formula [24], we can have $(\det\psi)_x = 0$, since $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$. The adjoint equations of (2.2) and the adjoint equations of (3.1) and (3.2) read

$$i\tilde{\phi}_x = \tilde{\phi}U, \ i\tilde{\phi}_t = \tilde{\phi}V, \tag{3.3}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P, \ i\tilde{\psi}_t = \lambda^3[\tilde{\psi}, \Omega] + \tilde{\psi}Q, \tag{3.4}$$

respectively.

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial and temporal spectral problems (3.1) and (3.2), associated with an eigenvalue λ. Then, $C\psi^{-1}(x,t,\lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. Actually, under the nonlocal group reductions in (2.9), we can have

$$\begin{aligned}
& i[\psi^\dagger(-x,-t,-\lambda^*)C]_x = i[-(\psi_x)^\dagger(-x,-t,-\lambda)C] \\
&= i\{-i(-\lambda^*)[\Lambda, \psi(-x,-t,-\lambda^*)] - \check{P}(-x,-t)\psi(-x,-t,-\lambda^*)\}^\dagger C \\
&= i\{i(-\lambda)[\psi^\dagger(-x,-t,-\lambda^*), \Lambda] - \psi^\dagger(-x,-t,-\lambda^*)\check{P}^\dagger(-x,-t)\}C \\
&= \lambda[\psi^\dagger(-x,-t,-\lambda^*)C, \Lambda] + \psi^\dagger(-x,-t,-\lambda^*)C[-C^{-1}P^\dagger(-x,-t)C] \\
&= \lambda[\psi^\dagger(-x,-t,-\lambda^*)C, \Lambda] + \psi^\dagger(-x,-t,-\lambda^*)CP(x,t),
\end{aligned}$$

and similarly, we have

$$i[\psi^\dagger(-x,-t,-\lambda^*)C]_t = \lambda^3[\psi^\dagger(-x,-t,-\lambda^*)C, \Omega] + \psi^\dagger(-x,-t,-\lambda^*)CQ(x,t).$$

It then follows that

$$\tilde{\psi}(x,t,\lambda) = \psi^\dagger(-x,-t,-\lambda^*)C \tag{3.5}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ, i.e., $\psi^\dagger(-x,-t,-\lambda^*)C$ solves the adjoint spectral problems in (3.4).

Therefore, upon observing the asymptotic properties for ψ at infinity of x or t, the uniqueness of solutions tells that

$$\psi^\dagger(-x,-t,-\lambda^*) = C\psi^{-1}(x,t,\lambda)C^{-1}, \tag{3.6}$$

if $\psi \to I_{n+1}$, x or $t \to +\infty$ or $-\infty$. It then follows that if λ is an eigenvalue of (3.1) and (3.2) (or (3.4)), then $-\lambda^*$ will be another eigenvalue of (3.1) and (3.2) (or (3.4)), and the property (3.6) holds.

3.2. Riemann-Hilbert problems

We establish a class of associated Riemann–Hilbert problems with the variable x. In order to facilitate the expression below, we assume that

$$\alpha = \alpha_1 - \alpha_2 < 0, \ \beta = \beta_1 - \beta_2 < 0. \tag{3.7}$$

In the scattering problem, let us now introduce the two matrix eigenfunctions $\psi^\pm(x,\lambda)$ of (3.1) with the following asymptotic conditions

$$\psi^\pm \to I_{n+1}, \text{ when } x \to \pm\infty, \tag{3.8}$$

respectively. It follows from $(\det\psi)_x = 0$ or directly from the generalized Liouville's formula [24] that $\det\psi^\pm = 1$ for all $x \in \mathbb{R}$. Because

$$\phi^\pm = \psi^\pm E, \ E = e^{i\lambda\Lambda x}, \tag{3.9}$$

are both matrix eigenfunctions of the x-part of (2.2), they must be linearly dependent, and consequently, we obtain

$$\psi^- E = \psi^+ E S(\lambda), \ \lambda \in \mathbb{R}, \tag{3.10}$$

where $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is called the scattering matrix. Note that $\det S(\lambda) = 1$, thanks to $\det\psi^\pm = 1$.

We point out that we can turn the x-part of (2.2) into the following Volterra integral equations for $\psi^\pm$ [30]:

$$\psi^-(\lambda,x) = I_{n+1} + \int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^-(\lambda,y)e^{i\lambda\Lambda(y-x)}\,dy, \tag{3.11}$$

$$\psi^+(\lambda,x) = I_{n+1} - \int_{x}^{\infty} e^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^+(\lambda,y)e^{i\lambda\Lambda(y-x)}\,dy, \tag{3.12}$$

where the asymptotic conditions (3.8) have been used. Now, the theory of Volterra integral equations shows that the eigenfunctions $\psi^{\pm}$ could exist and allow analytical continuations off the real line $\lambda \in \mathbb{R}$ as long as the integrals on the right hand sides converge.

First, in order to determine the two generalized matrix Jost solutions, T^+ and T^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ (the upper and lower half-planes) and continuous in $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$ (the closed upper and lower half-planes), respectively, we denote

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \ldots, \psi_{n+1}^{\pm}), \tag{3.13}$$

where $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ $(1 \le j \le n+1)$. Then we can take the generalized matrix Jost solution T^+ as

$$T^+ = T^+(x,\lambda) = (\psi_1^-, \psi_2^+, \ldots, \psi_{n+1}^+) = \psi^- H_1 + \psi^+ H_2, \tag{3.14}$$

which is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \bar{\mathbb{C}}^+$. Here the two matrices H_1 and H_2 are

$$H_1 = \mathrm{diag}(1, \underbrace{0, \ldots, 0}_{n}), \;\; H_2 = \mathrm{diag}(0, \underbrace{1, \ldots, 1}_{n}). \tag{3.15}$$

Second, to determine the other generalized matrix Jost solution T^-, i.e., the analytic counterpart of T^+ in the lower half-plane $\mathbb{C}^-$, we take advantage of the adjoint matrix spectral problems. Note that the inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ provide those matrix eigenfunctions to the two adjoint equations. Upon stating $\tilde{\psi}^{\pm}$ as

$$\tilde{\psi}^{\pm} = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \cdots, \tilde{\psi}^{\pm,n+1})^T, \tag{3.16}$$

where $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^{\pm}$ $(1 \le j \le n+1)$, we can show by similar arguments that the generalized matrix Jost solution T^- can be taken as the adjoint matrix solution of the x-part of (3.4), i.e.,

$$T^- = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \cdots, \tilde{\psi}^{+,n+1})^T = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{3.17}$$

which is analytic for $\lambda \in \mathbb{C}^-$ and continuous for $\lambda \in \bar{\mathbb{C}}^-$.

Based on $\det \psi^{\pm} = 1$ and the definitions of $T^{\pm}$ and $S(\lambda)$, we can know that

$$\det T^+(x,\lambda) = s_{11}(\lambda), \;\; \det T^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{3.18}$$

where $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. This leads to

$$\lim_{x\to+\infty} T^+(x,\lambda) = \begin{bmatrix} s_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \; \lambda \in \bar{\mathbb{C}}^+; \;\; \lim_{x\to-\infty} T^-(x,\lambda) = \begin{bmatrix} \hat{s}_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \; \lambda \in \bar{\mathbb{C}}^-. \tag{3.19}$$

Now we introduce two unimodular generalized matrix Jost functions

$$\begin{cases} G^+(x,\lambda) = T^+(x,\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \; \lambda \in \bar{\mathbb{C}}^+, \\ (G^-)^{-1}(x,\lambda) = \begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x,\lambda), \; \lambda \in \bar{\mathbb{C}}^-, \end{cases} \tag{3.20}$$

and then formulate the required matrix Riemann–Hilbert problems on the real line for the nonlocal complex reverse-spacetime mKdV equations (2.11) as follows:

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda), \;\; \lambda \in \mathbb{R}, \tag{3.21}$$

with the jump matrix

$$G_0(x,\lambda) = E\begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}\tilde{S}(\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}E^{-1}. \tag{3.22}$$

Here the matrix $\tilde{S}(\lambda) = (\tilde{s}_{jl})_{(n+1)\times(n+1)}$ has the triangular decomposition:

$$\tilde{S}(\lambda) = (H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda)H_2), \tag{3.23}$$

and can be worked out:

$$\tilde{s}_{1,j+1} = \hat{s}_{1,j+1}, \;\; \tilde{s}_{j+1,1} = s_{j+1,1}, \;\; 1 \le j \le n, \;\; \tilde{s}_{jj} = 1, \;\; 1 \le j \le n+1, \;\; \tilde{s}_{jl} = 0, \;\; \text{otherwise}. \tag{3.24}$$

We see that the jump matrix G_0 carries all basic scattering data from the scattering matrix $S(\lambda)$. Also, the Volterra integral equations (3.11) and (3.12) guarantee the canonical normalization conditions:

$$G^{\pm}(x,\lambda) \to I_{n+1}, \;\; \text{when } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty, \tag{3.25}$$

for the presented Riemann–Hilbert problems.

From the property of eigenfunctions in (3.6), we can have

$$(G^+)^\dagger(-x,-t,-\lambda^*) = C(G^-)^{-1}(x,t,\lambda)C^{-1}. \tag{3.26}$$

Based on this, we know that the jump matrix G_0 satisfies

$$G_0^\dagger(-x,-t,-\lambda^*) = CG_0(x,t,\lambda)C^{-1}. \tag{3.27}$$

3.3. Evolution of the scattering data

To complete the direct scattering transforms, let us compute the derivative of (3.10) with time t and use the temporal matrix spectral problem (3.2) that $\psi^\pm$ satisfy.

Then we can see that the scattering matrix S will satisfy an evolution law:

$$S_t = i\lambda^3[\Omega, S]. \tag{3.28}$$

This generates the time evolution for the time-dependent scattering coefficients:

$$\begin{cases} s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t},\ s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t},\ \cdots,\ s_{1,n+1} = s_{1,n+1}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t},\ s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t},\ \cdots,\ s_{n+1,1} = s_{n+1,1}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t}, \end{cases}$$

and all other scattering coefficients are independent of the time variable t.

3.4. Gelfand–Levitan–Marchenko type equations

To determine the generalized matrix Jost solutions, let us rewrite the Riemann–Hilbert problems in (3.21) as

$$\begin{cases} G^+ - G^- = G^- v,\ v = G_0 - I_{n+1},\ \text{on } \mathbb{R}, \\ G^\pm \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty. \end{cases} \tag{3.29}$$

Let $G(\lambda) = G^\pm(\lambda)$, if $\lambda \in \mathbb{C}^\pm$. Assume that G has simple poles off the real line $\mathbb{R}$: $\{\mu_j\}_{j=1}^R$, where R is an arbitrary natural number. We now set

$$\tilde{G}^\pm(\lambda) = G^\pm(\lambda) - \sum_{j=1}^R \frac{G_j}{\lambda-\mu_j},\ \lambda \in \bar{\mathbb{C}}^\pm;\ \tilde{G}(\lambda) = \tilde{G}^\pm(\lambda),\ \lambda \in \mathbb{C}^\pm; \tag{3.30}$$

where G_j is the residue of G at $\lambda = \mu_j$, i.e.,

$$G_j = \mathrm{res}(G(\lambda), \mu_j) = \lim_{\lambda\to\mu_j}(\lambda-\mu_j)G(\lambda),\ 1 \le j \le R. \tag{3.31}$$

Then, we can have

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v,\ \text{on } \mathbb{R}, \\ \tilde{G}^\pm \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty. \end{cases} \tag{3.32}$$

Directly applying the Sokhotski–Plemelj formula [9], we obtain the solutions

$$\tilde{G}(\lambda) = I_{n+1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G^- v(\xi)}{\xi-\lambda}\, d\xi,\ \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.33}$$

Further, computing the limit as $\lambda \to \mu_l$ leads to

$$\mathrm{lhs} = \lim_{\lambda\to\mu_l} \tilde{G} = F_l - \sum_{j\ne l}^R \frac{G_j}{\mu_l-\mu_j},\ \mathrm{rhs} = I_{n+1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi-\mu_l}\, d\xi,$$

where

$$F_l = \lim_{\lambda\to\mu_l} \frac{(\lambda-\mu_l)G(\lambda) - G_l}{\lambda-\mu_l},\ 1 \le l \le R, \tag{3.34}$$

and accordingly, we obtain the Gelfand–Levitan–Marchenko type equations

$$I_{n+1} - F_l + \sum_{j\ne l}^R \frac{G_j}{\mu_l-\mu_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi-\mu_l}\, d\xi = 0,\ 1 \le l \le R. \tag{3.35}$$

These equations determine solutions to the associated Riemann–Hilbert problems, namely the generalized matrix Jost solutions.

3.5. Recovery of the potential

To recover the potential matrix P from the generalized matrix Jost solutions, we expand

$$G^{+}(x,t,\lambda)=I_{n+1}+\frac{1}{\lambda}G_1^{+}(x,t)+O(\frac{1}{\lambda^2}),\ \ \lambda\to\infty. \tag{3.36}$$

Upon plugging this asymptotic expansion into the matrix spectral problems (3.1), the O(1) terms tell

$$P=\lim_{\lambda\to\infty}\lambda[G^{+}(\lambda),\Lambda]=-[\Lambda,G_1^{+}]. \tag{3.37}$$

We have to check an involution property for G_1^{+}:

$$(G_1^{+})^{\dagger}(-x,-t)=CG_1^{+}(x,t)C^{-1}. \tag{3.38}$$

This way, we obtain the solutions to the nonlocal complex reverse-spacetime mKdV equations (2.11):

$$p_j=-\alpha(G_1^{+})_{1,j+1},\ \ 1\le j\le n. \tag{3.39}$$

where $G_1^{+}=((G_1^{+})_{jl})_{(n+1)\times(n+1)}$.

This finishes the inverse scattering procedure from the scattering matrix $S(\lambda)$, through the jump matrix $G_0(\lambda)$ and the solution $\{G^{+}(\lambda),G^{-}(\lambda)\}$ to the associated Riemann–Hilbert problems, to the potential matrix P. The final potential P presents solutions to the nonlocal complex reverse-spacetime mKdV equations (2.11).

4. Soliton solutions

Let $N\in\mathbb{N}$ be arbitrary. Assume that s_{11} has N zeros $\{\lambda_k\in\mathbb{C},\ 1\le k\le N\}$, and $\hat{s}_{11}$ has other N zeros $\{\hat{\lambda}_k\in\mathbb{C},\ 1\le k\le N\}$. We also assume that all these zeros, λ_k and $\hat{\lambda}_k,\ 1\le k\le N$, are geometrically simple. Then, each of $\ker T^{+}(\lambda_k)$, $1\le k\le N$, contains only a single basis column vector, denoted by v_k, $1\le k\le N$; and each of $\ker T^{-}(\hat{\lambda}_k)$, $1\le k\le N$, a single basis row vector, denoted by $\hat{v}_k$, $1\le k\le N$:

$$T^{+}(\lambda_k)v_k=0,\ \ \hat{v}_kT^{-}(\hat{\lambda}_k)=0,\ \ 1\le k\le N. \tag{4.1}$$

The Riemann–Hilbert problems with the identity jump matrix, the canonical normalization conditions in (3.25) and the zero structures given in (4.1) can be solved explicitly [18,30], and consequently, we can recover the potential matrix P, which presents soliton solutions.

The choice of the identity jump matrix $G_0=I_{n+1}$ in the Riemann–Hilbert problems in (3.21) can be made under the conditions $s_{i1}=\hat{s}_{1i}=0,\ 2\le i\le n+1$, i.e., zero reflection coefficients in the scattering problem. Solutions to this kind of special Riemann–Hilbert problems can be formulated as follows:

$$G^{+}(\lambda)=I_{n+1}-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\ \ (G^{-})^{-1}(\lambda)=I_{n+1}+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_k}, \tag{4.2}$$

where $M=(m_{kl})_{N\times N}$ is a square matrix whose entries are determined by

$$m_{kl}=\begin{cases}\dfrac{\hat{v}_kv_l}{\lambda_l-\hat{\lambda}_k}, & \text{if } \lambda_l\ne\hat{\lambda}_k,\\ 0, & \text{if } \lambda_l=\hat{\lambda}_k,\end{cases}\qquad 1\le k,l\le N, \tag{4.3}$$

and an orthogonal condition

$$\hat{v}_kv_l=0,\ \ \text{if } \lambda_l=\hat{\lambda}_k,\ \ 1\le k,l\le N, \tag{4.4}$$

is required.

To satisfy the involution property (3.38), we arbitrarily take N zeros of $\det T^{+}(\lambda)$: $\{\lambda_k\in\mathbb{C}\}_{k=1}^{N}$, but determine N zeros of $\det T^{-}(\lambda)$ as follows:

$$\hat{\lambda}_k=\begin{cases}-\lambda_k^{*}, & \text{if } \lambda_k\notin i\mathbb{R},\ \ 1\le k\le N,\\ \text{any value}\in i\mathbb{R}, & \text{if } \lambda_k\in i\mathbb{R},\ \ 1\le k\le N.\end{cases} \tag{4.5}$$

Then, $\ker T^{+}(\lambda_k)$ and $\ker T^{-}(\lambda_k)$, $1\le k\le N$, can be determined by

$$v_k(x,t)=v_k(x,t,\lambda_k)=e^{i\lambda_k\Lambda x+i\lambda_k^3\Omega t}w_k,\ \ 1\le k\le N, \tag{4.6}$$

and

$$\hat{v}_k(x,t)=\hat{v}_k(x,t,\hat{\lambda}_k)=v_k^{\dagger}(-x,-t,-\lambda_k^{*})C=w_k^{\dagger}e^{-i\hat{\lambda}_k\Lambda x-i\hat{\lambda}_k^3\Omega t}C,\ \ 1\le k\le N, \tag{4.7}$$

respectively, where $w_k,\ 1\le k\le N$, are arbitrary column vectors but need to satisfy

$$w_k^{\dagger}Cw_l=0,\ \ \text{if } \lambda_l=\hat{\lambda}_k,\ \ 1\le k,l\le N, \tag{4.8}$$

This is a consequence of the orthogonal condition (4.4), and the case of $\lambda_k=\hat{\lambda}_k$ occurs only when $\lambda_k\in i\mathbb{R}$ and $\hat{\lambda}_k=-\lambda_k^{*}$.

Finally, under the orthogonal condition (4.8), we need to show that the solutions to the specific Riemann–Hilbert problems, determined by (4.2) and (4.3), satisfy (3.26), which implies that G_1^+ satisfies the reduction condition (3.38). Then, as a consequence, we can see that the nonlocal complex reverse-spacetime mKdV equations (2.11) possess the following N-soliton solutions:

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,j+1}, \ \ 1 \le j \le n, \tag{4.9}$$

where M is defined by (4.3), and $v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \dots, \hat{v}_{k,n+1})$ are determined by (4.6) and (4.7), respectively.

5. Concluding remarks

This paper proposed a class of nonlocal complex reverse-spacetime multicomponent modified Korteweg–de Vries (mKdV) equations from a kind of nonlocal group reductions, and constructed their inverse scattering transforms. The basic tool is the Riemann–Hilbert approach to matrix spectral problems. We determined solutions to the Riemann–Hilbert problems by applying the Sokhotski–Plemelj formula, and systematically presented soliton solutions to the nonlocal complex reverse-spacetime mKdV equations, based on the reflectionless transforms (or equivalently the Riemann–Hilbert problems with the identity jump matrix).

We point out that it would be very interesting to explore connections among soliton structures formulated via different approaches such as the Hirota direct method [15], the Wronskian technique [8,25] and the Darboux transformation [29]. It would also be important to generate different kinds of exact solutions in nonlinear dispersive waves, for example, lump solutions [27,28,35], Rossby wave solutions [36], solitonless solutions [20,21] and algebro-geometric solutions [12,19], through the Riemann–Hilbert technique.

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