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Riemann–Hilbert problems and N-soliton solutions for a coupled mKdV system

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ABSTRACT

A 3×3 matrix spectral problem is introduced and its associated AKNS integrable hierarchy with four components is generated. From this spectral problem, a kind of Riemann–Hilbert problems is formulated for a system of coupled mKdV equations in the resulting AKNS integrable hierarchy. N-soliton solutions to the coupled mKdV system are presented through a specific Riemann–Hilbert problem with an identity jump matrix.

1. Introduction

The Riemann–Hilbert method is one of the most powerful approaches to generate integrable systems and their soliton solutions [1]. Its basic starting point is a kind of matrix spectral problems, which possess bounded eigenfunctions analytically extendable to the upper or lower half-plane. There is a close connection with the inverse scattering method in soliton theory [2]. The asymptotics at infinity on the real axis necessary in constructing the scattering coefficients is used primarily in solving the corresponding Riemann–Hilbert problems [1]. Specific Riemann–Hilbert problems with the identity jump matrix lead to N-soliton solutions, which could contain rational solutions and periodic solutions.

To formulate a Riemann–Hilbert problem, we usually start with taking the following pair of equivalent matrix spectral problems

$$\psi_x = i[A(\lambda), \psi] + \check{P}(u, \lambda)\psi, \; \psi_t = i[B(\lambda), \psi] + \check{Q}(u, \lambda)\psi,$$

where $[\cdot, \cdot]$ is the matrix commutator, λ is a spectral parameter, u is a potential, A, B are two constant $n \times n$ matrices, $\check{P}$, $\check{Q}$ are trace-less $n \times n$ matrices and ψ is an $n \times n$ matrix eigenfunction. An important step is to explore the existence of analytical matrix eigenfunctions with the asymptotic conditions

$$\psi^{\pm} \to I_n, \;\; \text{when } x, t \to \pm\infty,$$

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where I_n stands for the identity matrix of size n, to determine two analytical matrix functions $P^{\pm}(x, t, \lambda)$, which are analytical in the upper and lower half-planes, respectively. Then formulate a Riemann–Hilbert problem on the real line:

$$P^{-}(x, t, \lambda)P^{+}(x, t, \lambda) = EG(\lambda)E^{-1}, \ \lambda \in \mathbb{R},$$

where $E = e^{iA(\lambda)x+iB(\lambda)t}$. Upon taking G to be the identity matrix I_n, the corresponding Riemann–Hilbert problem can be explicitly solved to present N-soliton solutions by checking asymptotic behaviors of the matrix functions at infinity of λ. We shall consider an example of coupled mKdV equations and generate its N-soliton solutions by a special Riemann–Hilbert problem.

The rest of the paper is structured as follows. In Section 2, with the aid of the zero-curvature formulation and the trace identity, we rederive the AKNS soliton hierarchy with four components and its bi-Hamiltonian structure from a new matrix spectral problem suited for the Riemann–Hilbert theory. In Section 3, taking a system of coupled mKdV equations as an example, we analyze analytical properties of matrix eigenfunctions for an equivalent spectral problem, and then formulate a kind of Riemann–Hilbert problems associated with the newly introduced spectral problem. In Section 4, we compute N-soliton solutions to the considered system of coupled mKdV equations from a specific Riemann–Hilbert problem, which possesses the identity jump matrix on the real axis. In the last section, we present a few concluding remarks, together with some discussions on other solution methods.

2. AKNS soliton hierarchy with four components

2.1. Soliton hierarchy

Let us first recall the zero curvature formulation and the trace identity [3]. We follow the standard procedure suited for Riemann–Hilbert problems, where we consistently use the unit imaginary number i. Let $U = U(u, \lambda)$ be a square spectral matrix belonging to a given matrix loop algebra, where u is a potential and λ is a spectral parameter. Assume that

$$W = W(u, \lambda) = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} W_k(u)\lambda^{-k} \tag{2.1}$$

solves the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.2}$$

Then introduce a series of Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (\lambda^r W)_+ + \Delta_r, \ r \ge 0, \tag{2.3}$$

where the subscript +denotes the operation of taking a polynomial part in λ and Δ_r, $r \ge 0$, are appropriate modification terms, to generate a soliton hierarchy

$$u_t = K_r(u) = K_r(x, t, u, u_x, \ldots), \ r \ge 0, \tag{2.4}$$

from a series of zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \ r \ge 0. \tag{2.5}$$

The two matrices U and $V^{[r]}$ are called a Lax pair [4] of the rth soliton equation in the hierarchy (2.4). The zero curvature equations in (2.5) are the compatibility conditions of the spatial and temporal spectral problems

$$-i\phi_x = U\phi = U(u, \lambda)\phi, \ -i\phi_t = V^{[r]}\phi = V^{[r]}(u, \lambda)\phi, \ r \ge 0, \tag{2.6}$$

where ϕ is the matrix eigenfunction.

An important task in soliton theory is to show the Liouville integrability of a soliton hierarchy. This can be achieved usually by establishing a bi-Hamiltonian structure [5]:

$$u_t = K_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u}, \ r \ge 1, \tag{2.7}$$

where J and M constitute a Hamiltonian pair and $\frac{\delta}{\delta u}$ denotes the variational derivative (see, e.g., [6]). The Hamiltonian structures can be often furnished through applying the trace identity [3]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^{\gamma}\mathrm{tr}(W\frac{\partial U}{\partial u})\Big], \ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{2.8}$$

or more generally, the variational identity [7]:

$$\frac{\delta}{\delta u}\int \langle W, \frac{\partial U}{\partial \lambda}\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle\Big], \ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra [8]. The bi-Hamiltonian structure guarantees that infinitely many Lie symmetries $\{K_n\}_{n=0}^{\infty}$ and conserved quantities $\{\tilde{H}_n\}_{n=0}^{\infty}$ commute:

$$[K_{n_1}, K_{n_2}] = K'_{n_1}[K_{n_2}] - K'_{n_2}[K_{n_1}] = 0, \tag{2.10}$$

$$\{\tilde{\mathcal{H}}_{n_1}, \tilde{\mathcal{H}}_{n_2}\}_N = \int \Big(\frac{\delta \tilde{\mathcal{H}}_{n_1}}{\delta u}\Big)^T N \frac{\delta \tilde{\mathcal{H}}_{n_2}}{\delta u}\, dx = 0, \tag{2.11}$$

where $n_1, n_2 \geq 0$, $N = J$ or M, and K' denotes the Gateaux derivative of K:

$$K'(u)[S] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u+\varepsilon S, u_x + \varepsilon S_x, \ldots).$$

It is known that for an evolution equation $u_t = K(u)$, $\tilde{H} = \int H\, dx$ is a conserved functional iff $\frac{\delta \tilde{H}}{\delta u}$ is an adjoint symmetry [9], and so, the Hamiltonian structures link conserved functionals to adjoint symmetries and further symmetries. When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations yield classical soliton hierarchies [10]; when semisimple, the associated zero curvature equations yield a collection of different soliton hierarchies; and when non-semisimple, we obtain hierarchies of integrable couplings [11], which require extra care in presenting soliton solutions.

2.2. AKNS hierarchy with four components

Let us consider a 3×3 matrix spectral problem

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ U = (U_{kl})_{3\times 3} = \begin{bmatrix} \alpha_1\lambda & p_1 & p_2 \\ q_1 & \alpha_2\lambda & 0 \\ q_2 & 0 & \alpha_2\lambda \end{bmatrix}, \tag{2.12}$$

where α_1 and α_2 are real constants, λ is a spectral parameter and u is a four-dimensional potential

$$u = (p, q^T)^T,\ p = (p_1, p_2),\ q = (q_1, q_2)^T. \tag{2.13}$$

The special reduction of $p_2 = q_2 = 0$ transforms (2.12) into the AKNS spectral problem [12], and thus it is called a four-component AKNS spectral problem. Since $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2)$ has a multiple eigenvalue, the spectral problem (2.12) is degenerate.

To derive the associated soliton hierarchy, we first solve the stationary zero curvature equation (2.2) corresponding to (2.12). We suppose that a solution W is given by

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.14}$$

where a is a scalar, b^T and c are two-dimensional columns, and d is a 2×2 matrix. Then the stationary zero curvature equation (2.2) becomes

$$a_x = i(pc - bq),\ b_x = i(\alpha\lambda b + pd - ap),\ c_x = i(-\alpha\lambda c + qa - dq),\ d_x = i(qb - cp), \tag{2.15}$$

where $\alpha = \alpha_1 - \alpha_2$. We seek a formal series solution as

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m},\ W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix},\ m \geq 0, \tag{2.16}$$

with $b^{[m]}$, $c^{[m]}$ and $d^{[m]}$ being assumed to be

$$b^{[m]} = (b_1^{[m]}, b_2^{[m]}),\ c^{[m]} = (c_1^{[m]}, c_2^{[m]})^T,\ d^{[m]} = (d_{kl}^{[m]})_{2\times 2},\ m \geq 0. \tag{2.17}$$

Thus, the system (2.15) equivalently leads to the following recursion relations:

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a_x^{[0]} = 0,\ d_x^{[0]} = 0, \tag{2.18a}$$

$$b^{[m+1]} = \frac{1}{\alpha}(-ib_x^{[m]} - pd^{[m]} + a^{[m]}p),\ m \geq 0, \tag{2.18b}$$

$$c^{[m+1]} = \frac{1}{\alpha}(ic_x^{[m]} + qa^{[m]} - d^{[m]}q),\ m \geq 0, \tag{2.18c}$$

$$a_x^{[m]} = i(pc^{[m]} - b^{[m]}q),\ d_x^{[m]} = i(qb^{[m]} - c^{[m]}p),\ m \geq 1. \tag{2.18d}$$

We choose the initial values as follows:

$$a^{[0]} = \beta_1,\ d^{[0]} = \beta_2 I_2, \tag{2.19}$$

where β_1, β_2 are arbitrary real constants and $I_2 = \mathrm{diag}(1, 1)$, and take constants of integration in (2.18d) to be zero, i.e., require

$$W_m|_{u=0} = 0,\ m \geq 1. \tag{2.20}$$

Therefore, with $a^{[0]}$ and $d^{[0]}$ given by (2.19), all matrices $W_m,\ m \geq 1$, will be uniquely determined. For instance, a direct computation, based on (2.18), tells that

$$b_k^{[1]} = \frac{\beta}{\alpha} p_k,\ c_k^{[1]} = \frac{\beta}{\alpha} q_k,\ a^{[1]} = 0,\ d_{kl}^{[1]} = 0; \tag{2.21a}$$

$$b_k^{[2]} = -\frac{\beta}{\alpha^2} i p_{k,x},\ c_k^{[2]} = \frac{\beta}{\alpha^2} i q_{k,x},\ a^{[2]} = -\frac{\beta}{\alpha^2}(p_1q_1 + p_2q_2),\ d_{kl}^{[2]} = \frac{\beta}{\alpha^2} p_l q_k; \tag{2.21b}$$

$$b_k^{[3]} = -\frac{\beta}{\alpha^3}[p_{k,xx} + 2(p_1q_1 + p_2q_2)p_k],\ c_k^{[3]} = -\frac{\beta}{\alpha^3}[q_{k,xx} + 2(p_1q_1 + p_2q_2)q_k], \tag{2.21c}$$

$$a^{[3]} = -\frac{\beta}{\alpha^3} i(p_1q_{1,x} - p_{1,x}q_1 + p_2q_{2,x} - p_{2,x}q_2),\ d_{kl}^{[3]} = -\frac{\beta}{\alpha^3} i(p_{l,x}q_k - p_l q_{k,x}); \tag{2.21d}$$

$$b_k^{[4]} = \frac{\beta}{\alpha^4} i[p_{k,xxx} + 3(p_1q_1 + p_2q_2)p_{k,x} + 3(p_{1,x}q_1 + p_{2,x}q_2)p_k], \tag{2.21e}$$

$$c_k^{[4]} = -\frac{\beta}{\alpha^4} i[q_{k,xxx} + 3(p_1q_1 + p_2q_2)q_{k,x} + 3(p_1q_{1,x} + p_2q_{2,x})q_k], \tag{2.21f}$$

$$a^{[4]} = \frac{\beta}{\alpha^4}[3(p_1q_1 + p_2q_2)^2 + p_1q_{1,xx} - p_{1,x}q_{1,x} + p_{1,xx}q_1 + p_2q_{2,xx} - p_{2,x}q_{2,x} + p_{2,xx}q_2], \tag{2.21g}$$

$$d_{kl}^{[4]} = -\frac{\beta}{\alpha^4}[3p_l(p_1q_1 + p_2q_2)q_k + p_{l,xx}q_k - p_{l,x}q_{k,x} + p_l q_{k,xx}]; \tag{2.21h}$$

where $\beta = \beta_1 - \beta_2$ and $1 \leq k, l \leq 2$. Based on (2.18d), we can obtain, from (2.18b) and (2.18c), a recursion relation for $b^{[m]}$ and $c^{[m]}$:

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix},\ m \geq 1, \tag{2.22}$$

where Ψ is a 4×4 matrix operator

$$\Psi = \frac{i}{\alpha} \begin{bmatrix} (\partial + \sum_{k=1}^{2} q_k \partial^{-1} p_k) I_2 + q\partial^{-1} p & -q\partial^{-1} q^T - (q\partial^{-1} q^T)^T \\ p^T \partial^{-1} p + (p^T \partial^{-1} p)^T & -(\partial + \sum_{k=1}^{2} p_k \partial^{-1} q_k) I_2 - p^T \partial^{-1} q^T \end{bmatrix}. \tag{2.23}$$

To get a soliton hierarchy, we introduce, for all integers $r \geq 0$, the following Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (V_{kl}^{[r]})_{3\times 3} = (\lambda^r W)_+ = \sum_{k=0}^{r} W_k \lambda^{r-k},\ r \geq 0, \tag{2.24}$$

where the modification terms are taken as zero. The compatibility conditions of (2.6), i.e., the zero curvature equations (2.5), generate the AKNS soliton hierarchy with four components

$$u_t = \begin{bmatrix} p^T \\ q \end{bmatrix}_t = K_r = i \begin{bmatrix} \alpha b^{[r+1]T} \\ -\alpha c^{[r+1]} \end{bmatrix},\ r \geq 0. \tag{2.25}$$

The first two nonlinear systems in the soliton hierarchy (2.25) read

$$p_{k,t} = -\frac{\beta}{\alpha^2} i[p_{k,xx} + 2(p_1q_1 + p_2q_2)p_k],\ 1 \leq k \leq 2, \tag{2.26a}$$

$$q_{k,t} = \frac{\beta}{\alpha^2} i[q_{k,xx} + 2(p_1q_1 + p_2q_2)q_k],\ 1 \leq k \leq 2, \tag{2.26b}$$

and

$$p_{k,t} = -\frac{\beta}{\alpha^3}[p_{k,xxx} + 3(p_1q_1 + p_2q_2)p_{k,x} + 3(p_{1,x}q_1 + p_{2,x}q_2)p_k],\ 1 \leq k \leq 2, \tag{2.27a}$$

$$q_{k,t} = -\frac{\beta}{\alpha^3}[q_{k,xxx} + 3(p_1q_1 + p_2q_2)q_{k,x} + 3(p_1q_{1,x} + p_2q_{2,x})q_k],\ 1 \leq k \leq 2, \tag{2.27b}$$

which are the four-component versions of the AKNS systems of coupled nonlinear Schrödinger equations and coupled mKdV equations, respectively. Under a symmetric reduction, the four-component AKNS equations (2.26b) can be reduced to the Manakov system [13], for which a decomposition into finite-dimensional integrable Hamiltonian systems was presented in [14], while as the four-component AKNS equations (2.27b) contain various mKdV equations, for which there exist different kinds of integrable decompositions under symmetry constraints (see, e.g., [15,16]).

The AKNS soliton hierarchy (2.25) with four components has a Hamiltonian structure [9], which can be generated using the trace identity [3], or more generally, the variational identity [7]. Actually, we have

$$-\,i\,\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = \alpha_1 a + \alpha_2 \mathrm{tr}(d) = \sum_{m=0}^{\infty}(\alpha_1 a^{[m]} + \alpha_2 d_{11}^{[m]} + \alpha_2 d_{22}^{[m]})\lambda^{-m},$$

and

$$-\,i\,\mathrm{tr}(W\frac{\partial U}{\partial u}) = \begin{bmatrix} c \\ b^T \end{bmatrix} = \sum_{m\geq 0} G_{m-1}\lambda^{-m}.$$

Inserting these expressions into the trace identity and considering the case of $m = 2$, we get $\gamma = 0$ and thus we obtain

$$\frac{\delta \tilde{H}_m}{\delta u} = iG_{m-1},\ \tilde{H}_m = -\frac{i}{m}\int(\alpha_1 a^{[m+1]} + \alpha_2 d_{11}^{[m+1]} + \alpha_2 d_{22}^{[m+1]})\,dx,\ G_{m-1} = \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix},\ m \geq 1. \tag{2.28}$$

A bi-Hamiltonian structure of the four-component AKNS equations (2.25) then follows:

$$u_t = K_r = JG_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u},\ r \geq 1, \tag{2.29}$$

where the Hamiltonian pair $(J, M = J\Psi)$ is given by

$$J = \begin{bmatrix} 0 & \alpha I_2 \\ -\alpha I_2 & 0 \end{bmatrix}, \tag{2.30a}$$

$$M = i\begin{bmatrix} p^T\partial^{-1}p + (p^T\partial^{-1}p)^T & -(\partial + \sum_{k=1}^{2} p_k\partial^{-1}q_k)I_2 - p^T\partial^{-1}q^T \\ -(\partial + \sum_{k=1}^{2} p_k\partial^{-1}q_k)I_2 - q\partial^{-1}p & q\partial^{-1}q^T + (q\partial^{-1}q^T)^T \end{bmatrix}. \tag{2.30b}$$

Adjoint symmetry constraints or equivalently symmetry constraints separate the four-component AKNS equations into two commuting finite-dimensional Liouville integrable Hamiltonian systems [9]. In the next section, we will focus on the system of coupled mKdV equations (2.27b).

3. Riemann–Hilbert problems

The spectral problems of the system of coupled mKdV equations (2.27b) are

$$-\,i\phi_x = U\phi,\quad -\,i\phi_t = V^{[3]}\phi, \tag{3.1}$$

with

$$U = \lambda\Lambda + P,\ V^{[3]} = \lambda^3\Omega + Q, \tag{3.2}$$

where $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2, \beta_2)$, and

$$P = \begin{bmatrix} 0 & p_1 & p_2 \\ q_1 & 0 & 0 \\ q_2 & 0 & 0 \end{bmatrix},\ Q = \begin{bmatrix} a^{[1]}\lambda^2 + a^{[2]}\lambda + a^{[3]} & b^{[1]}\lambda^2 + b^{[2]}\lambda + b^{[3]} \\ c^{[1]}\lambda^2 + c^{[2]}\lambda + c^{[3]} & d^{[1]}\lambda^2 + d^{[2]}\lambda + d^{[3]} \end{bmatrix}, \tag{3.3}$$

$a^{[m]}$, $b^{[m]}$, $c^{[m]}$ and $d^{[m]}$, $1 \leq m \leq 3$, being defined in (2.21h).

In this section, we present the scattering and inverse scattering methods for the coupled mKdV system (2.27b) using the Riemann–Hilbert formulation [1,17,18]. The resulting results will lay the groundwork for N-soliton solutions in the next section. Suppose that all the potentials rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$ and satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^m|t|^n(|p_1| + |p_2| + |q_1| + |q_2|)\,dxdt < \infty,\ m, n \geq 0. \tag{3.4}$$

For the sake of presentation, we also assume that $\alpha = \alpha_1 - \alpha_2 < 0$ and $\beta = \beta_1 - \beta_2 < 0$.

In the Riemann–Hilbert formulation, we treat ϕ in the spectral problems (3.1) as a fundamental matrix. From (3.1), we note, under (3.4), that when $x, t \to \pm\infty$, one has the asymptotic behavior: $\phi \sim E = \mathrm{e}^{i\lambda\Lambda x + i\lambda^3\Omega t}$. This motivates us to introduce the variable transformation

$$\psi = \phi\mathrm{e}^{-i\lambda\Lambda x - i\lambda^3\Omega t}, \tag{3.5}$$

to have the canonical normalization for the associated Riemann–Hilbert problem:

$$\psi \to I_3, \text{ when } x, t \to \pm\infty, \tag{3.6}$$

where $I_3 = \mathrm{diag}(1, 1, 1)$. This way, the spectral problems in (3.1) equivalently lead to

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.7}$$

$$\psi_t = i\lambda^3[\Omega, \psi] + \check{Q}\psi, \tag{3.8}$$

where $\check{P} = iP$ and $\check{Q} = iQ$. Noting $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$, we have

$$\det \psi = 1, \tag{3.9}$$

by Abel's formula.

Let us now consider the formulation of an associated Riemann–Hilbert problem with the variable x. In the scattering problem, we first introduce the matrix solutions $\psi^{\pm}(x, \lambda)$ of (3.7) with the asymptotic conditions

$$\psi^{\pm} \to I_3, \text{ when } x \to \pm\infty, \tag{3.10}$$

respectively. The subscripts above refer to which end of the x-axis the boundary conditions are required. Then, by (3.9), we have $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since $\phi^{\pm} = \psi^{\pm}E$ are both solutions of (3.1), they must be linearly related, and so we can have

$$\psi^{-}E = \psi^{+}ES(\lambda), \ \lambda \in \mathbb{R}, \tag{3.11}$$

where

$$S(\lambda) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}, \ \lambda \in \mathbb{R}, \tag{3.12}$$

is the scattering matrix. Note that $\det(S(\lambda)) = 1$ since $\det(\psi^{\pm}) = 1$. Using the method of variation in parameters as well as the boundary condition (3.10), we can turn the x-part of (3.1) into the following Volterra integral equations for $\psi^{\pm}$ [1]:

$$\psi^{-}(\lambda, x) = I_3 + \int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^{-}(\lambda, y)e^{i\lambda\Lambda(y-x)}\, dy, \tag{3.13}$$

$$\psi^{+}(\lambda, x) = I_3 - \int_{x}^{\infty} e^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^{+}(\lambda, y)e^{i\lambda\Lambda(y-x)}\, dy. \tag{3.14}$$

Thus, $\psi^{\pm}$ allows analytical continuations off the real axis $\lambda \in \mathbb{R}$ as long as the integrals on their right hand sides converge. It is direct to see that the integral equation for the first column of ψ^{-} contains only the exponential factor $e^{-i\alpha\lambda(x-y)}$, which, due to $y < x$ in the integral, decays when λ is in the upper half-plane $\mathbb{C}^{+} = \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\}$, and the integral equation for the last two columns of ψ^{+} contains only the exponential factor $e^{i\alpha\lambda(x-y)}$, which, due to $y > x$ in the integral, also decays when λ is in the upper half-plane $\mathbb{C}^{+}$. Thus, these three columns can be analytically continued to the upper half-plane $\lambda \in \mathbb{C}^{+}$. Similarly, we find that the last two columns of ψ^{-} and the first column of ψ^{+} can be analytically continued to the lower half-plane $\lambda \in \mathbb{C}^{-} = \{z \in \mathbb{C} |\, \mathrm{Im}(z) < 0\}$. Let us express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \psi_3^{\pm}), \tag{3.15}$$

that is, $\psi_k^{\pm}$ stands for the kth column of $\phi^{\pm}$ $(1 \le k \le 3)$. Then the matrix solution

$$P^{+} = P^{+}(x, \lambda) = (\psi_1^{-}, \psi_2^{+}, \psi_3^{+}) = \psi^{-}H_1 + \psi^{+}H_2 \tag{3.16}$$

is analytic in $\lambda \in \mathbb{C}^{+}$, and the matrix solution

$$(\psi_1^{+}, \psi_2^{-}, \psi_3^{-}) = \psi^{+}H_1 + \psi^{-}H_2 \tag{3.17}$$

is analytic in $\lambda \in \mathbb{C}^{-}$, where $H_1 = \mathrm{diag}(1, 0, 0)$ and $H_2 = \mathrm{diag}(0, 1, 1)$. In addition, from the Volterra integral equation (3.13), we find that

$$P^{+}(x, \lambda) \to I_3, \text{ when } \lambda \in \mathbb{C}^{+} \to \infty, \tag{3.18}$$

and

$$(\psi_1^{+}, \psi_2^{-}, \psi_3^{-}) \to I_3, \text{ when } \lambda \in \mathbb{C}^{-} \to \infty. \tag{3.19}$$

Next we construct the analytic counterpart of P^{+} in the lower half-plane $\mathbb{C}^{-}$. Note that the adjoint equation of the x-part of (3.1) and the adjoint equation of (3.7) read as

$$i\tilde{\phi}_x = \tilde{\phi}U, \tag{3.20}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P. \tag{3.21}$$

It is easy to see that the inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ solve these adjoint equations, respectively. If we express $\tilde{\psi}^{\pm}$ as follows:

$$\tilde{\psi}^{\pm} = \begin{bmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \tilde{\psi}^{\pm,3} \end{bmatrix}, \tag{3.22}$$

that is, $\tilde{\psi}^{\pm,k}$ stands for the kth row of $\tilde{\psi}^{\pm}$ ($1 \le k \le 3$). Then by similar arguments, we can show that adjoint matrix solution

$$P^- = \begin{bmatrix} \tilde{\psi}^{-,1} \\ \tilde{\psi}^{+,2} \\ \tilde{\psi}^{+,3} \end{bmatrix} = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1} \tag{3.23}$$

is analytic for $\lambda \in \mathbb{C}^-$, and the other matrix solution

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{-,3} \end{bmatrix} = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1} \tag{3.24}$$

is analytic for $\lambda \in \mathbb{C}^+$. In the same way, we see that

$$P^-(x,\lambda) \to I_3, \text{ when } \lambda \in \mathbb{C}^- \to \infty, \tag{3.25}$$

and

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{-,3} \end{bmatrix} \to I_3, \text{ when } \lambda \in \mathbb{C}^+ \to \infty. \tag{3.26}$$

Now we have constructed two matrix functions P^+ and P^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. It is direct to see that on the real line, the two matrix functions P^+ and P^- are related by

$$P^-(x,\lambda)P^+(x,\lambda) = G(x,\lambda), \ \lambda \in \mathbb{R}, \tag{3.27}$$

where

$$\begin{aligned} G(x,\lambda) &= E(H_1 + H_2S)(H_1 + S^{-1}H_2)E^{-1} \\ &= E\begin{bmatrix} 1 & s_{13}s_{32} - s_{12}s_{33} & s_{12}s_{23} - s_{13}s_{22} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{bmatrix}E^{-1}. \end{aligned} \tag{3.28}$$

Eqs. (3.27) and (3.28) are exactly the associated matrix Riemann–Hilbert problem we wanted to present. The asymptotics

$$P^{\pm}(x,\lambda) \to I_3, \text{ when } \lambda \in \mathbb{C}^{\pm} \to \infty, \tag{3.29}$$

provide the canonical normalization condition for the established Riemann–Hilbert problem.

To finish the direct scattering transform, we take the derivative of (3.11) with time t and use the vanishing conditions of the potentials, we can show that S satisfies

$$S_t = i\lambda^3[\Omega, S], \tag{3.30}$$

which gives the time evolution of the scattering coefficients:

$$\begin{cases} s_{11,t} = s_{22,t} = s_{33,t} = s_{23,t} = s_{32,t} = 0, \\ s_{12} = s_{12}(\lambda,0)e^{i\beta\lambda^3 t}, s_{13} = s_{13}(\lambda,0)e^{i\beta\lambda^3 t}, \\ s_{21} = s_{21}(\lambda,0)e^{-i\beta\lambda^3 t}, s_{31} = s_{31}(\lambda,0)e^{-i\beta\lambda^3 t}. \end{cases} \tag{3.31}$$

4. *N*-soliton solutions

The Riemann–Hilbert problems with zeros generate soliton solutions. The uniqueness of the associated Riemann–Hilbert problem (3.27) does not hold unless the zeros of $\det P^+$ and $\det P^-$ in the upper and lower half-planes are specified and the kernel structures of $P^{\pm}$ at these zeros are determined [19,20]. Following the definitions of $P^{\pm}$ as well as the scattering relation between ψ^{+} and ψ^{-}, we find that

$$\det P^{+}(x,\lambda)=s_{33}(\lambda),\ \ \det P^{-}(x,\lambda)=\hat{s}_{33}(\lambda), \tag{4.1}$$

where $\hat{s}_{33}=(S^{-1})_{33}=s_{11}s_{22}-s_{12}s_{21}$ due to $\det S=1$. Suppose that s_{33} has zeros $\{\lambda_k\in\mathbb{C}^{+},\ 1\le k\le N\}$, and $\hat{s}_{33}$ has zeros $\{\hat{\lambda}_k\in\mathbb{C}^{-},\ 1\le k\le N\}$. For simplicity, we assume that these zeros, λ_k and $\hat{\lambda}_k,\ 1\le k\le N$, are simple. Then, each of $\ker P^{+}(\lambda_k)$, $1\le k\le N$, contains only a single column vector, denoted by v_k, $1\le k\le N$; and each of $\ker P^{+}(\hat{\lambda}_k)$, $1\le k\le N$, a row vector, denoted by $\hat{v}_k$, $1\le k\le N$:

$$P^{+}(\lambda_k)v_k=0,\ \hat{v}_kP^{-}(\hat{\lambda}_k)=0,\ 1\le k\le N. \tag{4.2}$$

The Riemann–Hilbert problem (3.27) with the canonical normalization condition (3.29) and the zero structure (4.2) can be solved explicitly [1,21], and thus one can readily reconstruct the potential P as follows. Note that P^{+} is a solution to the spectral problem (3.7). Therefore, as long as we expand P^{+} at large λ as

$$P^{+}(x,\lambda)=I_3+\frac{1}{\lambda}P_1^{+}(x)+O(\frac{1}{\lambda^2}),\ \lambda\to\infty, \tag{4.3}$$

inserting this expansion into (3.7) and comparing $O(1)$ terms leads to

$$\check{P}=-i[\Lambda,P_1^{+}], \tag{4.4}$$

which implies that

$$P=-[\Lambda,P_1^{+}]=\begin{bmatrix}0 & -\alpha(P_1^{+})_{12} & -\alpha(P_1^{+})_{13}\\ \alpha(P_1^{+})_{21} & 0 & 0\\ \alpha(P_1^{+})_{31} & 0 & 0\end{bmatrix}, \tag{4.5}$$

where $P_1^{+}=((P_1^{+})_{kl})_{1\le k,l\le 3}$. Further, the potentials p_k and q_k, $k=1,2$, can be computed as

$$\begin{cases}p_1=-\alpha(P_1^{+})_{12},\ p_2=-\alpha(P_1^{+})_{13},\\ q_1=\alpha(P_1^{+})_{21},\ q_2=\alpha(P_1^{+})_{31},\end{cases} \tag{4.6}$$

To obtain soliton solutions, we set $G=I_3$ in the Riemann–Hilbert problem (3.27). This can be achieved if we assume $s_{12}=s_{13}=s_{21}=s_{31}=0$, which means that there is no reflection in the scattering problem. The solutions to this specific Riemann–Hilbert problem can be given as follows (see, e.g., [1,21]):

$$P^{+}(\lambda)=I_3-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\ P^{-}(\lambda)=I_3+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_l}, \tag{4.7}$$

where $M=(M_{kl})_{N\times N}$ is a square matrix whose entries read

$$M_{kl}=\frac{\hat{v}_kv_l}{\lambda_l-\hat{\lambda}_k},\ 1\le k,l\le N. \tag{4.8}$$

Noting that the zeros λ_k and $\hat{\lambda}_k$ are constants, i.e., space and time independent, we can easily find the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1\le k\le N$. For example, let us take the x-derivative of both sides of the equation $P^{+}(\lambda_k)v_k=0$. By using (3.7) and then $P^{+}(\lambda_k)v_k=0$, we get

$$P^{+}(\lambda_k,x)\Big(\frac{dv_k}{dx}-i\lambda_k\Lambda v_k\Big)=0,\ 1\le k\le N,$$

which implies

$$\frac{dv_k}{dx}=i\lambda_k\Lambda v_k,\ 1\le k\le N.$$

The time dependence of v_k:

$$\frac{dv_k}{dt}=i\lambda_k^3\Omega v_k,\ 1\le k\le N,$$

can be determined similarly through an associated Riemann–Hilbert problem with the variable t. Summing up, we obtain

$$v_k(x,t)=e^{i\lambda_k\Lambda x+i\lambda_k^3\Omega t}v_{k,0},\ 1\le k\le N, \tag{4.9}$$

$$\hat{v}_k(x,t)=\hat{v}_{k,0}e^{-i\hat{\lambda}_k\Lambda x-i\hat{\lambda}_k^3\Omega t},\ 1\le k\le N, \tag{4.10}$$

where $v_{k,0}$ and $\hat{v}_{k,0}$, $1\le k\le N$, are arbitrary constant vectors.

Finally, from (4.7), we get

$$P_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.11}$$

and thus by (4.6), the N-soliton solution to the system of coupled mKdV equations (2.27b):

$$\begin{cases} p_1 = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,2}, \; p_2 = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,3}, \\ q_1 = -\alpha \sum_{k,l=1}^{N} v_{k,2}(M^{-1})_{kl}\hat{v}_{l,1}, \; q_2 = -\alpha \sum_{k,l=1}^{N} v_{k,3}(M^{-1})_{kl}\hat{v}_{l,1}, \end{cases} \tag{4.12}$$

where $v_k = (v_{k,1}, v_{k,2}, v_{k,3})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3})$, $1 \le k \le N$, are arbitrary.

5. Concluding remarks

The paper is dedicated to development of Riemann–Hilbert problem representations and associated N-soliton solutions to integrable equations. The starting point is a kind of equivalent spectral problem, which guarantees the existence of analytical eigenfunctions in the upper or lower half-plane. We considered a 3×3 degenerate matrix spatial spectral problem and worked out its soliton hierarchy. Taking the system of coupled mKdV equations as an example, we computed its associated Riemann–Hilbert problems, together with an explicit formula for jump matrices. From the case of taking the identity jump matrix, we generated N-soliton solutions to the considered system of coupled mKdV equations.

We see the effectiveness of using the Riemann–Hilbert formulation to derive N-soliton solutions (see, [22–24], for other examples). We point out that lump solutions could be generated within the Riemann–Hilbert formulation. How about other solutions such as positon solutions and complexiton solutions [25,26]? About systems of coupled mKdV equations, there are other studies such as integrable couplings [27], super hierarchies [28] and fractional counterparts [29]. The Riemann–Hilbert method has been also generalized to solve initial–boundary value problems of integrable equations on the half-line [30]. There are many other approaches to soliton solutions in the field of integrable systems, which include the bilinear method [31,32], the Wronskian technique [33,34] and the Darboux transformation [35]. It should be interesting to study exact solutions to soliton equations, particularly rational solutions [36,37] and algebro-geometric solutions [38,39], using other solution techniques.

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