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# Long-time asymptotics of a three-component coupled nonlinear Schrödinger system

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ABSTRACT

Starting from a specific example of $4 \times 4$ matrix spectral problems, an integrable coupled hierarchy, which includes a three-component coupled nonlinear Schrödinger system as the first nonlinear one, is generated, and an associated oscillatory Riemann–Hilbert problem is formulated. With the nonlinear steepest descent method, the leading long-time asymptotics for the Cauchy problem of the three-component coupled nonlinear Schrödinger system is computed, through deforming the oscillatory Riemann–Hilbert problem into a model one which is solvable.

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## 1. Introduction

A wide variety of significant physical phenomena is modeled by ordinary or partial differential equations [27]. Generally, the corresponding Cauchy problems are not solvable explicitly, and thus asymptotic expansions or limiting behaviors of solutions are apparently important. Regular perturbation, even with multiple scales, keeps integrable properties invariant (see, e.g., [49,50]). More difficult and interesting situations often arise when a regular perturbation expansion does not work, for which singular perturbation methods are required [27]. Sometimes, simple expansions exist but expansions of different kinds are valid in different regions of space and/or time. In such cases, boundary layers analysis of matched asymptotic expansions should be developed. We will focus on leading long-time asymptotics for nonlinear integrable systems in the physically interesting region $x = O(t)$, where $t$ and $x$ are the time and space variables.

In soliton theory, significant work on long-time asymptotics of integrable systems was conducted by Manakov [60] and Ablowitz and Newell [2]. For the leading long-time asymptotics of the nonlinear Schrödinger equation (NLS) in the physically interesting region $x = O(t)$, Zakharov and Manakov [77] provided precise expressions, depending explicitly on initial data. The method of Zakharov and Manakov takes an ansatz for the asymptotic form of the solution and utilizes some techniques which are removed from the classical framework of Riemann–Hilbert (RH) problems. Ablowitz and Segur [3]

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presented a complete description for the leading asymptotics for the Cauchy problem of the Korteweg–de Vries (KdV) equation. Segur and Ablowitz [68] started from a similarity solution form to work out the leading two terms in each of the asymptotic expansions for the amplitude and phase for the nonlinear Schrödinger equation, using conservation laws. Its [25] began to use the stationary phase idea to conjugate the RH problem associated with the NLS equation, up to small errors which decay as $t \to \infty$, by an appropriate parametrix to a model RH problem solvable by the technique from the theory of isomonodromic deformations. Deift and Zhou [12] determined the long-time asymptotics of the modified KdV equation, by deforming an associated oscillatory RH problem systematically and vigorously, in the spirit of the stationary phase method. Their technique, further developed in [10,13] and also in [11], opens a nonlinear steepest descent method to evaluate long-time asymptotics of integrable systems through estimating solutions to oscillatory RH problems generating from matrix spectral problems. McLaughlin and Miller [62] generalized the steepest descent method to the case when the jump matrix fails to be analytic and their method is now called a nonlinear $\bar{\partial}$ steepest descent method. A crucial ingredient of the Deift–Zhou approach is the asymptotic analysis of singular integrals on contours by deformations.

There have been many applications of the nonlinear steepest descent method to long-time asymptotics of integrable systems, including the KdV equation [4,20], the NLS equation [9,26], the sine–Gordon equation [6], the derivative nonlinear Schrödinger equation [28], the Camassa–Holm equation [64], the Kundu–Eckhaus equation [72] and the Fokas–Lenells equation [76]. One important factor in the nonlinear steepest descent method is the order of involved spectral matrices in the formulation of RH problems or equivalently the inverse scattering transformation. However, only $2 \times 2$ spectral matrices and their RH problems have been systematically analyzed (see, e.g., [71]), which engender algebro-geometric solutions to integrable systems expressed by hyperelliptic functions [19]. There have been very few $3 \times 3$ spectral matrices, whose long-time asymptotics or oscillatory RH problems are considered (see, e.g., [18,65]) and whose associated inverse scattering transforms are successfully formulated (see, e.g., [45,74]). Associated trigonal curves show much more diverse asymptotic behaviors and algebro-geometric solutions than hyperelliptic curves [39,40]. To best of our knowledge, there has been no application example of the nonlinear steepest descent method to the 4th-order or higher-order matrix spectral problems.

In this paper, we would like to consider a specific example of $4 \times 4$ matrix spectral problems

$$-i\phi_x = U\phi, \;\; U = \begin{bmatrix} -k & p_1 & p_2 & p_3 \\ \sigma p_1^* & k & 0 & 0 \\ \sigma p_2^* & 0 & k & 0 \\ \sigma p_3^* & 0 & 0 & k \end{bmatrix}, \tag{1.1}$$

where $i = \sqrt{-1}$, $\sigma = \pm 1$, $k$ is the spectral parameter, $p = (p_1, p_2, p_3)$ is a vector potential, and the superscript $*$ denotes the complex conjugate. This spectral problem generates a three-component coupled NLS system:

$$ip_{j,t} + p_{j,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + |p_3|^2)p_j = 0, \;\; 1 \le j \le 3, \tag{1.2}$$

where $|p_j|^2 = p_j p_j^*, \;\; 1 \le j \le 3$. The two values $\sigma = \pm 1$ correspond to the focusing and defocusing cases, respectively. There exist rogue-wave solutions in the focusing case [78], but there do not in the defocusing case. We will compute the leading asymptotics of this coupled NLS system by analyzing an associated oscillatory RH problem in the physically interesting region $x = O(t)$. RH problems have been made for an unreduced coupled NLS system in [48]. We point out that this coupled NLS system (1.2) has a slightly different matrix spectral problem from the one for the multiple wave interaction equations [56].

By $|M|$, let us denote an equivalent matrix norm for a matrix $M$ (may not be square):

$$|M| = [\mathrm{tr}(M^\dagger M)]^{\frac{1}{2}}, \tag{1.3}$$

where $M^\dagger$ denotes the Hermitian transpose of $M$, and $\mathcal{S}^3 = \{(f_1, f_2, f_3) | f_i \in \mathcal{S}, \;\; 1 \le i \le 3\}$, where $\mathcal{S}$ denotes the Schwartz space. The primary result of the paper is as follows.

**Theorem 1.1.** *Let $p(x,t) = (p_1(x,t), p_2(x,t), p_3(x,t))$ solves the Cauchy problem of the three-component coupled NLS system (1.2) with the initial data in $\mathcal{S}^3$. Suppose that $\gamma(k) = (\gamma_1(k), \gamma_2(k), \gamma_3(k))$ is the reflection coefficient vector in $\mathcal{S}^3$ associated with the initial data and satisfies*

$$\gamma^*(-k^*) = \gamma(k), \;\; \sup_{k \in \mathbb{R}} |\gamma(k)| < \infty, \;\; \text{when } \sigma = 1,$$

*or*

$$\gamma^*(-k^*) = \gamma(k), \;\; \sup_{k \in \mathbb{R}} |\gamma(k)| < 1, \;\; \text{when } \sigma = -1.$$

*Then, in the physically interesting region $x = O(t)$, the leading asymptotics of the solution is given by*

$$\begin{aligned} p(x,t) &= (p_1(x,t), p_2(x,t), p_3(x,t)) \\ &= \frac{1}{2\sqrt{\pi t}} \nu \Gamma(i\nu)(8t)^{-i\nu} e^{4ik_0^2 t + 2\tilde{\chi}(k_0) + \frac{\pi\nu}{2} - \frac{i\pi}{4}} \gamma(k_0) + O(\frac{\ln t}{t}), \end{aligned} \tag{1.4}$$

*where $\Gamma(\cdot)$ is the Gamma function and*

$$\begin{cases} k_0 = -\frac{x}{4t},\ \nu = -\frac{1}{2\pi}\ln(1+\sigma|\gamma(k_0)|^2), \\ \tilde{\chi}(k_0) = \frac{1}{2\pi i}\Big[\int_{-\infty}^{k_0-1}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k_0}d\xi \\ \qquad + \int_{k_0-1}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)-\ln(1+\sigma|\gamma(k_0)|^2)}{\xi-k_0}d\xi\Big]. \end{cases} \tag{1.5}$$

The rest of the paper is organized as follows. In Section 2, within the zero-curvature formulation, we derive an integrable coupled hierarchy, which includes the three-component coupled NLS system (1.2) as the first nonlinear one, and furnish its bi-Hamiltonian structure, starting from the $4\times 4$ matrix spectral problem (1.1). In Section 3, taking the coupled NLS system (1.2) as an example, we formulate an associated oscillatory RH problem for it. In Section 4, we evaluate its leading long-time asymptotics through deforming the oscillatory RH problem by the nonlinear steepest descent method. In the last section, we summarize our main results and give some remarks.

## 2. An integrable three-component coupled hierarchy

### 2.1. Zero curvature formulation

We start to recall the zero curvature formulation for constructing integrable hierarchies [70]. Let $u$ be a vector potential, $k$ be a spectral parameter, and $I_n$ stand for the $n$th order identity matrix. Assume that $U = U(u,k)$ is a square spectral matrix in a given matrix loop algebra (see, e.g., [37]), and

$$W = W(u,k) = \sum_{m=0}^{\infty} W_m k^{-m} = \sum_{m=0}^{\infty} W_m(u) k^{-m} \tag{2.1}$$

solves the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.2}$$

Using this solution $W$, we define an infinite sequence of Lax matrices

$$V^{[r]} = V^{[r]}(u,k) = (k^r W)_+ + \Delta_r,\ r \geq 0, \tag{2.3}$$

where the subscript $+$ denotes the operation of taking a polynomial part in $k$, and $\Delta_r$, $r \geq 0$, are appropriate modification terms such that the spatial and temporal matrix spectral problems

$$-i\phi_x = U\phi = U(u,k)\phi,\ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,k)\phi,\ r \geq 0, \tag{2.4}$$

are compatible, where $\phi$ is the matrix eigenfunction. This is guaranteed to occur if we require an infinite sequence of zero curvature equations, the compatibility conditions of (2.4),

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\ r \geq 0, \tag{2.5}$$

which essentially generates an integrable hierarchy

$$u_t = K_r(u) = K_r(x, t, u, u_x, \ldots),\ r \geq 0. \tag{2.6}$$

The matrices $U$ and $V^{[r]}$ are called a Lax pair [29] of the $r$th integrable system in the hierarchy (2.6). We point out that only for computational convenience in formulating a Riemann–Hilbert problem later, we have input the unit imaginary number $i$ in the above statement.

To show the Liouville integrability of the hierarchy (2.6), we normally try to furnish a bi-Hamiltonian structure [59]:

$$u_t = K_r = J_1\frac{\delta \tilde{H}_{r+1}}{\delta u} = J_2\frac{\delta \tilde{H}_r}{\delta u},\ r \geq 1, \tag{2.7}$$

where $J_1$ and $J_2$ constitute a Hamiltonian pair and $\frac{\delta}{\delta u}$ is the variational derivative (see, e.g., [50]). The Hamiltonian structures can usually be achieved through the trace identity [70]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial k})dx = k^{-\gamma'}\frac{\partial}{\partial k}\Big[k^{\gamma'}\mathrm{tr}(W\frac{\partial U}{\partial u})\Big],\ \gamma' = -\frac{k}{2}\frac{d}{dk}\ln|\mathrm{tr}(W^2)|, \tag{2.8}$$

or more generally, the variational identity [47]:

$$\frac{\delta}{\delta u}\int \langle W, \frac{\partial U}{\partial k}\rangle dx = k^{-\gamma'}\frac{\partial}{\partial k}\Big[k^{\gamma'}\langle W, \frac{\partial U}{\partial u}\rangle\Big],\ \gamma' = -\frac{k}{2}\frac{d}{dk}\ln|\langle W, W\rangle|, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra [35,52]. The bi-Hamiltonian structure most often guarantees the existence of infinitely many commuting Lie symmetries $\{K_n\}_{n=0}^{\infty}$ and conserved quantities $\{\tilde{H}_n\}_{n=0}^{\infty}$:

$$[K_{n_1}, K_{n_2}] = K'_{n_1}[K_{n_2}] - K'_{n_2}[K_{n_1}] = 0, \tag{2.10}$$

and

$$\{\tilde{\mathcal{H}}_{n_1}, \tilde{\mathcal{H}}_{n_2}\}_N = \int \Big(\frac{\delta \tilde{\mathcal{H}}_{n_1}}{\delta u}\Big)^T N \frac{\delta \tilde{\mathcal{H}}_{n_2}}{\delta u}\, dx = 0, \tag{2.11}$$

where $n_1, n_2 \geq 0$, $N = J_1$ or $J_2$, and $K'$ stands for the Gateaux derivative of $K$ with respect to $u$:

$$K'(u)[S] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u+\varepsilon S, u_x + \varepsilon S_x, \ldots).$$

Such Abelian algebras of symmetries and conserved quantities can also be generated directly from Lax pairs and the associated Lie bracket for Lax matrices is given by (see, e.g., [32–34]):

$$[\![V_{n_1}, V_{n_2}]\!] = V'_{n_1}[K_{n_2}] - V'_{n_2}[K_{n_1}] + [V_{n_1}, V_{n_2}],\ \ n_1, n_2 \geq 0.. \tag{2.12}$$

It is also known that for a system of evolution equations , $\tilde{H} = \int H\, dx$ is conserved if and only if $\frac{\delta \tilde{H}}{\delta u}$ is an adjoint symmetry [38,57], and thus, Hamiltonian structures connect conserved functionals with adjoint symmetries and further symmetries. Pairs of adjoint symmetries and symmetries precisely correspond to conservation laws [38,41].

When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations produce classical integrable hierarchies [16,69]; when semisimple, the associated zero curvature equations generate a set of different integrable hierarchies; and when non-semisimple, we obtain hierarchies of integrable couplings [53], which require extra care in recognizing their specific structures.

### *2.2. A three-component coupled hierarchy*

Let us start from a $4 \times 4$ matrix spectral problem

$$-i\phi_x = U\phi = U(u,k)\phi,\ \ U = (U_{jl})_{4\times 4} = \begin{bmatrix} -k & p_1 & p_2 & p_3 \\ \sigma p_1^* & k & 0 & 0 \\ \sigma p_2^* & 0 & k & 0 \\ \sigma p_3^* & 0 & 0 & k \end{bmatrix}, \tag{2.13}$$

where $u = p^T = (p_1, p_2, p_3)^T$ is a three-component vector potential. Since the leading matrix diag$(-1, 1, 1, 1)$ has a multiple eigenvalue, this spectral problem (2.13) is degenerate, which is slightly different from the case of multiple wave interaction equations, where we do not have zeros [56].

To derive an associated integrable coupled hierarchy, we first solve the stationary zero curvature equation (2.2) associated with (2.13). A solution $W$ is assumed to be determined by

$$W = \begin{bmatrix} a & b \\ \sigma b^\dagger & d \end{bmatrix}, \tag{2.14}$$

where $a$ is a real scalar, $b$ is three-dimensional row, and $d$ is a $3\times 3$ Hermitian matrix. Obviously, the stationary zero curvature equation (2.2) is now equivalent to

$$a_x = i\sigma(pb^\dagger - bp^\dagger),\ \ b_x = i(-2kb + pd - ap),\ \ d_x = i\sigma(p^\dagger b - b^\dagger p). \tag{2.15}$$

We search for a formal series solution as follows:

$$W = \sum_{m=0}^{\infty} W_m k^{-m},\ \ W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ \sigma b^{[m]\dagger} & d^{[m]} \end{bmatrix},\ \ m \geq 0, \tag{2.16}$$

with $b^{[m]}$ and $d^{[m]}$ being assumed to be

$$b^{[m]} = (b_1^{[m]}, b_2^{[m]}, b_3^{[m]}),\ \ d^{[m]} = (d_{jl}^{[m]})_{3\times 3},\ \ m \geq 0. \tag{2.17}$$

It is easy to see that the system (2.15) presents the following equivalent recursion relations:

$$b^{[0]} = 0,\ \ a_x^{[0]} = 0,\ \ d_x^{[0]} = 0, \tag{2.18a}$$

$$b^{[m+1]} = \frac{1}{2}(ib_x^{[m]} + pd^{[m]} - a^{[m]}p),\ \ m \geq 0, \tag{2.18b}$$

$$a_x^{[m]} = i\sigma(pb^{[m]\dagger} - b^{[m]}p^\dagger),\ \ d_x^{[m]} = i\sigma(p^\dagger b^{[m]} - b^{[m]\dagger}p),\ \ m \geq 1. \tag{2.18c}$$

The three-component NLS system (1.2) can correspond to the special initial values:

$$a^{[0]}=-3,\ d^{[0]}=I_3. \tag{2.19}$$

We also set all constants of integration in (2.18c) to be zero, that is, require

$$W_m|_{u=0}=0,\ m\geq 1. \tag{2.20}$$

Then, with $a^{[0]}$ and $d^{[0]}$ given by (2.19), all matrices $W_m,\ m\geq 1$, are uniquely determined. For example, based on (2.18), a direct computation tells

$$b_j^{[1]}=2p_j,\quad a^{[1]}=0,\ d_{jl}^{[1]}=0; \tag{2.21a}$$

$$b_j^{[2]}=ip_{j,x},\ a^{[2]}=\sigma\sum_{l=1}^{3}|p_l|^2,\ d_{jl}^{[2]}=-\sigma p_l p_j^*; \tag{2.21b}$$

$$b_j^{[3]}=-\frac{1}{2}\big[p_{j,xx}+2\sigma(\sum_{l=1}^{3}|p_l|^2)p_j\big], \tag{2.21c}$$

$$a^{[3]}=-\frac{1}{2}i\sigma\sum_{l=1}^{3}(p_lp_{l,x}^*-p_{l,x}p_l^*),\ d_{jl}^{[3]}=-\frac{1}{2}i\sigma(p_{l,x}p_j^*-p_lp_{j,x}^*); \tag{2.21d}$$

$$b_j^{[4]}=-\frac{1}{4}i[p_{j,xxx}+3\sigma(\sum_{l=1}^{3}|p_l|^2)p_{j,x}+3\sigma(\sum_{l=1}^{3}p_{l,x}p_l^*)p_j], \tag{2.21e}$$

$$a^{[4]}=-\frac{1}{4}[3(\sum_{l=1}^{3}|p_l|^2)^2+\sigma\sum_{l=1}^{3}(p_lp_{l,xx}^*-p_{l,x}p_{l,x}^*+p_{l,xx}p_l^*)], \tag{2.21f}$$

$$d_{jl}^{[4]}=\frac{1}{4}[3p_l(\sum_{l=1}^{3}|p_l|^2)p_j^*+\sigma(p_{l,xx}p_j^*-p_{l,x}p_{j,x}^*+p_lp_{j,xx}^*)]; \tag{2.21g}$$

where $1\leq j,l\leq 3$. Based on (2.18b) and (2.18c), we can also obtain a recursion relation for $b^{[m]}$ and $c^{[m]}=\sigma b^{[m]\dagger}$:

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T}\end{bmatrix}=\Psi\begin{bmatrix} c^{[m]} \\ b^{[m]T}\end{bmatrix},\ m\geq 1, \tag{2.22}$$

where $\Psi$ is a $6\times 6$ matrix operator

$$\Psi=-\frac{i}{2}\begin{bmatrix} (\partial+\sum_{l=1}^{3}q_l\partial^{-1}p_l)I_3+q\partial^{-1}p & -q\partial^{-1}q^T-(q\partial^{-1}q^T)^T \\ p^T\partial^{-1}p+(p^T\partial^{-1}p)^T & -(\partial+\sum_{l=1}^{3}p_l\partial^{-1}q_l)I_3-p^T\partial^{-1}q^T \end{bmatrix}, \tag{2.23}$$

with $q=\sigma p^\dagger$. It is obvious that we actually have

$$b^{[m+1]T}=-\frac{i}{2}\big\{\sigma[p^T\partial^{-1}p+(p^T\partial^{-1}p)^T]b^{[m]\dagger}-[(\partial+\sigma\sum_{l=1}^{3}p_l\partial^{-1}p_l^*)I_3+\sigma p^T\partial^{-1}p^*]b^{[m]T}\big\},\ m\geq 1. \tag{2.24}$$

To generate an integrable three-component coupled hierarchy, it is now standard to take, for all integers $r\geq 0$, the following Lax matrices

$$V^{[r]}=V^{[r]}(u,\lambda)=(V_{jl}^{[r]})_{4\times 4}=(\lambda^r W)_+=\sum_{s=0}^{r}W_s\lambda^{r-s},\ r\geq 0, \tag{2.25}$$

where the modification terms, $\Delta_r$ ($r\geq 0$), are set to be zero. The compatibility conditions of the matrix spectral problems in (2.4), i.e., the zero curvature equations (2.5), generate the integrable three-component coupled hierarchy:

$$u_t=p_t^T=K_r=-2ib^{[r+1]T},\ r\geq 0. \tag{2.26}$$

The first two nonlinear systems in this integrable coupled hierarchy (2.26) are the three-component coupled NLS system (1.2) and the three-component coupled modified Korteweg–de Vries system

$$p_{j,t}=-\frac{1}{2}\big[p_{j,xxx}+3\sigma(|p_1|^2+|p_2|^2+|p_3|^2)p_{j,x}+3\sigma(p_1^*p_{1,x}+p_2^*p_{2,x}+p_3^*p_{3,x})p_j\big],\ 1\leq j\leq 3. \tag{2.27}$$

The three-component integrable coupled hierarchy (2.26) with an extended six-component potential $u=(p,q^T)^T=(p,\sigma p^*)^T$ possesses a Hamiltonian structure [46,57], which can be worked out through the trace identity [70], or more generally, the variational identity [47]. By a direct computation, we have

$$-i\,\mathrm{tr}(W\frac{\partial U}{\partial k})=-a+\mathrm{tr}(d)=\sum_{m=0}^{\infty}(-a^{[m]}+d_{11}^{[m]}+\mathrm{tr}\,d_{22}^{[m]})k^{-m},$$

and

$$-i\,\mathrm{tr}(W\frac{\partial U}{\partial u})=\begin{bmatrix} c \\ b^T \end{bmatrix}=\sum_{m\geq 0}G_{m-1}k^{-m},$$

where $W=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $c=\sigma b^\dagger$. Inserting these into the trace identity and considering the case of $m=2$ tell $\gamma'=0$, and thus

$$\frac{\delta\tilde{H}_m}{\delta u}=iG_{m-1},\ \tilde{H}_m=-\frac{i}{m}\int(-a^{[m+1]}+d_{11}^{[m+1]}+\mathrm{tr}\,d_{22}^{[m+1]})\,dx,\ G_{m-1}=\begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix},\ m\geq 1, \tag{2.28}$$

where $c^{[m]}=\sigma b^{[m]\dagger}$. Then it follows that the extended six-component coupled NLS systems, consisting of (2.26) and its conjugate compartment, possesses the following bi-Hamiltonian structure:

$$u_t=K_r=J_1G_r=J_1\frac{\delta\tilde{H}_{r+1}}{\delta u}=J_2\frac{\delta\tilde{H}_r}{\delta u},\ r\geq 1, \tag{2.29}$$

where the Hamiltonian pair $(J_1,J_2=J_1\Psi)$ is given by

$$J_1=\begin{bmatrix} 0 & -2I_3 \\ 2I_3 & 0 \end{bmatrix}, \tag{2.30a}$$

$$J_2=i\begin{bmatrix} p^T\partial^{-1}p+(p^T\partial^{-1}p)^T & -(\partial+\sum_{k=1}^{3}p_k\partial^{-1}q_k)I_3-p^T\partial^{-1}q^T \\ -(\partial+\sum_{k=1}^{3}p_k\partial^{-1}q_k)I_3-q\partial^{-1}p & q\partial^{-1}q^T+(q\partial^{-1}q^T)^T \end{bmatrix}, \tag{2.30b}$$

where $q=\sigma p^\dagger$ again. Adjoint symmetry constraints (or almost equivalently symmetry constraints) decompose a four-component coupled NLS system with $p_3=q_3=0$ into two commuting finite-dimensional Liouville integrable Hamiltonian systems in [57]. In what follows, we will concentrate on the three-component coupled NLS system (1.2).

## 3. An associated oscillatory Riemann–Hilbert problem

For a matrix $M=M(k;x,t)$, we define its $\mathcal{L}^p$-norm as follows:

$$\|M\|_p=\||M|\|_p, \tag{3.1}$$

where $|M|$ denotes the matrix norm of $M$ given by (1.3) and $\|f\|_p$ is the $\mathcal{L}^p$-norm of a function $f$ of $k$. We assume that

$$A\lesssim B \text{ means } \exists\, C>0 \text{ such that } |A|\leq CB. \tag{3.2}$$

When $C$ depends on a few parameters $\alpha_1,\alpha_2,\ldots,\alpha_n$, we write $A\lesssim_{\alpha_1,\alpha_2,\ldots,\alpha_n}B$, but where appropriate, we often suppress some of the parameters for simplicity.

For an oriented contour in the complex plane, we denote the left-hand side by $+$ and the right-hand side by $-$, as one travels on the contour in the direction of the arrow. A Riemann–Hilbert (RH) problem $(\Gamma,J)$ on an oriented contour $\Gamma\subset\mathbb{C}$ (open or closed) with a jump matrix $J$ defined on $\Gamma$ is defined by

$$\begin{cases} M_+(k)=M_-(k)J(k),\ k\in\Gamma, \\ M(k)\to J_0,\ \text{as } k\to\infty, \end{cases} \tag{3.3}$$

where $J_0$ is a given matrix determining a boundary condition at infinity, and $M_\pm$ are analytic in the $\pm$ side regions and continuous to $\Gamma$ from the $\pm$ sides, and $M=M_\pm$ in the $\pm$ side regions, respectively.

### 3.1. An equivalent matrix spectral problem

In Section 2, we have seen that the matrix spectral problems of the three-component coupled NLS system (1.2) read

$$-i\phi_x=U\phi=U(u,k)\phi,\ -i\phi_t=V^{[2]}\phi=V^{[2]}(u,k)\phi, \tag{3.4}$$

with the Lax pair being of the form

$$U(u,k)=k\Lambda+P,\ \ V^{[2]}(u,k)=k^2\Omega+Q, \tag{3.5}$$

where $u=p^T=(p_1,p_2,p_3)^T$ and

$$\Lambda=\mathrm{diag}(-1,1,1,1),\ \ \Omega=\mathrm{diag}(-3,1,1,1). \tag{3.6}$$

The other two matrices $P$ and $Q$ are given by

$$P=\left[\begin{array}{cc}0 & p\\ \sigma p^\dagger & 0\end{array}\right],\ Q=\left[\begin{array}{cc}a^{[1]}k+a^{[2]} & b^{[1]}k+b^{[2]}\\ \sigma b^{[1]\dagger}k+\sigma b^{[2]\dagger} & d^{[1]}k+d^{[2]}\end{array}\right]=\left[\begin{array}{cc}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{array}\right], \tag{3.7}$$

where $a^{[m]}$, $b^{[m]}$, $d^{[m]}$, $1\le m\le 2$, are defined in **??**, and thus

$$\begin{cases} Q_{11}=\sigma pp^\dagger,\ \ Q_{12}=2pk+ip_x,\\ Q_{21}=2\sigma p^\dagger k-i\sigma p_x^\dagger,\ \ Q_{22}=-\sigma p^\dagger p.\end{cases} \tag{3.8}$$

As usual, for the spectral problems in (3.4), upon making the variable transformation

$$\phi=\psi E_g,\ \ E_g=\mathrm{e}^{ik\Lambda x+ik^2\Omega t}, \tag{3.9}$$

we can impose the canonical normalization condition:

$$\psi_\pm\to I_4,\ \ \text{when } x,t\to\pm\infty. \tag{3.10}$$

The equivalent pair of matrix spectral problems are as follows:

$$\psi_x=ik[\Lambda,\psi]+\check{P}\psi, \tag{3.11}$$

$$\psi_t=ik^2[\Omega,\psi]+\check{Q}\psi, \tag{3.12}$$

where $\check{P}=iP$ and $\check{Q}=iQ$. Noting $\mathrm{tr}(\check{P})=\mathrm{tr}(\check{Q})=0$, we have

$$\det\psi=1, \tag{3.13}$$

by a generalized Liouville's formula [54].

Applying the method of variation in parameters and using the canonical normalization condition (3.10), we can transform the $x$-part of (3.4) into the following Volterra integral equations for $\psi_\pm$ [66]:

$$\psi_-(k,x)=I_4+\int_{-\infty}^{x}\mathrm{e}^{ik\Lambda(x-y)}\check{P}(y)\psi_-(k,y)\mathrm{e}^{ik\Lambda(y-x)}\,dy, \tag{3.14}$$

$$\psi_+(k,x)=I_4-\int_{x}^{\infty}\mathrm{e}^{ik\Lambda(x-y)}\check{P}(y)\psi_+(\lambda,y)\mathrm{e}^{ik\Lambda(y-x)}\,dy. \tag{3.15}$$

Similarly, we can turn the $t$-part of (3.4) into the following Volterra integral equations:

$$\psi_-(k,t)=I_4+\int_{-\infty}^{t}\mathrm{e}^{ik^2\Omega(t-s)}\check{Q}(s)\psi_-(k,s)\mathrm{e}^{ik^2\Omega(s-t)}\,ds, \tag{3.16}$$

$$\psi_+(k,t)=I_4-\int_{t}^{\infty}\mathrm{e}^{ik^2\Omega(t-s)}\check{Q}(s)\psi_+(k,s)\mathrm{e}^{ik^2\Omega(s-t)}\,ds. \tag{3.17}$$

Based on the structures of $\Lambda$ and $\Omega$, we can show that the first column of $\psi_-$ and the last three columns of $\psi_+$ consist of analytical functions in the upper half plane $\mathbb{C}_+$, and the first column of $\psi_+$ and the last three columns of $\psi_-$ consist of analytical functions in the lower half plane $\mathbb{C}_-$ (see also [43,44,46]). All this will help us formulate an associated RH problem for the three-component coupled NLS system (1.2).

### 3.2. An oscillatory Riemann–Hilbert problem

The scattering matrix $S$ is determined through

$$\psi_-=\psi_+\mathrm{e}^{ikx\hat{\Lambda}+ik^2t\hat{\Omega}}S(k), \tag{3.18}$$

where $\mathrm{e}^{\alpha\hat{M}}X=\mathrm{e}^{\alpha M}X\mathrm{e}^{-\alpha M}$ for a scalar $\alpha$ and two same order square matrices $M$ and $X$. For simplicity, we also write

$$M^\dagger(k^*)=(M(k))^\dagger,\ \ M^{-1}(k^*)=(M(k^*))^{-1}, \tag{3.19}$$

for a matrix $M$ depending on $k\in\mathbb{C}$. It is due to

$$U^\dagger(k^*)=T_\sigma U(k^*)T_\sigma^{-1},\ \ V^{[3]\dagger}(k^*)=T_\sigma V^{[3]}(k^*)T_\sigma^{-1},\ \ T_\sigma=\mathrm{diag}(1,\sigma,\sigma,\sigma), \tag{3.20}$$

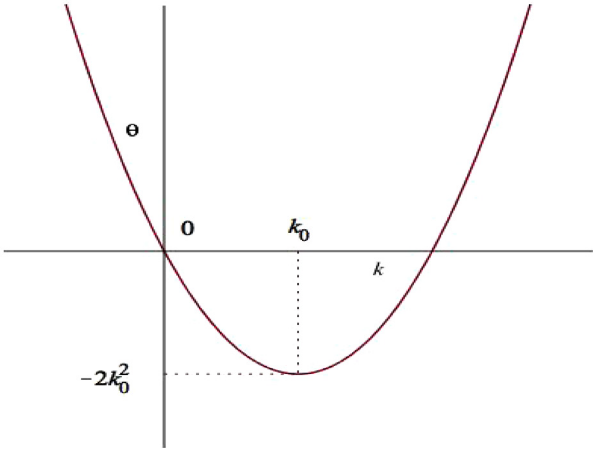


**Fig. 3.1.** Increasing and decreasing of $\theta$.

that we have

$$\psi^{\dagger}(k^*) = T_\sigma \psi^{-1}(k^*) T_\sigma^{-1}, \tag{3.21}$$

and

$$S^{\dagger}(k^*) = T_\sigma S^{-1}(k^*) T_\sigma^{-1}. \tag{3.22}$$

Note that $\det S(\lambda) = 1$ because of $\det \psi_\pm = 1$. It then follows from $\det S = 1$ and (3.22) that

$$S_{11}^*(k^*) = \det[S_{22}(k)],\ \ S_{21}^{\dagger}(k^*) = -\sigma S_{12}(k)\mathrm{adj}[S_{22}(k)],$$

where $\mathrm{adj}(M)$ denotes the adjoint matrix of $M$ and $S = (S_{jl})_{2\times 2}$ with $S_{22}$ being a $3 \times 3$ matrix block. Thus, the scattering matrix $S$ can be expressed as

$$S(k) = \begin{bmatrix} \det[\underline{a}^{\dagger}(k^*)] & \underline{b}(k) \\ -\sigma\,\mathrm{adj}[\underline{a}^{\dagger}(k^*)]\underline{b}^{\dagger}(k^*) & \underline{a}(k) \end{bmatrix}, \tag{3.23}$$

where $\underline{a}$ is a $3 \times 3$ matrix block.

Let us now introduce

$$M(k; x, t) = \begin{cases} (\frac{\psi_{-,L}(k)}{\det[\underline{a}^{\dagger}(k^*)]}, \psi_{+,R}(k)),\ \ k \in \mathbb{C}_+, \\ (\psi_{+,L}(k), \psi_{-,R}(k)\underline{a}^{-1}(k)),\ \ k \in \mathbb{C}_-, \end{cases} \tag{3.24}$$

where $\psi_{\pm,L}$ and $\psi_{\pm,R}$ denote the first column and the rest columns of $\psi_\pm$, respectively. Then, the matrix $M$ solves the following oscillatory RH problem

$$\begin{cases} M_+(k; x, t) = M_-(k; x, t)J(k; x, t),\ \ k \in \mathbb{R}, \\ M(k; x, t) \to I_4,\ \text{ as } k \to \infty, \end{cases} \tag{3.25}$$

where $M_\pm(k; x, t) = \lim_{\varepsilon\to 0^+} M(k \pm i\varepsilon, x, t)$ and the jump matrix is given by

$$J(k; x, t) = \begin{bmatrix} 1 + \sigma\gamma(k)\gamma^{\dagger}(k^*) & -e^{-2it\theta(k)}\gamma(k) \\ -\sigma e^{2it\theta(k)}\gamma^{\dagger}(k^*) & I_3 \end{bmatrix}, \tag{3.26}$$

with

$$\theta(k) = \theta(k; x, t) = \frac{kx}{t} + 2k^2,\ \ \gamma(k) = \underline{b}(k)\underline{a}^{-1}(k). \tag{3.27}$$

The behavior of the oscillatory factor $e^{2it\theta(k)}$ depends on the increasing and decreasing of $\theta(k)$ (see Fig. 3.1) and the signature of $\mathrm{Re}\, i\theta(k)$ (see Fig. 3.2). The associated oscillatory RH problem, defined by (3.25) and (3.26), is our starting point to explore long-time asymptotics for the three-component coupled NLS system (1.2).

In what follows, we assume that $\gamma$ lies in $\mathcal{S}^3$ and satisfies

$$\gamma^*(-k^*) = \gamma(k),\ \ 0 < 1 + \sigma \sup_{k\in\mathbb{R}} |\gamma(k)| < \infty, \tag{3.28}$$

where is equivalent to the conditions on $\gamma(k)$ in Theorem 1.1. The analysis on RH problems in [79] tells that the above RH problem (3.25) is essentially equivalent to a Fredholm integral equation of the second kind. For such Fredholm equations,

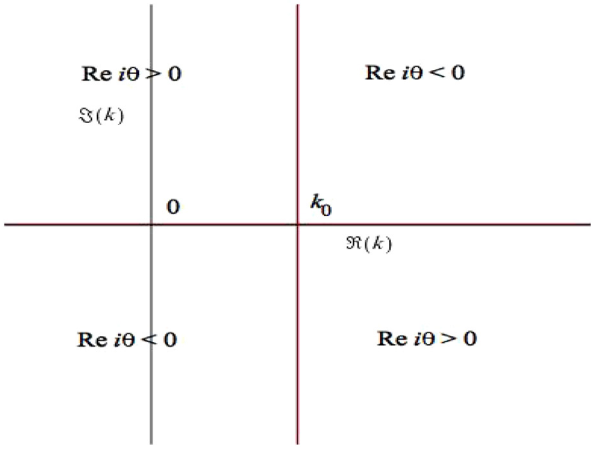


**Fig. 3.2.** The signature table of Re $i\theta$.

the existences and uniqueness of solutions can be guaranteed by satisfying the conditions in the vanishing lemma [1]. A direct computation also shows [42,46] that one can evaluate the potential $p(x,t)$ by using the solution $M(k;x,t)$ to the RH problem (3.25) as follows.

**Theorem 3.1.** *Assume that $\gamma$ lies in $\mathcal{S}^3$ and satisfies the conditions in (3.28). Then there exists a unique solution $M(k;x,t)$ to the Riemann–Hilbert problem (3.25), and the solution of the three-component coupled NLS system (1.2) is recovered via*

$$p(x,t) = (p_1(x,t), p_2(x,t), p_3(x,t)) = 2 \lim_{k\to\infty} (kM(k; ,x,t))_{12}, \tag{3.29}$$

*where $M$ is partitioned into a block matrix $M = (M_{jl})_{2\times 2}$, in which $M_{22}$ is a $3 \times 3$ matrix block.*

## 4. Long-time asymptotics

We will first deal with the Riemann–Hilbert (RH) problem presented by (3.25) and (3.26), and then compute the leading long-time aymptotics for the three-component coupled NLS system (1.2), through applying the nonlinear steepest descent method by Deift and Zhou [12]. We will focus on the physically interesting region $|\frac{x}{t}| \le C$, where $C$ is a constant.

### 4.1. Transformation of the RH problem

It is direct to see that the jump matrix $J$ has a upper-lower factorization and a lower-upper factorization:

$$J = \begin{bmatrix} 1 & -e^{-2it\theta(k)}\gamma(k) \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\sigma e^{2it\theta(k)}\gamma^\dagger(k^*) & I_3 \end{bmatrix}, \tag{4.1}$$

and

$$J = \begin{bmatrix} 1 & 0 \\ -\frac{\sigma e^{2it\theta(k)}\gamma^\dagger(k^*)}{1+\sigma\gamma(k)\gamma^\dagger(k^*)} & I_3 \end{bmatrix} J_c \begin{bmatrix} 1 & \frac{-e^{2it\theta(k)}\gamma(k)}{1+\sigma\gamma(k)\gamma^\dagger(k^*)} \\ 0 & I_3 \end{bmatrix}, \tag{4.2}$$

where

$$J_c = \begin{bmatrix} 1+\sigma\gamma(k)\gamma^\dagger(k^*) & 0 \\ 0 & (I_3+\sigma\gamma^\dagger(k^*)\gamma(k))^{-1} \end{bmatrix}.$$

We are going to introduce another RH problem to replace the upper-lower factorization by the lower-upper factorization to make the analytic and decay properties of the two factorizations to be consistent.

Obviously, the stationary point of $\theta$ is $k_0 = -\frac{x}{4t}$, by solving $\frac{d\theta}{dk}|_{k=k_0} = 0$. Let $\delta(k)$ solve the following RH problem

$$\begin{cases} \delta_+(k) = (I_3+\sigma\gamma^\dagger(k^*)\gamma(k))\delta_-(k), \ k<k_0, \\ \qquad\quad = \delta_-(k), \ k>k_0, \\ \delta(k)\to I_3, \ \text{as } k\to\infty, \end{cases} \tag{4.3}$$

which leads to a scalar RH problem

$$\begin{cases} \det \delta_+(k) = (1+\sigma|\gamma(k)|^2)\det\delta_-(k), \ \ k<k_0, \\ \qquad\qquad\ = \det\delta_-(k), \ \ k>k_0, \\ \det\delta(k) \to 1, \ \ \text{as } k\to\infty, \end{cases} \tag{4.4}$$

noting that

$$\det(I_3+\sigma\gamma^\dagger(k^*)\gamma(k)) = 1+\sigma\gamma(k)\gamma^\dagger(k^*) = 1+\sigma|\gamma(k)|^2,$$

where the norm of $\gamma$ is defined by (1.3).

It follows from (3.28) that the jump matrix $I_3+\sigma\gamma^\dagger(k^*)\gamma(k)$ is positive definite. Thus, the vanishing lemma (see, e.g., [1]) tells that the RH problem (4.3) has a unique solution $\delta(k)$. Also, the Plemelj formula [1] gives the unique solution of the above scalar RH problem (4.4):

$$\det\delta(k) = e^{\chi(k)}, \ \ \chi(k) = \frac{1}{2\pi i}\int_{-\infty}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k}d\xi. \tag{4.5}$$

Note that the above integral is singular as $k\to k_0$. We can actually express the integral as follows:

$$\begin{aligned} \int_{-\infty}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k}d\xi &= \int_{-\infty}^{k_0-1}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k}d\xi \\ &\quad + \ln(1+\sigma|\gamma(k_0)|^2)\int_{k_0-1}^{k_0}\frac{1}{\xi-k}d\xi \\ &\quad + \int_{k_0-1}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)-\ln(1+\sigma|\gamma(k_0)|^2)}{\xi-k}d\xi \\ &= \int_{-\infty}^{k_0-1}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k}d\xi \\ &\quad + \ln(1+\sigma|\gamma(k_0)|^2)\ln(k-k_0) \\ &\quad - \ln(1+\sigma|\gamma(k_0)|^2)\ln(k-k_0+1) \\ &\quad + \int_{k_0-1}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)-\ln(1+\sigma|\gamma(k_0)|^2)}{\xi-k}d\xi, \end{aligned}$$

where all the terms, with an exception $\ln(k-k_0)$, are analytic for $k$ in a neighborhood of $k_0$. Therefore, $\det\delta$ can be written as follows:

$$\det\delta(k) = (1+\frac{1}{k-k_0})^{i\nu}e^{\tilde{\chi}(k)}, \tag{4.6}$$

where

$$\nu = -\frac{1}{2\pi}\ln(1+\sigma|\gamma(k_0)|^2),$$

and

$$\begin{aligned} \tilde{\chi}(k_0) = \frac{1}{2\pi i}\Big[ &\int_{-\infty}^{k_0-1}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k_0}d\xi \\ &+ \int_{k_0-1}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)-\ln(1+\sigma|\gamma(k_0)|^2)}{\xi-k_0}d\xi\Big]. \end{aligned}$$

By the uniqueness for (4.3), we obtain

$$\delta(k) = (\delta^\dagger(k^*))^{-1}, \tag{4.7}$$

which implies that

$$\delta_\pm(k) = ((\delta_\mp(k))^\dagger)^{-1}.$$

It then follows that

$$|\delta_+(k)|^2 = \begin{cases} 3+\sigma|\gamma(k)|^2, \ \ k<k_0, \\ 3, \ \ k>k_0, \end{cases} \tag{4.8}$$

$$|\delta_-(k)|^2 = \begin{cases} 3-\frac{\sigma|\gamma(k)|^2}{1+\sigma|\gamma(k)|^2}, \ \ k<k_0, \\ 3, \ \ k>k_0, \end{cases} \tag{4.9}$$


**Fig. 4.1.** The oriented contour on $\mathbb{R}$.

and

$$|\det \delta_+(k)| \le \begin{cases} 1+|\gamma(k)|^2 < \infty, \ \sigma = 1, \\ 1, \ \sigma = -1, \end{cases} \tag{4.10}$$

$$|\det \delta_-(k)| \le \begin{cases} 1, \ \sigma = 1, \\ (1-|\gamma(k)|^2)^{-1} < \infty, \ \sigma = -1. \end{cases} \tag{4.11}$$

Therefore, by the maximum principle for analytic functions, we determine the boundedness

$$|\delta(k)| < \infty, \ |\det \delta(k)| < \infty, \ k \in \mathbb{C}, \tag{4.12}$$

from the canonical normalization condition of the above two RH problems.

Let us now define

$$\Delta(k) = \begin{bmatrix} \det \delta(k) & 0 \\ 0 & \delta^{-1}(k) \end{bmatrix}, \ k \in \mathbb{C}, \tag{4.13}$$

and a vector-valued spectral induced function

$$\rho(k) = \begin{cases} -\frac{\gamma(k)}{1+\sigma\gamma(k)\gamma^\dagger(k^*)}, & k < k_0, \\ \gamma(k), & k \ge k_0. \end{cases} \tag{4.14}$$

By $M^\Delta$, we denote

$$M^\Delta(k;x,t) = M(k;x,t)\Delta^{-1}, \tag{4.15}$$

and reverse the orientation for $k > k_0$ as shown in Fig. 4.1. Then, $M^\Delta$ solves the following RH problem on $\mathbb{R}$ oriented as in Fig. 4.1:

$$\begin{cases} M^\Delta_+(k;x,t) = M^\Delta_-(k;x,t)J^\Delta(k;x,t), \ k \in \mathbb{R}, \\ M^\Delta(k;x,t) \to I_4, \ \text{as } k \to \infty, \end{cases} \tag{4.16}$$

whose jump matrix is given by

$$\begin{aligned} J^\Delta(k;x,t) &= \Delta_-(k)J(k;x,t)\Delta_+^{-1}(k) \\ &= \begin{bmatrix} 1 & 0 \\ \frac{\sigma e^{2it\theta(k)}\delta_-^{-1}(k)\rho^\dagger(k^*)}{\det \delta_-(k)} & I_3 \end{bmatrix} \begin{bmatrix} 1 & e^{-2it\theta(k)}[\det \delta_+(k)]\rho(k)\delta_+(k) \\ 0 & I_3 \end{bmatrix}. \end{aligned} \tag{4.17}$$

We will deform this RH problem to evaluate the long-time asymptotics of the three-component coupled NLS system (1.2).

### *4.2. Decomposition of the spectral induced function*

In order to determine the required deformation, we first make a decomposition of the spectral induced function $\rho(k)$ defined by (4.14).

Let $L$ denote the contour

$$L : \{k = k_0 + we^{\frac{3\pi i}{4}} : -\infty < w < \infty\} \tag{4.18}$$

and $L^*$, the complex conjugate of $L$. Define $\Sigma$ as the contour

$$\Sigma = L \cup L^* \cup \mathbb{R} \tag{4.19}$$

with the orientation in Fig. 4.2. We will focus on the contour $\Sigma$, though any similar contour, the part $L$ of which locates in the region where $\mathrm{Re}\, i\theta(k)$ is positive, will work.

**Proposition 4.1.** *The vector-valued spectral induced function $\rho(k)$ has a decomposition on the real axis:*

$$\rho(k) = R(k) + h_1(k) + h_2(k), \ k \in \mathbb{R}, \tag{4.20}$$

*where $R(k)$ is piecewise rational, $h_2(k)$ has an analytic continuation from $\mathbb{R}$ to $L$ in the region where $\mathrm{Re}\, i\theta(k) > 0$, and where $R$, $h_1$ and $h_2$ satisfy*

$$R(k) = O((1+|k-k_0|^2)^{-1}), \ k \in \mathbb{C}, \tag{4.21}$$

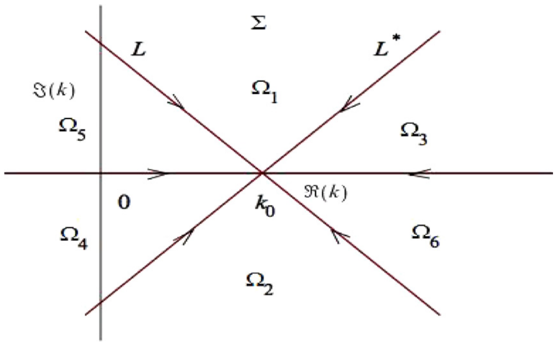


**Fig. 4.2.** The oriented jump contour $\Sigma$.

$$h_1(k)+h_2(k)=O((k-k_0)^{l+1}),\ \ k\in\mathbb{R}, \tag{4.22}$$

*and $h_1$ and $h_2$ have the estimates as $t\to\infty$:*

$$|e^{-2it\theta(k)}h_1(k)|\lesssim\frac{1}{(1+|k-k_0|^2)t^l},\ \ k\in\mathbb{R}, \tag{4.23}$$

$$|e^{-2it\theta(k)}h_2(k)|\lesssim\frac{1}{(1+|k-k_0|^2)t^l},\ \ k\in L, \tag{4.24}$$

*in which $l$ is an arbitrary positive integer. Taking the Hermitian conjugate*

$$\rho^\dagger(k^*)=R^\dagger(k^*)+h_1^\dagger(k^*)+h_2^\dagger(k^*) \tag{4.25}$$

*yields the same estimates for $e^{2it\theta(k)}h_1^\dagger(k^*)$ and $e^{2it\theta(k)}h_2^\dagger(k^*)$ on $\mathbb{R}$ and $L^*$, respectively.*

**Proof.** We consider the Fourier transform with respect to $\theta\in(-2k_0^2,\infty)$. Because $k\mapsto\theta(k)$ is one-to-one in $k<k_0$ or $k>k_0$ (see Fig. 3.1). In the proof, we adopt the differential notation $\bar{d}s=\frac{1}{\sqrt{2\pi}}ds$ for brevity, while dealing with the Fourier transform. We only consider the physically interesting region $x=O(t)$ and so $k_0$ is bounded. Let $r$ be a fixed positive integer:

$$r=4q+1,\ \ q\in\mathbb{Z}_+, \tag{4.26}$$

with an even number $q$.

(a) First, we consider the case of $k<k_0$. In this case, we have $\rho(k)=-\gamma(k)(1-\gamma(k)\gamma^\dagger(k^*))^{-1}$. By Taylor's theorem with the integral form of the remainder, we have

$$(k-k_0+i)^{r+6}\rho(k)=\sum_{j=0}^{r}\mu_j^-(k-k_0)^j+\frac{1}{r!}\int_{k_0}^{k}((\cdot-k_0+i)^{r+6}\rho(\cdot))^{(r+1)}(\xi)(k-\xi)^r d\xi. \tag{4.27}$$

Define

$$R(k)=\frac{1}{(k-k_0+i)^{r+6}}\sum_{j=0}^{r}\mu_j^-(k-k_0)^j, \tag{4.28}$$

and express

$$\rho(k)=h(k)+R(k),\ \ k<k_0. \tag{4.29}$$

Then, we have

$$\left.\frac{d^j\rho(k)}{dk^j}\right|_{k=k_0}=\left.\frac{d^jR(k)}{dk^j}\right|_{k=k_0},\ \ 0\le j\le r, \tag{4.30}$$

and the coefficients

$$\mu_j^-=\mu_j^-(k_0)=\frac{1}{j!}\frac{d^j}{dk^j}\big[(k-k_0+i)^{r+6}\rho(k)\big]\Big|_{k=k_0},\ \ 0\le j\le r, \tag{4.31}$$

decay rapidly as $k_0\to\infty$.

By the characteristic of $R$ in (4.28), we have

$$\frac{d^j h(k)}{dk^j}\Big|_{k=k_0} = 0,\ \ 0 \le j \le r. \tag{4.32}$$

Based on this property, we will try to split $h$ into two parts

$$h(k) = h_1(k) + h_2(k), \tag{4.33}$$

where $h_1$ is small and $h_2$ has an analytic continuation from $\mathbb{R}$ to $L$ in the region where $\mathrm{Re}\, i\theta(k) > 0$. This way, we obtain the required splitting of $\rho$. The properties of $R$ and $h$ in (4.21) and (4.22) are direct consequences of (4.28) and (4.32).

Let us introduce

$$\alpha(k) = \frac{(k-k_0)^q}{(k-k_0+i)^{q+2}}. \tag{4.34}$$

We consider the Fourier transform with respect to $\theta \in (-2k_0^2, \infty)$. Because $k \mapsto \theta(k)$ is one-to-one in $k < k_0$ (see Fig. 3.1), we define

$$\begin{cases} (h/\alpha)(k) = h(k(\theta))/\alpha(k(\theta)), & -2k_0^2 = \theta(k_0) < \theta < \infty, \\ \qquad\qquad = 0, & \theta \le -2k_0^2. \end{cases} \tag{4.35}$$

Based on (4.32), we have

$$(h/\alpha)(\theta) = \mathrm{O}((k(\theta)-k_0)^{r+1-q}),\ \ \theta \to \theta(k_0) = -2k_0^2; \tag{4.36}$$

and as $dk/d\theta = [4(k(\theta)-k_0)]^{-1}$, we see that

$$h/\alpha \in H^j(-\infty < \theta < \infty),\ \ 0 \le j \le 3q+2, \tag{4.37}$$

where the $H^j$s are Hilbert spaces. Now from the Fourier inversion theorem, we have

$$(h/\alpha)(k) = \int_{-\infty}^{\infty} \mathrm{e}^{is\theta(k)} (\widehat{h/\alpha})(s) \bar{d}s,\ \ k < k_0, \tag{4.38}$$

where $\widehat{h/\alpha}$ is the Fourier transform

$$(\widehat{h/\alpha})(s) = -\int_{-\infty}^{k_0} \mathrm{e}^{-is\theta(k)} (h/\alpha)(k) \bar{d}\theta(k),\ \ s \in \mathbb{R}. \tag{4.39}$$

By the formulae (4.27), (4.29) and (4.34), we have

$$(h/\alpha)(k) = \frac{(k-k_0)^{3q+2}}{(k-k_0+i)^{3q+5}} f(k, k_0), \tag{4.40}$$

where

$$f(k, k_0) = \frac{1}{r!} \int_0^1 \left((\cdot - k_0 + i)^{r+6} \rho(\cdot)\right)^{(r+1)} (k_0 + w(k-k_0))(1-w)^r dw, \tag{4.41}$$

from which we also have

$$\left|\frac{d^j f(k, k_0)}{dk^j}\right| \lesssim 1,\ \ k < k_0,\ \ j \ge 0.$$

Then, it follows that

$$\begin{aligned} &\int_{-\infty}^{k_0} \left|\left(\frac{d}{d\theta}\right)^j \frac{h}{\alpha}(k)\right|^2 |\bar{d}\theta(k)| \\ &= \int_{-\infty}^{k_0} \left|\left(\frac{1}{4(k-k_0)} \frac{d}{dk}\right)^j \frac{h}{\alpha}(k)\right|^2 |4(k-k_0)| \bar{d}k \\ &\lesssim \int_{-\infty}^{k_0} \left|\frac{(k-k_0)^{3q+2-2j}}{(k-k_0+i)^{3q+5}}\right|^2 (k-k_0) \bar{d}k \\ &\lesssim 1,\ \ 0 \le j \le \frac{3}{2} q + 1, \end{aligned} \tag{4.42}$$

from which, by the Plancherel theorem, we know

$$\int_{-\infty}^{\infty} (1+s^2)^j |(\widehat{h/\alpha})(s)|^2 ds \lesssim 1,\ \ 0 \le j \le \frac{3}{2} q + 1. \tag{4.43}$$

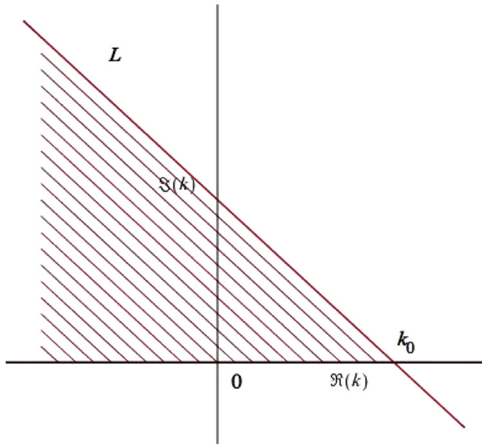


**Fig. 4.3.** Part of $\mathrm{Re}\, i\theta(k) > 0$ in the upper half $k$-plane.

Now make a splitting for $h$ as follows:

$$\begin{aligned} h(k) &= \alpha(k)\int_t^{\infty} \mathrm{e}^{is\theta(k)}(\widehat{h/\alpha})(s)\bar{d}s \\ &\quad + \alpha(k)\int_{-\infty}^{t} \mathrm{e}^{is\theta(k)}(\widehat{h/\alpha})(s)\bar{d}s \\ &= h_1(k) + h_2(k). \end{aligned} \tag{4.44}$$

On one hand, based on (4.43), for $k < k_0$, we have

$$\begin{aligned} |\mathrm{e}^{-2it\theta(k)}h_1(k)| &\le |\alpha(k)|\int_t^{\infty} |(\widehat{h/\alpha})(s)|\bar{d}s \\ &\le |\alpha(k)|\Big(\int_t^{\infty}(1+s^2)^{-j}\bar{d}s\Big)^{\frac12}\Big(\int_t^{\infty}(1+s^2)^j|(\widehat{h/\alpha})(s)|\bar{d}s\Big)^{\frac12} \\ &\le |\alpha(k)|\Big(\int_t^{\infty} s^{-2j}\bar{d}s\Big)^{\frac12}\Big(\int_{-\infty}^{\infty}(1+s^2)^j|(\widehat{h/\alpha})(s)|\bar{d}s\Big)^{\frac12} \\ &\lesssim \frac{1}{|k-k_0+i|^2}\Big(\int_t^{\infty} s^{-2j}\bar{d}s\Big)^{\frac12} \le \frac{1}{(1+|k-k_0|^2)t^{j-\frac12}},\ \ 1\le j\le \frac32 q+1. \end{aligned} \tag{4.45}$$

On the other hand, we know that $\mathrm{Re}\, i\theta(k)$ is positive in the hatched region in Fig. 4.3, and thus that $h_2(k)$ has an analytic continuation from $\mathbb{R}$ to the line $L$ in the upper half $k$-plane. Let $k$ be on the part of the line $L$ with $w \ge 0$. Then, we can have that

$$\begin{aligned} |\mathrm{e}^{-2it\theta(k)}h_2(k)| &\le |\alpha(k)|\mathrm{e}^{-t\,\mathrm{Re}\, i\theta(k)}\int_{-\infty}^{t} \mathrm{e}^{(s-t)\,\mathrm{Re}\, i\theta(k)}|(\widehat{h/\alpha})(s)|\bar{d}s \\ &\lesssim \frac{w^q}{1+|k-k_0|^2}\mathrm{e}^{-t\,\mathrm{Re}\, i\theta(k)}\Big(\int_{-\infty}^{t}\frac{\bar{d}s}{1+s^2}\Big)^{\frac12}\Big(\int_{-\infty}^{t}(1+s^2)|(\widehat{h/\alpha})(s)|^2\bar{d}s\Big)^{\frac12} \\ &\lesssim \frac{1}{1+|k-k_0|^2}w^q\mathrm{e}^{-t\,\mathrm{Re}\, i\theta(k)}, \end{aligned} \tag{4.46}$$

again based on (4.43). But $\mathrm{Re}\, i\theta(k) = 2w^2$. Therefore, we have

$$|\mathrm{e}^{-2it\theta(k)}h_2(k)| \lesssim \frac{1}{(1+|k-k_0|^2)t^{\frac q2}}t^{\frac q2}w^q\mathrm{e}^{-2tw^2} \lesssim \frac{1}{(1+|k-k_0|^2)t^{\frac q2}}.$$

This completes the proof for the case $k < k_0$.

(b) Second, we consider the case of $k \ge k_0$. In this case, we have $\rho(k) = \gamma(k),\ k \ge k_0$. Again, we use Taylor's theorem to obtain

$$(k-k_0-i)^{r+6}\rho(k) = \sum_{j=0}^{r}\mu_j^+(k-k_0)^j + \frac{1}{r!}\int_{k_0}^{k}((\cdot-k_0-i)^{r+6}\rho(\cdot))^{(r+1)}(\xi)(k-\xi)^r d\xi. \tag{4.47}$$

Similarly define

$$R(k)=\frac{1}{(k-k_0-i)^{r+6}}\sum_{j=0}^{r}\mu_j^+(k-k_0)^j, \tag{4.48}$$

and set

$$h(k)=\rho(k)-R(k). \tag{4.49}$$

As before, we know that

$$\frac{d^jh(k)}{dk^j}\Big|_{k=k_0}=0,\ 0\le j\le r, \tag{4.50}$$

and

$$\mu_j^+=\mu_j^+(k_0)=\frac{1}{j!}\frac{d^j}{dk^j}\big[(k-k_0-i)^{r+6}\rho(k)\big]\Big|_{k=k_0},\ 0\le j\le r, \tag{4.51}$$

decay rapidly as $k_0\to\infty$. The properties of $R$ and $h$ in (4.21) and (4.22) are direct consequences of (4.48) and (4.50).

Let us similarly introduce

$$\beta(k)=\frac{(k-k_0)^q}{(k-k_0-i)^{q+2}}. \tag{4.52}$$

Following the Fourier inversion theorem, we have

$$(h/\beta)(k)=(h/\beta)(k(\theta))=\int_{-\infty}^{\infty}\mathrm{e}^{is\theta(k)}(\widehat{h/\beta})(s)\bar{d}s,\ k\ge k_0, \tag{4.53}$$

where $\widehat{h/\beta}$ is the Fourier transform

$$(\widehat{h/\beta})(s)=\int_{k_0}^{\infty}\mathrm{e}^{-is\theta(k)}(h/\beta)(k)\bar{d}\theta(k)=\int_{-2k_0^2}^{\infty}\mathrm{e}^{-is\theta}(h/\beta)(k(\theta))\bar{d}\theta,\ s\in\mathbb{R}. \tag{4.54}$$

Based on the formulae (4.47), (4.49) and (4.52), we see that

$$(h/\beta)(k)=\frac{(k-k_0)^{3q+2}}{(k-k_0-i)^{3q+5}}g(k,k_0), \tag{4.55}$$

where

$$g(k,k_0)=\frac{1}{r!}\int_0^1\big((\cdot-k_0-i)^{r+6}\rho(\cdot)\big)^{(r+1)}(k_0+w(k-k_0))(1-w)^r dw, \tag{4.56}$$

from which it follows that

$$\Big|\frac{d^jg(k,k_0)}{dk^j}\Big|\lesssim 1,\ k\ge k_0,\ j\ge 0.$$

Now, we can similarly compute that

$$\begin{aligned}&\int_{k_0}^{\infty}\Big|\Big(\frac{d}{d\theta}\Big)^j\Big(\frac{h}{\beta}\Big)(k)\Big|^2\bar{d}\theta(k)\\ &=\int_{k_0}^{\infty}\Big|\Big(\frac{1}{4(k-k_0)}\frac{d}{dk}\Big)^j\Big(\frac{h}{\beta}\Big)(k)\Big|^2[4(k-k_0)]\bar{d}k\\ &\lesssim 1,\ 0\le j\le\frac{3}{2}q+1.\end{aligned} \tag{4.57}$$

By the Plancherel theorem,

$$\int_{-\infty}^{\infty}(1+s^2)^j|(\widehat{h/\beta})(s)|^2ds<\infty,\ 0\le j\le\frac{3}{2}q+1. \tag{4.58}$$

Again, we split

$$\begin{aligned}h(k)&=\beta(k)\int_t^{\infty}\mathrm{e}^{is\theta(k)}(\widehat{h/\beta})(s)\bar{d}s\\ &\quad+\beta(k)\int_{-\infty}^{t}\mathrm{e}^{is\theta(k)}(\widehat{h/\beta})(s)\bar{d}s\\ &=h_1(k)+h_2(k).\end{aligned} \tag{4.59}$$

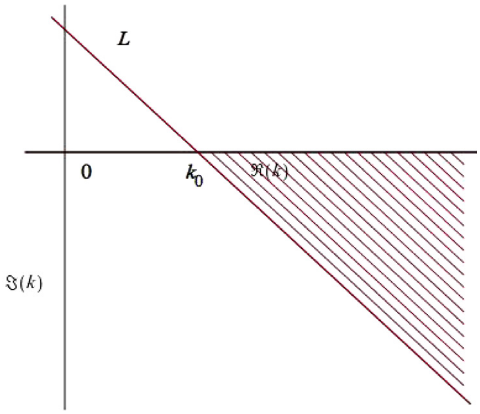


**Fig. 4.4.** Part of $\operatorname{Re} i\theta(k) > 0$ in the lower half $k$-plane.

On one hand, for $k \geq k_0$, similarly as in the previous example (4.45), we see that

$$|e^{-2it\theta(k)}h_1(k)| \lesssim \frac{1}{|k-k_0-i|^2 t^{j-\frac{1}{2}}},\ \ 0 \leq j \leq \frac{3}{2}q+1.$$

On the other hand, $h_2(k)$ has an analytic continuation from $\mathbb{R}$ to the part of the line $L$:

$$k(w) = k_0 + we^{-\frac{\pi i}{4}},\ \ w \geq 0, \tag{4.60}$$

in the lower half $k$-plane, as shown in Fig. 4.4. Let $k$ be on the ray (4.60). Similarly as in the previous case $k < k_0$, we can show that

$$|e^{-2it\theta(k)}h_2(k)| \lesssim \frac{w^q e^{-t\operatorname{Re} i\theta(k)}}{|k-k_0-i|^{q+2}}.$$

But we know $\operatorname{Re} i\theta(k) = 2w^2$, and thus, we have

$$|e^{-2it\theta(k)}h_2(k)| \lesssim \frac{(t^{\frac{q}{2}}w)^q e^{-2tw^2}}{(1+|k-k_0|^2)t^{\frac{q}{2}}} \lesssim \frac{1}{(1+|k-k_0|^2)t^{\frac{q}{2}}},$$

since $k_0$ is bounded. This completes the proof for the case $k \geq k_0$. □

We are now ready to make a deformation of the RH problem (4.16).

### 4.3. Deformation of the RH problem

Let us first state what a deformation of a RH problem is. Suppose that we have a RH problem $(\Gamma, J)$ on an oriented contour $\Gamma$ (see Fig. 4.5):

$$\begin{cases} M_+(k) = M_-(k)J(k),\ \ k \in \Gamma, \\ M(k) \to J_0,\ \ \text{as } k \to \infty, \end{cases} \tag{4.61}$$

and that on a part (which could be the whole contour $\Gamma$) of $\Gamma$ from $k_1$ to $k_2$ in the direction of $\Gamma$, denoted by $\Gamma_{k_1k_2}$, the jump matrix $J$ has a factorization

$$J(k) = b_-^{-1}(k)J_1(k)b_+(k),\ \ k \in \Gamma_{k_1k_2}, \tag{4.62}$$

where $b_\pm$ have invertible and analytic continuations to the $\pm$ sides of a region $D$ (see Fig. 4.5) supported by $k_1$ and $k_2$, respectively. We introduce an extended oriented contour $\Gamma'$ (see Fig. 4.5):

$$\Gamma' = \Gamma \cup \partial D,\ \ \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_{k_1k_2},\ \ \partial D = B_+ \cup B_-, \tag{4.63}$$

and an extended jump matrix $J'$:

$$\begin{cases} J'(k) = J(k),\ \ k \in \Gamma \setminus \Gamma_{k_1k_2}, \\ J'(k) = J_1(k),\ \ k \in \Gamma_{k_1k_2}, \\ J'(k) = b_+(k),\ \ k \in B_+, \\ J'(k) = b_-^{-1}(k),\ \ k \in B_-. \end{cases} \tag{4.64}$$

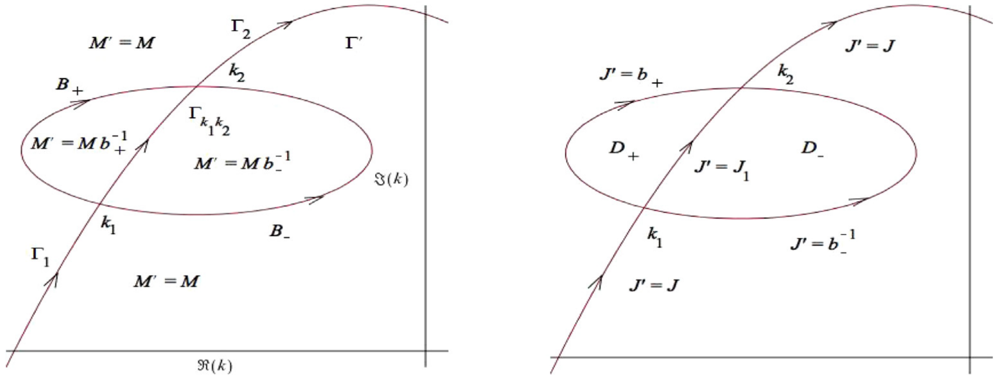


**Fig. 4.5.** Deformation of a RH problem.

Obviously, the original RH problem (4.61) on $\Gamma$ is equivalent to the following RH problem on $\Gamma'$:

$$\begin{cases} M'_+(k) = M'_-(k)J'(k), \ k \in \Gamma', \\ M'(k) \to J_0, \ \text{as } k \to \infty, \end{cases} \tag{4.65}$$

and the relation between the two solutions is given by

$$\begin{cases} M'(k) = M(k), \ k \in \mathbb{C} \setminus D, \\ M'(k) = M(k)b_+^{-1}(k), \ k \in D_+, \\ M'(k) = M(k)b_-^{-1}(k), \ k \in D_-. \end{cases} \tag{4.66}$$

It is clear that one RH problem is solvable if and only if the other RH problem is solvable, and the solution to the one problem gives the solution to the other problem precisely by (4.66). We call $(\Gamma', J')$ a deformation of the RH problem $(\Gamma, J)$. When $D$ is not bounded, we have to require that $b_+(k)$ and $b_-(k)$ tend to the identity matrix as $k \to \infty$, in order to keep the same normalization condition.

We now deform the RH problem (4.16) from $\mathbb{R}$ to the augmented contour $\Sigma$. To the end, we observe that the jump matrix $J^\Delta(k; , x, t)$ can be factorized as

$$J^\Delta(k; x, t) = (b_-)^{-1} b_+, \ b_\pm = I_4 \pm \omega_\pm, \ k \in \mathbb{R}, \tag{4.67}$$

where

$$\omega_+ = \begin{bmatrix} 0 & e^{-2it\theta(k)}[\det \delta_+(k)]\rho(k)\delta_+(k) \\ 0 & 0 \end{bmatrix}, \ \omega_- = \begin{bmatrix} 0 & 0 \\ \frac{\sigma e^{2it\theta(k)}\delta_-^{-1}(k)\rho^\dagger(k^*)}{\det \delta_-(k)} & 0 \end{bmatrix}. \tag{4.68}$$

Moreover, using the decomposition of $\rho$ (4.20), we can make

$$\begin{aligned} \omega_+ &= \omega_+^o + \omega_+^a \\ &= \begin{bmatrix} 0 & e^{-2it\theta(k)}[\det \delta_+(k)]h_1(k)\delta_+(k) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e^{-2it\theta(k)}[\det \delta_+(k)][h_2(k) + R(k)]\delta_+(k) \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{4.69}$$

$$\omega_- = \omega_-^o + \omega_-^a = \begin{bmatrix} 0 & 0 \\ \frac{\sigma e^{2it\theta(k)}\delta_-^{-1}(k)h_1^\dagger(k^*)}{\det \delta_-(k)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{\sigma e^{2it\theta(k)}\delta_-^{-1}(k)[h_2^\dagger(k^*)+R^\dagger(k^*)]}{\det \delta_-(k)} & 0 \end{bmatrix}, \tag{4.70}$$

and hence, we have the factorizations for $b_\pm$:

$$\begin{aligned} b_+ &= b_+^o b_+^a = (I_4 + \omega_+^o)(I_4 + \omega_+^a) \\ &= \begin{bmatrix} 1 & e^{-2it\theta(k)}[\det \delta_+(k)]h_1(k)\delta_+(k) \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} 1 & e^{-2it\theta(k)}[\det \delta_+(k)][h_2(k) + R(k)]\delta_+(k) \\ 0 & I_3 \end{bmatrix}, \end{aligned} \tag{4.71}$$

$$\begin{aligned} b_- &= b_-^o b_-^a = (I_4 - \omega_-^o)(I_4 - \omega_-^a) \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{\sigma e^{2it\theta(k)}\delta_-^{-1}(k)h_1^\dagger(k^*)}{\det \delta_-(k)} & I_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{\sigma e^{2it\theta(k)}\delta_-^{-1}(k)[h_2^\dagger(k^*)+R^\dagger(k^*)]}{\det \delta_-(k)} & I_3 \end{bmatrix}. \end{aligned} \tag{4.72}$$

Further based on (4.67), the jump matrix $J^\Delta$ has the following factorization:

$$J^\Delta(k; x, t) = (b_-^a)^{-1}[(b_-^o)^{-1} b_+^o] b_+^a = (b_-^a)^{-1} J_1 b_+^a, \ k \in \mathbb{R}. \tag{4.73}$$

We point out that we can accurately find that

$$b^o_\pm = \omega_\pm|_{\rho=h_1},\ b^a_\pm = \omega_\pm|_{\rho=h_2+R}.$$

It is now a standard procedure to introduce

$$M^\sharp(k;x,t) = \begin{cases} M^\Delta(k;x,t), & k\in \Omega_1\cup\Omega_2,\\ M^\Delta(k;x,t)(b^a_-)^{-1}, & k\in\Omega_3\cup\Omega_4,\\ M^\Delta(k;x,t)(b^a_+)^{-1}, & k\in\Omega_5\cup\Omega_6,\end{cases} \tag{4.74}$$

and deform the original RH problem (4.16) on $\mathbb{R}$ to the following new RH problem on $\Sigma$:

$$\begin{cases} M^\sharp_+(k;x,t) = M^\sharp_-(k;x,t)J^\sharp(k;x,t),\ k\in\Sigma,\\ M^\sharp(k;x,t)\to I_4,\ k\to\infty,\end{cases} \tag{4.75}$$

whose jump matrix reads

$$J^\sharp = (b^\sharp_-)^{-1}b^\sharp_+,\ b^\sharp_+ = \begin{cases} b^o_+, & k\in\mathbb{R},\\ b^a_+, & k\in L,\\ I_4, & k\in L^*,\end{cases} \quad b^\sharp_- = \begin{cases} b^o_-, & k\in\mathbb{R},\\ I_4, & k\in L,\\ b^a_-, & k\in L^*.\end{cases} \tag{4.76}$$

The canonical normalization condition in (4.75) can be verified, indeed. For example, $(b^a_+)^{-1}$ converges to $I_4$ as $k\to\infty$ in $\Omega_6$, because we observe that for fixed $x,t$, by the definition of $h_2$ in (4.59) and the boundedness of $\det\delta$ and $\delta$ in (4.12), we have

$$\begin{aligned} |e^{-2it\theta(k)}[\det\delta(k)]h_2(k)\delta(k)| &\lesssim |\beta(k)|e^{-t\,\mathrm{Re}\,i\theta(k)}\int_{-\infty}^t e^{(s-t)\,\mathrm{Re}\,i\theta(k)}|(\widehat{h/\beta})(s)|\bar{d}s\\ &\le \frac{|k-k_0|^q}{|k-k_0-i|^{q+2}}\int_{-\infty}^t |(\widehat{h/\beta})(s)|\bar{d}s \lesssim \frac{1}{|k-k_0-i|^2};\end{aligned}$$

and by the definition of $R$ in (4.48) and the boundedness of $\det\delta$ and $\delta$ in (4.12), we have

$$|e^{-2it\theta(k)}[\det\delta(k)]R(k)\delta(k)| \lesssim \frac{|\sum_{j=0}^r \mu_j^+(k-k_0)^j|}{|k-k_0-i|^{r+6}} \lesssim \frac{1}{|k-k_0-i|^6};$$

both of which converge to 0 as $k\to\infty$ in $\Omega_6$.

It is known that the above RH problem (4.75) can be solved by using the Cauchy operators as follows (see [5,8]). Let

$$(C_\pm f)(k) = \int_\Sigma \frac{f(\xi)}{\xi-k_\pm}\frac{d\xi}{2\pi i},\ k\in\Sigma,\ f\in\mathcal{L}^2(\Sigma) \tag{4.77}$$

denote the two Cauchy operators on $\Sigma$. As is well known, the two operators $C_\pm$ are bounded from $\mathcal{L}^2(\Sigma)$ to $\mathcal{L}^2(\Sigma)$, and $C_+ - C_- = 1$. Define

$$C_{\omega^\sharp}f = C_+(f\omega^\sharp_-) + C_-(f\omega^\sharp_+) \tag{4.78}$$

for a $4\times 4$ matrix-valued function $f$, where

$$\omega^\sharp_\pm = \pm(b^\sharp_\pm - I_4),\ \omega^\sharp = \omega^\sharp_+ + \omega^\sharp_-. \tag{4.79}$$

Assume that $\mu^\sharp = \mu^\sharp(k;x,t)\in\mathcal{L}^2(\Sigma)+\mathcal{L}^\infty(\Sigma)$ solves a basic inverse equation

$$\mu^\sharp = I_4 + C_{\omega^\sharp}\mu^\sharp. \tag{4.80}$$

Then

$$M^\sharp(k;x,t) = I_4 + \int_\Sigma \frac{\mu^\sharp(k;x,t)\omega^\sharp(k;x,t)}{\xi-k}\frac{d\xi}{2\pi i},\ k\in\mathbb{C}\setminus\Sigma, \tag{4.81}$$

presents the unique solution of the RH problem (4.75).

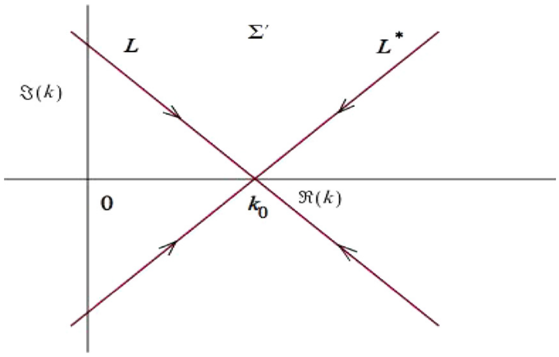


**Fig. 4.6.** The oriented contour $\Sigma'$.

**Theorem 4.2.** *The solution $p(x,t)$ of the three-component coupled NLS system (1.2) is expressed by*

$$\begin{aligned} &p(x,t) = (p_1(x,t), p_2(x,t), p_3(x,t)) \\ &= 2\lim_{k\to\infty}(kM^\sharp(k;,x,t))_{12} \\ &= i(\int_\Sigma \mu^\sharp(\xi;x,t)\omega^\sharp(\xi)\frac{d\xi}{\pi})_{12} \\ &= i(\int_\Sigma ((1-C_{\omega^\sharp})^{-1}I_4)(\xi)\omega^\sharp(\xi)\frac{d\xi}{\pi})_{12}. \end{aligned} \tag{4.82}$$

**Proof.** This statement can be shown by Theorem 3.1, (4.15) and (4.74). □

*4.4. Second contour deformation and reduction*

Let $\Sigma'$ be the reduced contour $\Sigma' = \Sigma \setminus \mathbb{R}$ with the orientation as in Fig. 4.6. We further deform the RH problem (4.75) from $\Sigma$ to $\Sigma'$, and estimate the difference between the two RH problems, one on $\Sigma$ and the other on $\Sigma'$.

Let $\omega^e : \Sigma \to M(4,\mathbb{C})$ be a sum of two terms

$$\omega^e = \omega^a + \omega^b, \tag{4.83}$$

where $\omega^a = \omega^\sharp \restriction \mathbb{R}$ is supported on $\mathbb{R}$ and is composed of the contribution to $\omega^\sharp$ from terms of type $h_1(k)$ and $h_1^\dagger(k^*)$ and $\omega^b = \omega^\sharp \restriction L \cup L^*$ is supported on $L \cup L^*$ and is composed of terms of type $h_2(k)$ and $h_2^\dagger(k^*)$.

Define

$$\omega' = \omega^\sharp - \omega^e. \tag{4.84}$$

Obviously $\omega' = 0$ on $\mathbb{R}$, and thus, $\omega'$ is supported on $\Sigma'$ and is composed of the contribution to $\omega^\sharp$ from terms of type $R(k)$ and $R^\dagger(k^*)$.

**Lemma 4.3.** *For an arbitrary positive integer $l$, we have*

$$\|\omega^a\|_{\mathcal{L}^1(\mathbb{R})\cap\mathcal{L}^2(\mathbb{R})\cap\mathcal{L}^\infty(\mathbb{R})} \lesssim t^{-l}, \tag{4.85}$$

$$\|\omega^b\|_{\mathcal{L}^1(L\cup L^*)\cap\mathcal{L}^2(L\cup L^*)\cap\mathcal{L}^\infty(L\cup L^*)} \lesssim t^{-l}, \tag{4.86}$$

$$\|\omega'\|_{\mathcal{L}^2(\Sigma)} \lesssim t^{-\frac{1}{4}}, \quad \|\omega'\|_{\mathcal{L}^1(\Sigma)} \lesssim t^{-\frac{1}{2}}. \tag{4.87}$$

**Proof.** The estimates (4.85) and (4.86). can be derived, similarly to Proposition 4.1. Based on the definition of $R(k)$, we can see that

$$|R(k)| \lesssim (1+|k-k_0|^6)^{-1}$$

on the line $L = \{k = k_0 + w e^{\frac{3\pi i}{4}} : -\infty < w < \infty\}$. Then using (4.12), we find that

$$|e^{-2it\theta(k)}[\det\delta(k)]R(k)\delta(k)| \lesssim e^{-4w^2 t}(1+|k-k_0|^6)^{-1},$$

since $\mathrm{Re}\, i\theta(k) = 2w^2$ on the line $L$. It then follows from a direct computation that the estimate (4.87) holds. □

Similarly to Proposition 2.23 in [12], we can get the following estimate.

**Proposition 4.4.** *As $t \to \infty$, the inverse of the operator $1 - C_{\omega'} : \mathcal{L}^2(\Sigma) \to \mathcal{L}^2(\Sigma)$ exists and is uniformly bounded:*

$$\|(1 - C_{\omega'})^{-1}\|_{\mathcal{L}^2(\Sigma)} \lesssim 1. \tag{4.88}$$

**Corollary 4.5.** *As $t \to \infty$, the inverse of the operator $1 - C_{\omega^\sharp} : \mathcal{L}^2(\Sigma) \to \mathcal{L}^2(\Sigma)$ exists and is uniformly bounded:*

$$\|(1 - C_{\omega^\sharp})^{-1}\|_{\mathcal{L}^2(\Sigma)} \lesssim 1. \tag{4.89}$$

**Proof.** By $C_{\omega^\sharp} = C_{\omega'} + C_{\omega^e}$, we have

$$\|C_{\omega^\sharp} - C_{\omega'}\|_{\mathcal{L}^2(\Sigma)} = \|C_{\omega^e}\|_{\mathcal{L}^2(\Sigma)} \lesssim \|\omega^e\|_{\mathcal{L}^\infty(\Sigma)}.$$

Also, it follows from Lemma 4.3 that

$$\|\omega^e\|_{\mathcal{L}^\infty(\Sigma)} \le \|\omega^a\|_{\mathcal{L}^\infty(\Sigma)} + \|\omega^b\|_{\mathcal{L}^\infty(\Sigma)} \lesssim t^{-l}.$$

Therefore, we obtain

$$\|C_{\omega^\sharp} - C_{\omega'}\|_{\mathcal{L}^2(\Sigma)} \lesssim t^{-l}. \tag{4.90}$$

Now first from $1 - C_{\omega^\sharp} = (1 - C_{\omega'}) - (C_{\omega^\sharp} - C_{\omega'})$, we know, based on (4.90), that $(1 - C_{\omega^\sharp})^{-1}$ exists. Second, the second resolvent identity implies

$$(1 - C_{\omega'})^{-1}(C_{\omega^\sharp} - C_{\omega'})(1 - C_{\omega^\sharp})^{-1} = (1 - C_{\omega'})^{-1} - (1 - C_{\omega^\sharp})^{-1}. \tag{4.91}$$

Again based on (4.90), we see, using this identity, that the estimate in the corollary follows from Proposition 4.4. □

**Theorem 4.6.** *As $t \to \infty$, we have*

$$\int_\Sigma \big((1 - C_{\omega^\sharp})^{-1} I_4\big)(\xi)\omega^\sharp(\xi)d\xi = \int_\Sigma \big((1 - C_{\omega'})^{-1} I_4\big)(\xi)\omega'(\xi)d\xi + O(t^{-l}). \tag{4.92}$$

**Proof.** First, from (4.91), we can obtain

$$\begin{aligned} \big((1 - C_{\omega^\sharp})^{-1} I_4\big)\omega^\sharp =& \big((1 - C_{\omega'})^{-1} I_4\big)\omega' + \omega^e + \big((1 - C_{\omega'})\big)^{-1}\big(C_{\omega^e} I_4\big)\omega^\sharp \\ &+ \big((1 - C_{\omega'})^{-1}(C_{\omega'} I_4)\big)\omega^e \\ &+ \big((1 - C_{\omega'})^{-1} C_{\omega^e}(1 - C_{\omega^\sharp})^{-1}\big)(C_{\omega^\sharp} I_4)\omega^\sharp. \end{aligned} \tag{4.93}$$

Second, directly using Lemma 4.3, we can compute that

$$\begin{aligned} &\|\omega'\|_{\mathcal{L}^1(\Sigma)} \le \|\omega^a\|_{\mathcal{L}^1(\Sigma)} + \|\omega^b\|_{\mathcal{L}^1(L\cup L^*)} \lesssim t^{-l}, \\ &\|\big((1 - C_{\omega'})^{-1}(C_{\omega^e} I_4)\big)\omega^\sharp\|_{\mathcal{L}^1(\Sigma)} \le \|(1 - C_{\omega'})^{-1}\|_{\mathcal{L}^2(\Sigma)}\|C_{\omega^e} I_4\|_{\mathcal{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathcal{L}^2(\Sigma)} \\ &\quad \lesssim \|\omega^e\|_{\mathcal{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathcal{L}^2(\Sigma)} \lesssim \|\omega^e\|_{\mathcal{L}^2(\Sigma)}(\|\omega^e\|_{\mathcal{L}^2(\Sigma)} + \|\omega'\|_{\mathcal{L}^2(\Sigma)}) \lesssim t^{-l-\frac{1}{4}}, \\ &\|\big((1 - C_{\omega'})^{-1}(C_{\omega^e} I_4)\big)\omega^e\|_{\mathcal{L}^1(\Sigma)} \le \|(1 - C_{\omega'})^{-1}\|_{\mathcal{L}^2(\Sigma)}\|C_{\omega'} I_4\|_{\mathcal{L}^2(\Sigma)}\|\omega^e\|_{\mathcal{L}^2(\Sigma)} \\ &\quad \lesssim \|\omega'\|_{\mathcal{L}^2(\Sigma)}\|\omega^e\|_{\mathcal{L}^2(\Sigma)} \lesssim t^{-l-\frac{1}{4}}, \\ &\|\big((1 - C_{\omega'})^{-1} C_{\omega^e}(1 - C_{\omega^\sharp})^{-1}\big)(C_{\omega^\sharp} I_4)\omega^\sharp\|_{\mathcal{L}^1(\Sigma)} \\ &\quad \le \|(1 - C_{\omega'})^{-1}\|_{\mathcal{L}^2(\Sigma)}\|C_{\omega^e}\|_{\mathcal{L}^2(\Sigma)}\|(1 - C_{\omega^\sharp})^{-1}\|_{\mathcal{L}^2(\Sigma)}\|C_{\omega^\sharp} I_4\|_{\mathcal{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathcal{L}^2(\Sigma)} \\ &\quad \lesssim \|\omega^e\|_{\mathcal{L}^\infty(\Sigma)}\|\omega^\sharp\|^2_{\mathcal{L}^2(\Sigma)} \lesssim t^{-l-\frac{1}{2}}. \end{aligned}$$

Now, together with (4.93), this completes the proof of the theorem. □

Note that as $k \in \mathbb{R}$, $\omega'(k) = 0$, we can reduce $C_{\omega'}$ from $\mathcal{L}^2(\Sigma)$ to $\mathcal{L}^2(\Sigma')$, and for simplicity, we still denote this reduced operator by $C_{\omega'}$. Thus

$$\int_\Sigma \big((1 - C_{\omega'})^{-1} I_4\big)(\xi)\omega'(\xi)d\xi = \int_{\Sigma'} \big((1 - C_{\omega'})^{-1} I_4\big)(\xi)\omega'(\xi)d\xi,$$

and then from Theorems 4.2 and 4.6, we obtain the following theorem.

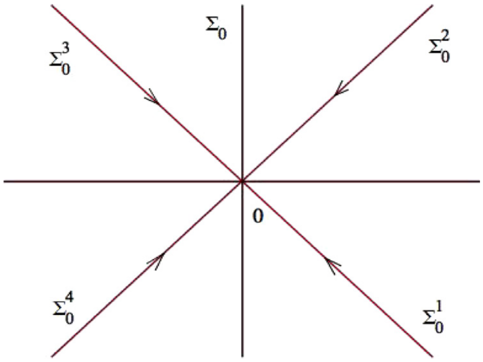


**Fig. 4.7.** Oriented contour $\Sigma_0$.

**Theorem 4.7.** *As $t \to \infty$, we have*

$$p(x,t) = i\Big(\int_{\Sigma'} \big((1-C_{\omega'})^{-1}I_4\big)(\xi)\omega'(\xi)\frac{d\xi}{\pi}\Big)_{12} + O(t^{-l}). \tag{4.94}$$

Denote $\mu' = (1-C_{\omega'})^{-1}I_4$, and then the matrix

$$M'(k;x,t) = I_4 + \int_{\Sigma'} \frac{\mu'(k;x,t)\omega'(k;x,t)}{\xi - k}\frac{d\xi}{2\pi i} \tag{4.95}$$

presents a solution to the following RH problem

$$\begin{cases} M'_+(k;x,t) = M'_-(k;x,t)J'(k;x,t), \ k \in \Sigma', \\ M'(k;x,t) \to I_4, \ \text{as } k \to \infty, \end{cases} \tag{4.96}$$

whose jump matrix is given by

$$J' = (b'_-)^{-1}b'_+, \ b'_\pm = I_4 \pm \omega'_\pm, \ \omega' = \omega'_+ + \omega'_-, \tag{4.97}$$

$$\omega'_+(k) = \begin{bmatrix} 0 & e^{-2i\theta(k)}[\det\delta(k)]R(k)\delta(k) \\ 0 & 0 \end{bmatrix}, \ \omega'_-(k) = 0, \ k \in L, \tag{4.98}$$

$$\omega'_+(k) = 0, \ \omega'_-(k) = \begin{bmatrix} 0 & 0 \\ \frac{\sigma e^{2i\theta(k)}\delta^{-1}(k)R^\dagger(k^*)}{\det\delta(k)} & 0 \end{bmatrix}, \ k \in L^*. \tag{4.99}$$

*4.5. Rescaling and second reduction*

Let $\Sigma_0$ denote the contours $\{k = we^{\pm\frac{\pi i}{4}} : w \in \mathbb{R}\}$ oriented inward as in $\Sigma'$, (see Fig. 4.7). Motivated by the method of stationary phase, define the scaling transformations

$$N : \mathcal{L}^2(\Sigma') \to \mathcal{L}^2(\Sigma_0), \ f \mapsto Nf, \ (Nf)(k) = f(k_0 + \frac{k}{\sqrt{8t}}), \tag{4.100}$$

and denote

$$\hat\omega = N\omega'. \tag{4.101}$$

A direct change-of-variable argument tells that

$$C_{\omega'} = N^{-1}C_{\hat\omega}N, \tag{4.102}$$

where $C_{\hat\omega}$ is a bounded operator from $\mathcal{L}^2(\Sigma_0)$ into $\mathcal{L}^2(\Sigma_0)$. On the line

$$\hat L = \{k = \alpha e^{\frac{3\pi i}{4}} : -\infty < \alpha < \infty\} \tag{4.103}$$

we have

$$\hat\omega(k) = \hat\omega_+(k) = \begin{bmatrix} 0 & (Ns_1)(k) \\ 0 & 0 \end{bmatrix}, \tag{4.104}$$

and on the conjugate line $\hat L^*$, we have

$$\hat\omega(k) = \hat\omega_-(k) = \begin{bmatrix} 0 & 0 \\ (Ns_2)(k) & 0 \end{bmatrix}, \tag{4.105}$$

where

$$s_1(k)=\mathrm{e}^{-2it\theta(k)}[\det\delta(k)]R(k)\delta(k),\ \ s_2(k)=-\frac{\sigma\,\mathrm{e}^{2it\theta(k)}\delta^{-1}(k)R^{\dagger}(k^*)}{\det\delta(k)}. \tag{4.106}$$

**Lemma 4.8.** *As $t\to\infty$, we have the estimates*

$$|(N\tilde{\delta})(k)|\lesssim t^{-l},\ \ k\in\hat{L}, \tag{4.107}$$

*where* $\tilde{\delta}(k)=e^{-2it\theta(k)}[R(k)\delta(k)-(\det\delta(k))R(k)]$, *and*

$$|(N\hat{\delta})(k)|\lesssim t^{-l},\ \ k\in\hat{L}^*, \tag{4.108}$$

*where* $\hat{\delta}(k)=-\sigma e^{2it\theta(k)}[\delta^{-1}(k)R^{\dagger}(k^*)-(\det\delta)^{-1}(k)R^{\dagger}(k^*)]$.

**Proof.** We only prove the estimate (4.107) and the proof of the other estimate is completely similar.

It follows directly from (4.3) and (4.4) that $\tilde{\delta}$ satisfies the following RH problem

$$\begin{cases}\tilde{\delta}_+(k)=\tilde{\delta}_-(k)(1+\sigma|\gamma(k)|^2)+\mathrm{e}^{-2it\theta(k)}f(k),\ \ k<k_0,\\ \tilde{\delta}_+(k)=\tilde{\delta}_-(k),\ \ k>k_0,\\ \tilde{\delta}(k)\to 0,\ \ \text{as } k\to\infty,\end{cases} \tag{4.109}$$

where

$$f(k)=R(k)(\gamma^{\dagger}(k^*)\gamma(k)-|\gamma(k)|^2I_3)\delta_-(k). \tag{4.110}$$

The solution of this vector RH problem can be determined by

$$\tilde{\delta}(k)=X(k)\int_{-\infty}^{k_0}\frac{\mathrm{e}^{-2it\theta(\xi)}f(\xi)}{X_+(\xi)(\xi-k)}\frac{d\xi}{2\pi i}, \tag{4.111}$$

with

$$X(k)=\mathrm{e}^{\frac{1}{2\pi i}\int_{-\infty}^{k_0}\frac{\ln(1+\sigma|\gamma(\xi)|^2)}{\xi-k}d\xi}. \tag{4.112}$$

Observing that

$$R\gamma^{\dagger}\gamma-R|\gamma|^2I_3=(R-\rho)\gamma^{\dagger}\gamma-(R-\rho)|\gamma|^2I_3=(h_1+h_2)\gamma^{\dagger}\gamma-(h_1+h_2)|\gamma|^2I_3,$$

we can have $f(k)=\mathrm{O}((k-k_0)^{l+1})$ when $k\to k_0$, upon noting the definition of $h=h_1+h_2$, and decompose $f(k)$ into two parts: $f(k)=f_1(k)+f_2(k)$, where $f_1(k)$ satisfies

$$|\mathrm{e}^{-2it\theta(k)}f_1(k)|\lesssim\frac{1}{(1+|k-k_0+\frac{1}{t}|^2)t^l},\ \ k\in\mathbb{R}, \tag{4.113}$$

and $f_2(k)$ has an analytical continuation from $\mathbb{R}$ to $L_t$ (see Fig. 4.8):

$$L_t=\{k=k_0-\frac{1}{t}+w\mathrm{e}^{\frac{3\pi i}{4}}:0\le w<\infty\} \tag{4.114}$$

and satisfies

$$|\mathrm{e}^{-2it\theta(k)}f_2(k)|\lesssim\frac{1}{(1+|k-k_0+\frac{1}{t}|^2)t^l},\ \ k\in L_t. \tag{4.115}$$

Let $k\in\hat{L}$, and we decompose

$$\begin{aligned}(N\tilde{\delta})(k)=&X(k_0+\frac{k}{\sqrt{8t}})\int_{k_0-\frac{1}{t}}^{k_0}\frac{\mathrm{e}^{-2it\theta(\xi)}f(\xi)}{X_+(\xi)(\xi-k_0-\frac{k}{\sqrt{8t}})}\frac{d\xi}{2\pi i}\\ &+X(k_0+\frac{k}{\sqrt{8t}})\int_{-\infty}^{k_0-\frac{1}{t}}\frac{\mathrm{e}^{-2it\theta(\xi)}f_1(\xi)}{X_+(\xi)(\xi-k_0-\frac{k}{\sqrt{8t}})}\frac{d\xi}{2\pi i}\\ &+X(k_0+\frac{k}{\sqrt{8t}})\int_{-\infty}^{k_0-\frac{1}{t}}\frac{\mathrm{e}^{-2it\theta(\xi)}f_2(\xi)}{X_+(\xi)(\xi-k_0-\frac{k}{\sqrt{8t}})}\frac{d\xi}{2\pi i}\\ :=&T_1+T_2+T_3.\end{aligned}$$

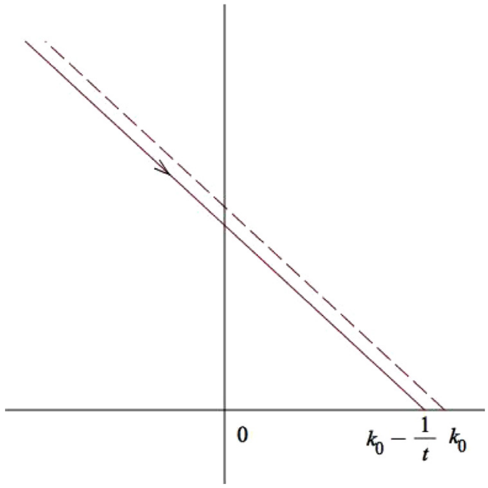


**Fig. 4.8.** The oriented contour $L_t$.

For the first and second terms, we can have

$$|T_1| \lesssim \int_{k_0-\frac{1}{t}}^{k_0} \frac{|f(\xi)|}{|\xi - k_0 - \frac{k}{\sqrt{8t}}|} d\xi \lesssim t^{-l} \frac{k}{\sqrt{t}} \ln|1 - \frac{2\sqrt{2}}{kt^{\frac{1}{2}}}| \lesssim t^{-l-1}, \ k \neq 0 \ (k = 0, \ \text{obvious}),$$
$$|T_2| \lesssim \int_{-\infty}^{k_0-\frac{1}{t}} \frac{|e^{-2it\theta(\xi)} f_1(\xi)|}{|\xi + k_0 - \frac{k}{\sqrt{8t}}|} d\xi \lesssim t^{-l}\sqrt{2}\,t \int_{-\infty}^{k_0-\frac{1}{t}} \frac{1}{1 + |\xi - k_0 + \frac{1}{t}|^2} d\xi \lesssim t^{-l+1},$$

which are due to $f(k) = O((k - k_0)^{l+1})$ and

$$|\xi - k_0 - \frac{k}{\sqrt{8t}}| \geq \frac{1}{\sqrt{2}\,t}, \ \xi \in (-\infty, k_0 - \frac{1}{t}), \ k \in \hat{L},$$

respectively. By using Cauchy's Theorem, we can have a similar estimate $|T_3| \lesssim t^{-l+1}$ through computing the integral in $T_3$ along the contour $L_t$ instead of the interval $(-\infty, k_0 - \frac{1}{t})$. Finally, combining those three estimates, we obtain the estimate (4.107). The proof is finished. □

Express the jump matrix $J^0$ as

$$J^0 = (I_4 - \omega_-^0)^{-1}(I_4 + \omega_+^0), \tag{4.116}$$

where

$$\omega^0 = \omega_+^0 = \begin{cases} \begin{bmatrix} 0 & \delta_0^2 k^{2\nu i} e^{-\frac{ik^2}{2}} \gamma(k_0) \\ 0 & 0 \end{bmatrix}, & k \in \Sigma_0^1, \\ \begin{bmatrix} 0 & -\delta_0^2 k^{2\nu i} e^{-\frac{ik^2}{2}} \frac{\gamma(k_0)}{1+\sigma|\gamma(k_0)|^2} \\ 0 & 0 \end{bmatrix}, & k \in \Sigma_0^3, \end{cases} \tag{4.117}$$

$$\omega^0 = \omega_-^0 = \begin{cases} \begin{bmatrix} 0 & 0 \\ -\sigma \delta_0^{-2} k^{-2\nu i} e^{\frac{ik^2}{2}} \gamma^\dagger(k_0) & 0 \end{bmatrix}, & k \in \Sigma_0^2, \\ \begin{bmatrix} 0 & 0 \\ \sigma \delta_0^{-2} k^{-2\nu i} e^{\frac{ik^2}{2}} \frac{\gamma^\dagger(k_0)}{1+\sigma|\gamma(k_0)|^2} & 0 \end{bmatrix}, & k \in \Sigma_0^4, \end{cases} \tag{4.118}$$

with

$$\delta_0 = (8t)^{-\frac{1}{2}i\nu} e^{2ik_0^2 t + \tilde{\chi}(k_0)}. \tag{4.119}$$

Based on (4.107), (4.108) and Lemma 3.35 in [12], we can obtain

$$\|\omega' - \omega^0\|_{\mathcal{L}^\infty(\Sigma_0)\cap\mathcal{L}^1(\Sigma_0)\cap\mathcal{L}^2(\Sigma_0)} \lesssim_{k_0} \frac{\ln t}{\sqrt{t}}. \tag{4.120}$$

Therefore, we have

$$\begin{aligned}
&\int_{\Sigma'} \big((1-C_{\omega'})^{-1}I_4\big)(\xi)\omega'(\xi)d\xi \\
&= \int_{\Sigma'} \big(N^{-1}(1-C_{\hat\omega})^{-1}NI_4\big)(\xi)\omega'(\xi)d\xi \\
&= \int_{\Sigma'} \big((1-C_{\hat\omega})^{-1}I_4\big)((\xi-k_0)\sqrt{8t})N\omega'((\xi-k_0)\sqrt{8t})d\xi \\
&= \frac{1}{\sqrt{8t}}\int_{\Sigma_0} \big((1-C_{\hat\omega})^{-1}I_4\big)(\xi)\hat\omega(\xi)d\xi \\
&= \frac{1}{\sqrt{8t}}\int_{\Sigma_0} \big((1-C_{\omega^0})^{-1}I_4\big)(\xi)\omega^0(\xi)d\xi + O(\frac{\ln t}{t}).
\end{aligned}$$

It now follows that

$$p(x,t) = \frac{i}{\sqrt{8t}}\Big(\int_{\Sigma_0} \big((1-C_{\omega^0})^{-1}I_4\big)(\xi)\omega^0(\xi)\frac{d\xi}{\pi}\Big)_{12} + O(\frac{\ln t}{t}). \tag{4.121}$$

For $k \in \mathbb{C}\setminus\Sigma_0$, we set

$$M^0(k) = I_4 + \int_{\Sigma_0} \frac{\big((1-C_{\omega^0})^{-1}I_4\big)(\xi)\omega^0(\xi)}{\xi-k}\frac{d\xi}{2\pi i}, \tag{4.122}$$

which solves the following RH problem

$$\begin{cases} M^0_+(k;x,t) = M^0_-(k;x,t)J^0(k;x,t), \ k\in\Sigma_0, \\ M^0(k;x,t)\to I_4, \ \text{as } k\to\infty. \end{cases} \tag{4.123}$$

Particularly, from an asymptotic expansion

$$M^0(k) = I_4 + \frac{M^0_1}{k} + O(k^{-2}), \ k\to\infty,$$

we get

$$M^0_1 = -\int_{\Sigma_0} \big((1-C_{\omega^0})^{-1}I_4\big)(\xi)\omega^0(\xi)\frac{d\xi}{2\pi i}.$$

Now from (4.121), we obtain

$$p(x,t) = \frac{1}{\sqrt{2t}}(M^0_1)_{12} + O(\frac{\ln t}{t}). \tag{4.124}$$

### *4.6. A model RH problem with an explicit solution*

To determine $(M^0_1)_{12}$ explicitly, we solve a model RH problem

$$\Psi_+(k) = \Psi_-(k)w(k_0), \ w(k_0) = k^{i\nu\hat\Lambda}e^{-\frac14 ik^2\hat\Lambda}\delta_0^{\hat\Lambda}J^0. \tag{4.125}$$

The solution to this RH problem is given by

$$\Psi(k) = H(k)k^{-i\nu\Lambda}e^{\frac14 ik^2\Lambda}, \ H(k) = \delta_0^{\hat\Lambda}M^0 = \delta_0^{\Lambda}M^0(k)\delta_0^{-\Lambda}, \tag{4.126}$$

where $\delta_0^{\Lambda} = \mathrm{diag}(\delta_0^{-1}, \delta_0 I_3)$ and $\delta_0^{-\Lambda} = (\delta_0^{\Lambda})^{-1}$.

Since the jump matrix $w(k_0)$ is independent of $k$ along each ray of $\Sigma_0$, we obtain

$$\frac{d\Psi_+(k)}{dk} = \frac{d\Psi_-(k)}{dk}w(k_0). \tag{4.127}$$

Together with (4.125), this implies that $\frac{d\Psi(k)}{dk}\Psi^{-1}(k)$ has no jump discontinuity along any of the rays of $\Sigma_0$. Directly from the solution (4.126), we obtain

$$\begin{aligned}
\frac{d\Psi(k)}{dk}\Psi^{-1}(k) &= \frac{dH(k)}{dk}H^{-1}(k) + \frac12 ikH(k)\Lambda H^{-1}(k) - \frac{i\nu}{k}H(k)\Lambda H^{-1}(k) \\
&= O(\frac1k) + \frac12 ik\Lambda - \frac12 i\delta_0^{\Lambda}[\Lambda, M^0_1]\delta_0^{-\Lambda}.
\end{aligned}$$

Then Liouville's theorem tells that

$$\frac{d\Psi(k)}{dk} - \frac12 ik\Lambda\Psi(k) = \beta\Psi(k), \tag{4.128}$$

where

$$\beta = -\frac{1}{2} i\delta_0^{\Lambda}[\Lambda, M_1^0]\delta_0^{-\Lambda} = \begin{bmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{bmatrix}. \tag{4.129}$$

In particular, we obtain

$$(M_1^0)_{12} = -i\delta_0^2 \beta_{12}. \tag{4.130}$$

Let us partition $\Psi(k)$ into the following form

$$\Psi(k) = \begin{bmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{21}(k) & \Psi_{22}(k) \end{bmatrix}, \tag{4.131}$$

where $\Psi_{11}(k)$ is a scalar and $\Psi_{22}(k)$ is a $3 \times 3$ matrix block. From the differential equation (4.128) for $\Psi$, we get

$$\begin{aligned}
&\frac{d^2\Psi_{11}(k)}{dk^2} = \big(\beta_{12}\beta_{21} - \frac{i}{2} - \frac{k^2}{4}\big)\Psi_{11}(k),\\
&\beta_{12}\Psi_{21}(k) = \frac{d\Psi_{11}(k)}{dk} + \frac{i}{2} k\Psi_{11}(k),\\
&\frac{d^2\beta_{12}\Psi_{22}(k)}{dk^2} = \big(\beta_{12}\beta_{21} + \frac{i}{2} - \frac{k^2}{4}\big)\beta_{12}\Psi_{22}(k),\\
&\Psi_{12}(k) = \frac{1}{\beta_{12}\beta_{21}}\big(\frac{d\beta_{12}\Psi_{22}(k)}{dk} - \frac{i}{2} k\beta_{12}\Psi_{22}(k)\big),
\end{aligned}$$

where

$$\beta_{12}\beta_{21} = \nu > 0 \tag{4.132}$$

provided that $\gamma(k_0) \neq 0$ (note that the case of $\gamma(k_0) = 0$ is, of course, trivial).

As is well known, the following Weber's equation

$$\frac{d^2 g(\zeta)}{d\zeta^2} + (a + \frac{1}{2} - \frac{\zeta^2}{4})g(\zeta) = 0$$

has a general solution

$$g(\zeta) = c_1 D_a(\zeta) + c_2 D_a(-\zeta),$$

where $c_1, c_2$ are arbitrary constants and $D_a(\zeta)$ denotes the standard (entire) parabolic-cylinder function and satisfies

$$\frac{dD_a(\zeta)}{d\zeta} + \frac{\zeta}{2} D_a(\zeta) - aD_{a-1}(\zeta) = 0, \tag{4.133}$$

$$D_a(\pm\zeta) = \frac{\Gamma(a+1)e^{\frac{i\pi a}{2}}}{\sqrt{2\pi}} D_{-a-1}(\pm i\zeta) + \frac{\Gamma(a+1)e^{-\frac{i\pi a}{2}}}{\sqrt{2\pi}} D_{-a-1}(\mp i\zeta), \tag{4.134}$$

where $\Gamma(\cdot)$ stands for the Gamma function. From the textbook [73] (see pp. 347–349), we know that as $\zeta \to \infty$,

$$D_a(\zeta) = \begin{cases} \zeta^a e^{-\frac{\zeta^2}{4}}\big(1 + O(\zeta^{-2})\big), & |\arg \zeta| < \frac{3\pi}{4},\\ \big(\zeta^a e^{-\frac{\zeta^2}{4}} - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{a\pi i}\zeta^{-a-1} e^{\frac{\zeta^2}{4}}\big)\big(1 + O(\zeta^{-2})\big), & \frac{\pi}{4} < \arg \zeta < \frac{5\pi}{4},\\ \big(\zeta^a e^{-\frac{\zeta^2}{4}} - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{-a\pi i}\zeta^{-a-1} e^{\frac{\zeta^2}{4}}\big)\big(1 + O(\zeta^{-2})\big), & -\frac{5\pi}{4} < \arg \zeta < -\frac{\pi}{4}. \end{cases} \tag{4.135}$$

Denote $a = i\beta_{12}\beta_{21}$ and then

$$\beta_{12}\beta_{21} \pm \frac{i}{2} = \pm i(\mp a + \frac{1}{2}).$$

Thus we find

$$\begin{aligned}
&\Psi_{11}(k) = d_1 D_a(e^{-\frac{3\pi i}{4}} k) + d_2 D_a(e^{\frac{\pi i}{4}} k),\\
&\beta_{12}\Psi_{22}(k) = d_3 D_{-a}(e^{\frac{3\pi i}{4}} k) + d_4 D_{-a}(e^{-\frac{\pi i}{4}} k),
\end{aligned}$$

where $d_1$ and $d_2$ are constants, and $d_3$ and $d_4$ are row vectors of constants. Note that as $\arg k \in (-\frac{3\pi}{4}, \frac{\pi}{4})$ and $k \to \infty$, we have

$$\Psi_{11}(k) k^{-\nu i} e^{\frac{ik^2}{4}} \to 1,\ \ \Psi_{22}(k) k^{\nu i} e^{-\frac{ik^2}{4}} \to I_3.$$

Therefore, for $\arg k \in (-\frac{\pi}{4}, \frac{\pi}{4})$, we have

$$\Psi_{11}(k) = e^{\frac{\pi\nu}{4}} D_a(e^{\frac{\pi i}{4}} k),\ \ \beta_{12}\Psi_{22}(k) = \beta_{12} e^{\frac{\pi\nu}{4}} D_{-a}(e^{-\frac{\pi i}{4}} k),$$

and further,

$$\beta_{12}\Psi_{21} = a\mathrm{e}^{\frac{\pi(\nu+i)}{4}} D_{a-1}(\mathrm{e}^{\frac{\pi i}{4}}k),\ \ \Psi_{12} = \beta_{12}\mathrm{e}^{\frac{\pi(\nu-3i)}{4}} D_{-a-1}(\mathrm{e}^{-\frac{\pi i}{4}}k).$$

For $\arg k \in (-\frac{3\pi}{4}, -\frac{\pi}{4})$, we have

$$\Psi_{11}(k) = \mathrm{e}^{\frac{\pi\nu}{4}} D_a(\mathrm{e}^{\frac{\pi i}{4}}k),\ \ \beta_{12}\Psi_{22}(k) = \beta_{12}\mathrm{e}^{-\frac{3\pi\nu}{4}} D_{-a}(\mathrm{e}^{\frac{3\pi i}{4}}k),$$

and further,

$$\beta_{12}\Psi_{21} = a\mathrm{e}^{\frac{\pi(\nu+i)}{4}} D_{a-1}(\mathrm{e}^{\frac{\pi i}{4}}k),\ \ \Psi_{12} = \beta_{12}\mathrm{e}^{\frac{\pi(i-3\nu)}{4}} D_{-a-1}(\mathrm{e}^{\frac{3\pi i}{4}}k).$$

Along the ray $\arg k = -\frac{\pi}{4}$, we have

$$\Psi_+(k) = \Psi_-(k)\begin{bmatrix} 1 & \gamma(k_0) \\ 0 & I_3 \end{bmatrix},$$

from which it follows that

$$\beta_{12}\mathrm{e}^{\frac{\pi(i-3\nu)}{4}} D_{-a-1}(\mathrm{e}^{\frac{3\pi i}{4}}k) = \mathrm{e}^{\frac{\pi\nu}{4}} D_a(\mathrm{e}^{\frac{\pi i}{4}}k)\gamma(k_0) + \beta_{12}\mathrm{e}^{\frac{\pi(\nu-3i)}{4}} D_{-a-1}(\mathrm{e}^{-\frac{\pi i}{4}}k). \tag{4.136}$$

Also, based on (4.134), we obtain

$$D_a(\mathrm{e}^{\frac{\pi i}{4}}k) = \frac{\Gamma(a+1)}{\sqrt{2\pi}}\mathrm{e}^{\frac{\pi a i}{2}} D_{-a-1}(\mathrm{e}^{\frac{3\pi i}{4}}k) + \frac{\Gamma(a+1)}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\pi a i}{2}} D_{-a-1}(\mathrm{e}^{-\frac{\pi i}{4}}k). \tag{4.137}$$

Now plugging (4.137) into (4.136), we separate the coefficients of the two independent special functions to get

$$\beta_{12} = \frac{\Gamma(a+1)}{\sqrt{2\pi}}\mathrm{e}^{\frac{\pi\nu}{2}-\frac{\pi i}{4}}\gamma(k_0) = \frac{\nu\Gamma(i\nu)}{\sqrt{2\pi}}\mathrm{e}^{\frac{\pi\nu}{2}+\frac{\pi i}{4}}\gamma(k_0), \tag{4.138}$$

since $a = i\nu$. Finally, we conclude that (1.4) is a consequence of (4.124), (4.130) and (4.138).

## 5. Concluding remarks

We have determined the leading long-time asymptotics for the Cauchy problem of the three-component coupled nonlinear Schrödinger (NLS) equation, based on an associated oscillatory Riemann–Hilbert (RH) problem. The essential analysis is that via the nonlinear steepest descent method, we deformed the associated oscillatory RH problem to a model one which is solvable explicitly, and guaranteed small errors between solutions to the different deformed RH problems. Our result is an application of the nonlinear steepest descent method to long-time asymptotics for integrable systems associated with $4 \times 4$ matrix spectral problems.

There are more and more studies on long-time asymptotics for integrable systems (see, e.g., [7,10,18,22,23,30]) and even nonlocal integrable systems (see, e.g., [67]). Moreover, it has been generalized to evaluate long-time asymptotics of initial–boundary value problems of integrable systems on the half-line (see, e.g., [22,30,63]), and asymptotics of integrable systems whose RH problems possess rational phases (see, e.g., [76]) or jump matrices of lower regularity (see, e.g., [31,62]).

Various solution approaches also exist in the field of integrable systems, some of which are the Hirota direct method [24], the generalized bilinear technique [36], the Darboux transformation [61], and the Wronskian technique [17,55]. Connections, similarities and differences between different approaches would be very interesting. There are many studies on counterparts of integrable systems, such as integrable couplings [75], super hierarchies [15] and fractional analogous systems [14,21]. It will be an important topic for further study to explore long-time asymptotics of those generalized integrable counterparts via the nonlinear steepest descent method. Particularly, it will be physically important to determine limiting behaviors of solutions incorporating features of other exact solutions, such as lump solutions [51,58], from the perspective of steepest descent based on RH problems. Such problems may show different asymptotic features in different regions of space and/or time. Boundary layer theory [27] can be used to match different asymptotics explored in different regions.

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