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Abundant lumps and their interaction solutions of (3+1)-dimensional linear PDEs

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The paper aims to explore the existence of diverse lump and interaction solutions to linear partial differential equations in (3+1)-dimensions. The remarkable richness of exact solutions to a class of linear partial differential equations in (3+1)-dimensions will be exhibited through Maple symbolic computations, which yields exact lump, lump-periodic and lump–soliton solutions. The results expand the understanding of lump, freak wave and breather solutions and their interaction solutions in soliton theory.

1. Introduction

Lump solutions are a particular kind of exact solutions, which describe various important nonlinear phenomena in nature [1,2]. More specifically, such solutions can be generated from solitons by taking long wave limits [3]. There are also positons and complexitons to integrable equations, enriching the diversity of solitons [4,5]. Interaction solutions between two different kinds of exact solutions exhibit more diverse nonlinear phenomena [6].

Soliton solutions are exponentially localized in all directions in space and time, and lump solutions, rationally localized in all directions in space. Through a Hirota bilinear form:

$$P(D_x, D_t)f \cdot f = 0, \tag{1.1}$$

where P is a polynomial and D_x and D_t are Hirota's bilinear derivatives, an N-soliton solution in (1+1)-dimensions can be defined by

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\Big), \tag{1.2}$$

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where

$$\begin{cases} \xi_i = k_i x - \omega_i t + \xi_{i,0}, \ 1 \le i \le N, \\ e^{a_{ij}} = -\dfrac{P(k_i - k_j, \omega_j - \omega_i)}{P(k_i + k_j, \omega_j + \omega_i)}, \ 1 \le i < j \le N, \end{cases} \tag{1.3}$$

with k_i and ω_i satisfying the dispersion relation and $\xi_{i,0}$ being arbitrary shifts. The KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.4}$$

has a lump solution [7]:

$$u = 2(\ln f)_{xx}, \ f = \left(a_1 x + a_2 y + a_3 t + a_4\right)^2 + \left(a_5 x + a_6 y + a_7 t + a_8\right)^2 + a_9, \tag{1.5}$$

where

$$a_3 = \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2\, a_2 a_5 a_6}{{a_1}^2 + {a_5}^2}, \ a_7 = \frac{2a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2}, \ a_9 = \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \tag{1.6}$$

and the other parameters a_i's are arbitrary but need to satisfy $a_1 a_6 - a_2 a_5 \neq 0$, which guarantees rational localization in all directions in the (x, y)-plane. Other integrable equations, possessing lump solutions, include the three-dimensional three-wave resonant interaction [8], the BKP equation [9,10], the Davey–Stewartson equation II [3], the Ishimori-I equation [11] and many others [12,13].

It is recognized by making symbolic computations that many nonintegrable equations possess lump solutions as well, including (2+1)-dimensional generalized KP, BKP and Sawada–Kotera equations [14–16]. Moreover, various studies show the existence of interaction solutions between lumps and another kind of exact solutions to nonlinear integrable equation in (2+1)-dimensions, which contain lump–soliton interaction solutions (see, e.g., [17–20]) and lump–kink interaction solutions (see, e.g., [21–24]). Nevertheless, in the (3+1)-dimensional case, only lump-type solutions are presented for the integrable Jimbo–Miwa equations, which are rationally localized in almost all but not all directions in space. All presented analytical rational solutions to the (3+1)-dimensional Jimbo–Miwa equation in [25–27] and the (3+1)-dimensional Jimbo–Miwa like equation in [28] are not rationally localized in all directions in space. It is absolutely very interesting and important to explore lump and interaction solutions to partial differential equations in (3+1)-dimensions.

This paper aims at showing that there do exist abundant lump solutions and their interaction solutions to linear partial differential equations in (3+1)-dimensions. A class of particular examples in (3+1)-dimensions will be considered to exhibit such solution phenomena. We will explicitly generate lump solutions and mixed lump-periodic and lump–soliton solutions for a specially chosen class of (3+1)-dimensional linear partial differential equations. Based on Maple symbolic computations, sufficient conditions and examples of lump and interaction solutions will be provided, together with three-dimensional plots and contour plots of special examples of the presented solutions. Some concluding remarks will be given in the final section.

2. Abundant lump and interaction solutions

Let $u = u(x, y, z, t)$ be a real function of $x, y, z, t \in \mathbb{R}$. We consider a class of linear partial differential equations (PDEs) in (3+1)-dimensions:

$$\alpha_1 u_{xy} + \alpha_2 u_{xz} + \alpha_3 u_{xt} + \alpha_4 u_{yz} + \alpha_5 u_{yt} + \alpha_6 u_{zt} + \alpha_7 u_{xx} + \alpha_8 u_{yy} + \alpha_9 u_{zz} + \alpha_{10} u_{tt} = 0, \tag{2.1}$$

where $\alpha_i, \ 1 \le i \le 10$, are real constants, and the subscripts denote partial differentiation.

We search for a kind of exact solutions

$$u = v(\xi_1, \xi_2, \xi_3, \xi_4) \tag{2.2}$$

where v is an arbitrary real function, and $\xi_i, \ 1 \le i \le 4$, are four wave variables:

$$\xi_i = a_i x + b_i y + c_i z + d_i t + e_i, \ 1 \le i \le 4, \tag{2.3}$$

in which a_i, b_i, c_i, d_i and e_i, $1 \le i \le 4$, are real constants to be determined. Then, the linear PDE (2.1) becomes

$$\sum_{i=1}^{4} \sum_{j=i}^{4} w_{ij} v_{\xi_i \xi_j} = 0, \tag{2.4}$$

where w_{ij}, $1 \le i \le j \le 4$, are quadratic functions of the parameters a_i, b_i, c_i and d_i, $1 \le i \le 4$. Upon setting all coefficients of the ten second partial derivatives of v to be zero, we obtain a system of equations on the parameters:

$$\begin{cases} \alpha_1 a_i b_i + \alpha_2 a_i c_i + \alpha_3 a_i d_i + \alpha_4 b_i c_i + \alpha_5 b_i d_i \\ \quad + \alpha_6 c_i d_i + \alpha_7 a_i^2 + \alpha_8 b_i^2 + \alpha_9 c_i^2 + \alpha_{10} d_i^2 = 0, \ 1 \le i \le 4, \\ \alpha_1 (a_i b_j + a_j b_i) + \alpha_2 (a_i c_j + a_j c_i) + \alpha_3 (a_i d_j + a_j d_i) + \alpha_4 (b_i c_j + b_j c_i) + \alpha_5 (b_i d_j + b_j d_i) \\ \quad + \alpha_6 (c_i d_j + c_j d_i) + 2\alpha_7 a_i a_j + 2\alpha_8 b_i b_j + 2\alpha_9 c_i c_j + 2\alpha_{10} d_i d_j = 0, \ 1 \le i < j \le 4. \end{cases} \tag{2.5}$$

Direct symbolic computations with Maple can determine a bunch of solutions to this system of quadratic equations. The interesting two ones are stated as follows:

$$\Big\{ b_2 = \frac{a_2}{a_4}\gamma_1,\ b_3 = \frac{a_3}{a_4}\gamma_1,\ b_4 = \gamma_1,\ c_3 = \frac{a_3c_4}{a_4},\ d_2 = \frac{a_2d_4}{a_4},\ d_3 = \frac{a_3d_4}{a_4}, \\ \alpha_1 = \gamma_2,\ \alpha_2 = -\frac{b_1\alpha_4 + d_1\alpha_6}{a_1},\ \alpha_7 = \gamma_3,\ \alpha_8 = -\frac{\alpha_4(\alpha_4\alpha_{10} - \alpha_5\alpha_6)}{\alpha_6^2},\ \alpha_9 = 0 \Big\}, \tag{2.6}$$

where

$$\begin{cases} \gamma_1 = \dfrac{a_4b_1\alpha_4 + (a_4d_1 - a_1d_4)\alpha_6}{a_1\alpha_4}, \\ \gamma_2 = \dfrac{a_1\alpha_3\alpha_4\alpha_6 + b_1\alpha_4(2\alpha_4\alpha_{10} - \alpha_5\alpha_6) + d_1\alpha_6(2\alpha_4\alpha_{10} - \alpha_5\alpha_6)}{a_1\alpha_6^2}, \\ \gamma_3 = -\dfrac{a_1\alpha_3\alpha_6(b_1\alpha_4 + d_1\alpha_6) + \alpha_{10}(b_1\alpha_4 + d_1\alpha_6)^2}{a_1^2\alpha_6^2}; \end{cases}$$

and

$$\Big\{ b_2 = \frac{a_2b_4}{a_4},\ b_3 = \frac{a_3b_4}{a_4},\ c_1 = 0,\ d_2 = \gamma_1,\ d_3 = \gamma_2,\ \alpha_1 = -\frac{d_1}{a_1}\alpha_5,\ \alpha_4 = \frac{a_4d_1 - a_1d_4}{a_1c_4}\alpha_5, \\ \alpha_6 = \gamma_3,\ \alpha_7 = \frac{d_1(b_1\alpha_5 - a_1\alpha_3)}{2a_1^2},\ \alpha_8 = 0,\ \alpha_9 = \gamma_4,\ \alpha_{10} = -\frac{a_1\alpha_3 + b_1\alpha_5}{2d_1} \Big\}, \tag{2.7}$$

where

$$\begin{cases} \gamma_1 = \dfrac{a_1c_2d_4 + a_2c_4d_1 - a_4c_2d_1}{a_1c_4}, \\ \gamma_2 = \dfrac{a_1c_3d_4 + a_3c_4d_1 - a_4c_3d_1}{a_1c_4},\ \gamma_3 = \dfrac{b_1(a_1d_4 - a_4d_1)\alpha_5 - a_1^2c_4\alpha_2}{a_1c_4d_1}, \\ \gamma_4 = \dfrac{(a_1d_4 + a_4d_1)^2(a_1\alpha_3 - b_1\alpha_5) + 2a_1^2c_4(a_1d_4 - a_4d_1)\alpha_2 - 4a_1a_4d_1d_4(a_1\alpha_3 - b_1\alpha_5)}{2a_1^2c_4^2d_1}. \end{cases}$$

In each set of the two solutions listed above, the parameters not determined in the set are arbitrary provided that all expressions in the set are meaningful. Though those two choices engender lumps and their interaction solutions, we remark that all the resulting solutions satisfy a determinant identity

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0. \tag{2.8}$$

From those two solutions, we can derive the corresponding two following results.

Case 1.

Upon taking $a_1 = b_1 = -d_1$ and $a_4 = c_4 = d_4$, we can know that the following reduced linear PDE:

$$u_{xy} + u_{xt} + u_{yz} + u_{yt} + u_{zt} + u_{tt} = 0 \tag{2.9}$$

possesses a class of exact solutions

$$u = (\ln f)_{xx},\ f = \xi_1^{2n_1} + \xi_2^{2n_2} + \xi_3^{2n_3} + g(\xi_4), \tag{2.10}$$

where $n_i,\ 1 \le i \le 3$, are arbitrary natural numbers, the wave variables $\xi_i,\ 1 \le i \le 4$, are defined by

$$\begin{cases} \xi_1 = a_1x + a_1y + c_1z - a_1t + e_1, \\ \xi_2 = a_2x - a_2y + c_2z + a_2t + e_2, \\ \xi_3 = a_3x - a_3y + a_3z + a_3t + e_3, \\ \xi_4 = a_4x - a_4y + a_4z + a_4t + e_4, \end{cases} \tag{2.11}$$

and the function g is arbitrary. Therefore, upon taking

$$g(\xi_4) = \beta_1,\ \beta_2 + \beta_3\cos\xi_4,\ \text{ or } \beta_4\cosh\xi_4, \tag{2.12}$$

where $\beta_i,\ 1 \le i \le 4$, are proper constants to fulfill the positivity of f, we can generate lump solutions, and interaction solutions: lump-periodic and lump–soliton solutions to the linear PDE (2.9). The resulting solution with $n_1 = n_2 = n_3 = 1$ is

$$u = \frac{f_{xx}f - f_x^2}{f^2} = \frac{2a_1^2 + 2a_2^2 + 2a_3^2 + a_4^2g''(\xi_4)}{f} - \frac{(2a_1\xi_1 + 2a_2\xi_2 + 2a_3\xi_3 + a_4g'(\xi_4))^2}{f^2}. \tag{2.13}$$

Case 2. Upon taking $a_1 = -b_1 = -d_1,\ a_4 = b_4 = d_4$ and $c_4 = 2d_4$, we can know that the following reduced linear PDE:

$$u_{xy} - u_{yz} + u_{yt} + u_{zt} + \frac{1}{2}u_{xx} - \frac{1}{2}u_{zz} - \frac{1}{2}u_{tt} = 0 \tag{2.14}$$

possesses a class of exact solutions

$$u = (\ln f)_{xx},\ f = \xi_1^{2n_1} + \xi_2^{2n_2} + \xi_3^{2n_3} + g(\xi_4), \tag{2.15}$$

where $n_i,\ 1 \le i \le 3$, are arbitrary natural numbers, the wave variables $\xi_i,\ 1 \le i \le 4$, are defined by

$$\begin{cases} \xi_1 = a_1x - a_1y - a_1t + e_1, \\ \xi_2 = a_2x + a_2y + c_2z + (c_2 - a_2)t + e_2, \\ \xi_3 = a_3x + a_3y + c_3z + (c_3 - a_3)t + e_3, \\ \xi_4 = d_4x + d_4y + 2d_4z + d_4t + e_4, \end{cases} \tag{2.16}$$

and the function g is arbitrary. Therefore, upon taking

$$g(\xi_4) = \beta_1,\ \beta_2 + \beta_3 \sin \xi_4,\ \text{or } \beta_4 \cosh \xi_4, \tag{2.17}$$

where $\beta_i,\ 1 \le i \le 4$, are proper constants to fulfill the positivity of f, we can generate lump solutions, and interaction solutions: lump-periodic and lump–soliton solutions to the linear PDE (2.14). The resulting solution with $n_1 = n_2 = n_3 = 1$ is

$$u = \frac{f_{xx}f - f_x^2}{f^2} = \frac{2a_1^2 + 2a_2^2 + 2a_3^2 + d_4^2 g''(\xi_4)}{f} - \frac{(2a_1\xi_1 + 2a_2\xi_2 + 2a_3\xi_3 + d_4g'(\xi_4))^2}{f^2}. \tag{2.18}$$

Specially taking

$$\begin{cases} a_1 = -1,\ a_2 = a,\ c_2 = 1, \\ a_3 = 2,\ c_3 = 3,\ d_4 = 1, \\ n_1 = n_2 = n_3 = 1, \\ \beta_1 = 1,\ \beta_2 = 15,\ \beta_3 = 16,\ \beta_4 = 25, \end{cases} \tag{2.19}$$

we get the three special solutions to the reduced PDE (2.14):

$$\begin{cases} u_1 = \dfrac{12f_1 - (12x + 8y + 14z + 2t)^2}{f_1^2}, \\ f_1 = (-x + y + t)^2 + (x + y + z)^2 + (2x + 2y + 3z + t)^2 + 1, \end{cases} \tag{2.20}$$

$$\begin{cases} u_2 = \dfrac{(12 - 15 \sin \xi_4)f_2 - (12x + 8y + 14z + 2t + 15 \cos \xi_4)^2}{f_2^2}, \\ f_2 = (-x + y + t)^2 + (x + y + z)^2 + (2x + 2y + 3z + t)^2 + 15 \sin \xi_4 + 16, \end{cases} \tag{2.21}$$

and

$$\begin{cases} u_3 = \dfrac{(12 + 25 \cosh \xi_4)f_3 - (12x + 8y + 14z + 2t + 25 \sinh \xi_4)^2}{f_3^2}, \\ f_3 = (-x + y + t)^2 + (x + y + z)^2 + (2x + 2y + 3z + t)^2 + 25 \cosh \xi_4, \end{cases} \tag{2.22}$$

where $\xi_4 = x + y + 2z + t$. Three three-dimensional plots and contour plots of those solutions are presented in Figs. 1–3.

All above results enrich the existing theories of soliton solutions and dromion-type solutions through basic approaches, including the Hirota perturbation technique and symmetry constraints (see, e.g., [29–34]).

3. Concluding remarks

We have considered a class of linear partial differential equations in (3+1)-dimensions to explore abundant lump solutions and their interaction solutions: lump-periodic solutions and lump–soliton solutions, which amends soliton theory of nonlinear integrable equations. A class of particular lump and interaction solutions were explicitly worked out through Maple symbolic computations, and three-dimensional plots and contour plots of three specially chosen solutions were made with Maple.

We remark that the presented solutions in (2.13) and (2.18) with $g = 0$ are all lump solutions, rationally localized in all directions in the (x, y, z)-space, but we failed to generate any rational solution to the considered class of (3+1)-dimensional linear partial differential equations, which is localized in all directions in the whole (x, y, z, t)-space. All the obtained lump, lump-periodic and lump–soliton solutions supplement exact solutions generated from various kinds of combinations [35–37]. It is also interesting to search for lump and interaction solutions to other generalized bilinear and tri-linear differential equations involving generalized bilinear derivatives [38,39]. The corresponding interaction solutions will generally not be

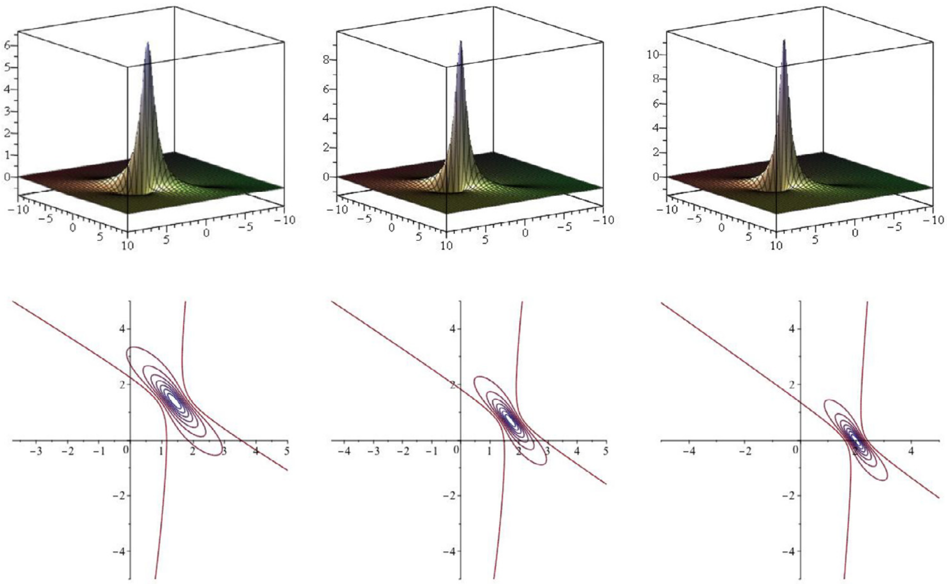

Fig. 1. Profiles of u_1 when $t = 0, 1, 2$ and $z = -2$: 3d plots (top) and contour plots (bottom).

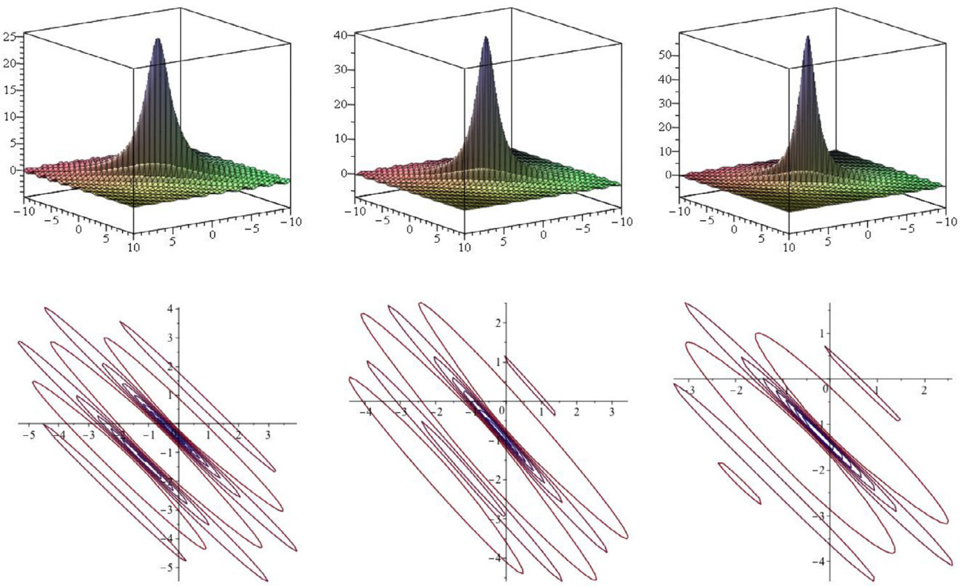

Fig. 2. Profiles of u_2 when $t = 0, 0.5, 1$ and $z = 1$: 3d plots (top) and contour plots (bottom).

resonant solutions generated through the linear superposition principle [40,41]. Integrable equations determined in terms of generalized bilinear derivatives [38,39] will have different interaction solutions, including lump-periodic, lump–kink and lump–soliton solutions, but lump solutions derived from quadratic functions remain the same as in the Hirota derivative case (see [42] for more details).

The richness of interaction solutions should imply that there exist diverse Lie–Bäcklund symmetries, which amends symmetry theories on partial differential equations, particularly integrable equations. It is well known that the Wronskian technique can solve integrable equations, and therefore, our study creates an interesting question: how can one generalize Wronskian solutions by introducing matrix entries of new type? Also, there is no doubt that it is important to establish a fundamental theory of lump solutions and their interactions for difference–differential equations. All those interesting problems deserve our further investigation and effort.

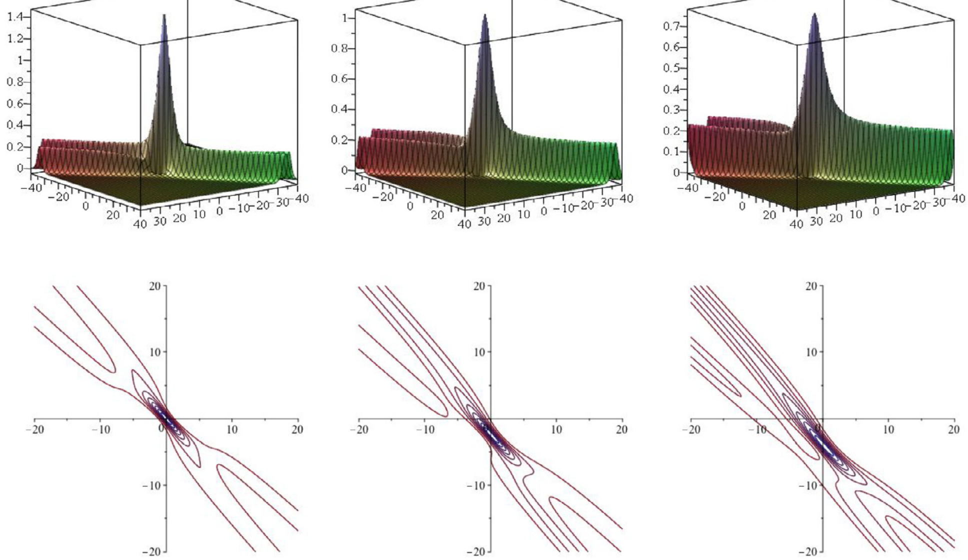

Fig. 3. Profiles of u_3 when $t = 0, 0.5, 1$ and $z = 0$: 3d plots (top) and contour plots (bottom).

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