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RESEARCH

The First Negative-order Matrix AKNS Flow and its Darboux Transformation

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Abstract
This paper presents the first negative-order matrix AKNS flow, derived by associating a Lax pair containing a first-order pole in the spectral parameter with the matrix AKNS spectral problem. The corresponding Darboux transformation is constructed within the AKNS framework. Starting from a seed solution, a class of exact and explicit solutions to the nonlinear negative-order model is generated via a single application of the derived Darboux transformation.

Keywords Lax pair · Matrix AKNS flows · Darboux transformation

Mathematics Subject Classification (2010) 37K10 · 35Q51 · 37K40 · 02.30.Ik · 05.45.Yv

1 Introduction

The Darboux transformation is a powerful algebraic method for generating new solutions from known ones in integrable systems, especially in soliton theory [1]. Applied to the Lax pair of an integrable system, it preserves integrability and enables the construction of multi-soliton solutions, rational solutions, rogue waves, and other complex structures. The method has been extended to matrix, higher-dimensional, noncommutative, and supersymmetric settings (see, e.g., [2–4]). Consider the zero-curvature formulation for integrable systems, in which a system of PDEs

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$$u_t = K(u) = K(x, t, u, u_x, \cdots), \tag{1}$$

is represented by the zero-curvature condition:

$$U_t - V_x + [U, V] = 0, \tag{2}$$

where U and V are matrix-valued functions (the Lax pair) belonging to a matrix loop algebra [5]. This zero-curvature equation arises as the compatibility condition of the matrix spectral problems

$$\phi_x = U\phi = U(u, \lambda)\phi, \ \phi_t = V\phi = V(u, \lambda)\phi, \tag{3}$$

where λ is the spectral parameter and ϕ is the vector eigenfunction.

A Darboux transformation consists of a gauge transformation $\phi' = D\phi$ and a new potential $u' = u'(u)$, with $D = D(u, \lambda)$ a matrix function of λ, such that the transformed function ϕ' solves a pair of matrix spectral problems of the same structure:

$$\phi'_x = U'\phi' = U(u', \lambda)\phi', \ \phi'_t = V'\phi' = V(u', \lambda)\phi', \tag{4}$$

where U' and V' retain the structure of the original Lax pair. The Darboux matrix D must satisfy the two compatibility conditions:

$$U'D = DU + D_x, \ V'D = DV + D_t. \tag{5}$$

One of important tasks in the field of integrable systems is to construct such Darboux transformations explicitly. Suppose that U and V are $N \times N$ matrices. A first-order transformation can often be chosen in the form:

$$D(\lambda) = \lambda I_N - S, \tag{6}$$

where I_N is the identity matrix of order N and S is an $N \times N$ matrix independent of λ. Given N eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$, let $\phi^{[j]}$ denote the corresponding eigenfunctions that satisfy:

$$\phi^{[j]}_x = U(u, \lambda_j)\phi^{[j]}, \ \phi^{[j]}_t = V(u, \lambda_j)\phi^{[j]}, \ 1 \leq j \leq N, \tag{7}$$

for some fixed solution u of (1). Then, a standard choice for the matrix S is:

$$S = HAH^{-1}, H = (\phi^{[1]}, \cdots, \phi^{[N]}), \tag{8}$$

where $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, as described in [2, 6]. Verifying the compatibility conditions (5) ensures the validity of the transformation.

In this letter, we begin with the matrix AKNS spectral problem and propose the first negative-order matrix AKNS flow. We construct its associated Darboux transformation within the AKNS framework and apply it once to generate a class of explicit solutions. The final section offers concluding remarks and discusses directions for future research.

2 The Lax Pair and the Matrix Model

Let α_1 and α_2 be two distinct constants, and let *m* and *n* be natural numbers. We consider the AKNS spectral matrix [7] of the form:

$$U = i\lambda\Lambda + Q,\ \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix},\ Q = \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}, \tag{9}$$

where *q* and *r* are the potential matrices of sizes $m \times n$ and $n \times m$, respectively. We define the Lax operator by

$$V = -\frac{1}{\alpha\lambda}W,\ \alpha = \alpha_1 - \alpha_2, \tag{10}$$

where

$$W = W_1 + W_2,\ W_1 = -i\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix},\ W_2 = i\begin{bmatrix} 0 & -q_t \\ r_t & 0 \end{bmatrix}, \tag{11}$$

with *v* and *w* being two square potential matrices of orders *m* and *n*, respectively.

The zero-curvature equation

$$U_t - V_x + [U, V] = 0 \tag{12}$$

is equivalent to

$$-\alpha\lambda U_t - W_x + [U, W] = 0. \tag{13}$$

By equating coefficients of powers of λ, we obtain the system:

$$\begin{cases} -\alpha Q_t + i[\Lambda, W] = 0, \\ -W_x + [Q, W] = 0. \end{cases} \tag{14}$$

The first equation is satisfied identically due to the choices:

$$\left[\Lambda, W_1\right] = 0,\ \left[\Lambda, W_2\right] = -i\alpha Q_t \tag{15}$$

For the second equation, we compute

$$\left[Q, W_1\right] = -i\begin{bmatrix} 0 & qw - vq \\ rv - wr & 0 \end{bmatrix},\ \left[Q, W_2\right] = i\begin{bmatrix} (qr)_t & 0 \\ 0 & -(rq)_t \end{bmatrix}. \tag{16}$$

Thus, the second equation yields:

$$\begin{cases} q_{tx} - qw + vq = 0, \\ r_{tx} + rv - wr = 0, \\ v_x + (qr)_t = 0, \\ w_x - (rq)_t = 0. \end{cases} \tag{17}$$

Since V contains a simple pole of λ, this system represents the first negative-order matrix AKNS flow. It should be noted that this represents a special case of the system introduced in [6], corresponding to the parameter choices

$$n = 2,\ p_1 = x,\ x_2 = t,\ A_1 = -U,\ a_2 = -\alpha\lambda,\ A_2 = W. \tag{18}$$

All functions involved in the Lax pair U and V, including q and r, depends solely on x and t. Consequently, there is no dependence on x_1 and p_2.

If we take

$$v = -\frac{a}{2} - \partial_x^{-1}(qr)_t,\ w = \frac{a}{2} + \partial_x^{-1}(rq)_t, \tag{19}$$

where a is an arbitrary constant, then the system simplifies to:

$$\begin{cases} q_{tx} - q\partial_x^{-1}(rq)_t - \partial_x^{-1}(qr)_t q = aq, \\ r_{tx} - r\partial_x^{-1}(qr)_t - \partial_x^{-1}(rq)_t r = ar. \end{cases} \tag{20}$$

When $a = 1$, this system corresponds to the first negative-order AKNS system studied in [8]. Furthermore, in the scalar case $m = n = 1$, it reduces to the first negative-order scalar AKNS system presented in [9, 10]. There are other negative-order AKNS flows (see, e.g., [11–13]). It is worth noting that the resulting equations do not depend explicitly on the constants α_1 and α_2.

Let

$$m = n = 1,\ w = -v. \tag{21}$$

Then, the system becomes

$$\begin{cases} q_{tx} + 2vq = 0, \\ r_{tx} + 2vr = 0, \\ v_x + (qr)_t = 0. \end{cases} \tag{22}$$

With $r = q$, this reduces to

$$\begin{cases} q_{tx} + 2vq = 0, \\ v_x + (q^2)_t = 0. \end{cases} \tag{23}$$

Let

$$m = n = 1,\ r = q(-x, t),\ w = v(-x, t), \tag{24}$$

then the system becomes a nonlocal model:

$$\begin{cases} q_{tx} + (v - v(-x,t))q = 0, \\ v_x + (qq(-x,t))_t = 0. \end{cases} \tag{25}$$

3 Darboux Transformation

3.1 Compatibility Conditions

We assume that he Darboux matrix takes the form:

$$D(\lambda) = \lambda I_{m+n} - S. \tag{26}$$

To satisfy the Darboux transformation conditions, this matrix must fulfill the spatial requirement

$$U'D = DU + D_x, \tag{27}$$

where U is given by (9), and the transformed operator is

$$U' = i\lambda\Lambda + Q'. \tag{28}$$

Substituting into the requirement yields:

$$(i\lambda\Lambda + Q')(\lambda I_{m+n} - S) = (\lambda I_{m+n} - S)(i\lambda\Lambda + Q) - S_x,$$

which implies:

$$Q' = Q + i[\Lambda, S], \tag{29}$$

and

$$S_x = Q'S - SQ = [Q + i\Lambda S, S]. \tag{30}$$

Next, we consider the temporal compatibility condition:

$$V'D = DV + D_t, \tag{31}$$

where

$$V' = -\frac{1}{\alpha\lambda}W'. \tag{32}$$

This gives:

$$W'(\lambda I_{m+n} - S) = (\lambda I_{m+n} - S)W + \alpha\lambda S_t,$$

which implies:

$$W' = W + \alpha S_t,\ W'S - SW = 0.$$

Thus, we have

$$W' = SWS^{-1}, \tag{33}$$

and

$$S_t = \frac{1}{\alpha}(SWS^{-1} - W). \tag{34}$$

3.2 Construction of the Darboux Matrix

Following the general framework (see, e.g., [2, 6]), we formulate the matrix S as:

$$S = HAH^{-1}, \tag{35}$$

where

$$H = (\phi^{[1]}, \cdots, \phi^{[m+n]}),\ A = \text{diag}(\lambda_1, \cdots, \lambda_{m+n}), \tag{36}$$

and each column vector $\phi^{[j]}$ satisfies the matrix spectral problems:

$$\phi_x^{[j]} = U(u, \lambda_j)\phi^{[j]},\ \phi_t^{[j]} = V(u, \lambda_j)\phi^{[j]},\ 1 \leq j \leq m+n. \tag{37}$$

It follows that

$$H_x = i\Lambda HA + QH,\ H_t = WHB, \tag{38}$$

where

$$B = \text{diag}(-\frac{1}{\alpha\lambda_1}, \cdots, -\frac{1}{\alpha\lambda_{m+n}}). \tag{39}$$

We can then compute:

$$
\begin{aligned}
S_x =& H_x A H^{-1} - HA(H^{-1}H_x H^{-1}) \\
=& i\Lambda H A^2 H^{-1} + QHAH^{-1} - HAH^{-1}(i\Lambda HA + QH)H^{-1} \\
=& i\Lambda H A^2 H^{-1} + QHAH^{-1} - iHAH^{-1}\Lambda HAH^{-1} - HAH^{-1}Q \\
=& QS - SQ + i\Lambda S^2 - iS\Lambda S,
\end{aligned}
$$

which confirms that the spatial condition (30) is satisfied.

Similarly, from the second equality in (38), we obtain:

$$
S_t = H_t A H^{-1} - HA(H^{-1}H_t H^{-1}) = WHBAH^{-1} - HA(H^{-1}WHBH^{-1}).
$$

Using the identity $\alpha BA = -I_{m+n}$, it simplifies to:

$$
\begin{aligned}
\alpha S_t =& WH(\alpha BA)H^{-1} - HAH^{-1}WH(\alpha B)H^{-1} \\
=& - WHH^{-1} + HAH^{-1}WHA^{-1}H^{-1} \\
=& - W + SWS^{-1} = -W + W',
\end{aligned}
$$

confirming the temporal condition (34).

3.3 The Darboux Transformation

We thus obtain the Darboux transformation:

$$
\phi' = (\lambda I_{m+n} - S)\phi, \; U' = i\lambda\Lambda + Q', \; V' = -\frac{1}{\alpha\lambda}W', \tag{40}
$$

with

$$
Q' = Q + i[\Lambda, S], \; W' = SWS^{-1}, \tag{41}
$$

where *S* is given by (35). Based on the structure of *U* and *W* in (9) and (11), this transformation gives the following expressions for the transformed quantities:

$$
\begin{cases} q' = q + i[\Lambda, S]_{12}, \; r' = r + i[\Lambda, S]_{21}, \\ v' = i(SWS^{-1})_{11}, \; w' = i(SWS^{-1})_{22}, \end{cases} \tag{42}
$$

where M_{jk} denotes the (*j*, *k*)-th block of the matrix *M*.

3.4 Explicit Solutions

As an application, consider the seed solution

$$
q = r = 0, \; v = v_0, \; w = w_0, \tag{43}
$$

where v_0 and w_0 are constant matrices. Then, the eigenfunctions take the form:

$$\phi^{[j]} = (\phi_1^{[j]T}, \phi_2^{[j]T})^T, \tag{44}$$

with

$$\begin{cases} \phi_1^{[j]} = \exp(i\alpha_1\lambda_j I_m x + \frac{i}{\alpha\lambda_j} v_0 t)\mu_1^{[j]}, \\ \phi_2^{[j]} = \exp(i\alpha_2\lambda_j I_n x + \frac{i}{\alpha\lambda_j} w_0 t)\mu_2^{[j]}, \end{cases} \quad 1 \le j \le m+n, \tag{45}$$

where $\mu_1^{[j]}$ and $\mu_2^{[j]}$ are arbitrary constant vectors of sizes m and n, respectively. Then, the following explicit solutions to the system (17) are obtained:

$$q' = i[\Lambda, S]_{12},\ r' = i[\Lambda, S]_{21},\ v' = i\alpha(SWS^{-1})_{11},\ w' = i(SWS^{-1})_{22}, \tag{46}$$

where $S = HAH^{-1}$ with $H = (\phi^{[1]}, \cdots, \phi^{[m+n]})$ and $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_{m+n})$.

4 Concluding Remarks

We have constructed the first negative-order matrix AKNS flow and its associated Darboux transformation, along with a class of explicit solutions generated from a single application of the transformation. Iterating this process N times yields a higher-order Darboux transformation. Other developments on Darboux transformations for negative-order flows can be found in the literature (see, e.g., [14–16]).

An intriguing question is whether negative-order flows form commuting hierarchies, analogous to their well-known positive-order counterparts. A broader generalization involves Lax operators with higher-order poles in the spectral parameter, and a major open problem is to characterize Darboux transformations in this extended, higher negative-order framework.

Constrained Lax pairs are of particular interest, especially when reductions are imposed on the general spectral matrix U (see, e.g., [17, 18]). Such reductions often lead to nonlocal integrable models, which themselves form commuting hierarchies [19]. A central challenge is to develop systematic methods for constructing Darboux transformations in these constrained matrix settings.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval Not applicable.

Competing interests The author declares no competing interests.

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