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# A combined integrable hierarchy with four potentials and its recursion operator and bi-Hamiltonian structure

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**Abstract:** Based on a specific matrix Lie algebra, we propose a spectral matrix with four potentials and generate its associated Liouville integrable Hamiltonian hierarchy. The zero curvature formulation and the trace identity are the basic tools. The Liouville integrability of the resulting hierarchy is shown by determining its recursion operator and bi-Hamiltonian structure. Two illustrative examples of generalized combined nonlinear Schrödinger equations and modified Korteweg-de Vries equations are explicitly presented. The success lies in introducing a specific $4 \times 4$ spectral matrix which keads to an integrable hierarchy.

**Keywords:** Matrix eigenvalue problem; Zero curvature equation; Integrable hierarchy; NLS equations; mKdV equations

**Mathematics Subject Classification:** 37K15; 35Q55; 37K40

## 1. Introduction

Integrable models comes in hierarchies [1, 2] and are generated from Lax pairs of matrix eigenvalue problems [3]. Lax pairs can also be used to establish Hamiltonian structures, which connect symmetries with conserved quantities. Integrable models have various applications in physical and engineering sciences, including fluid dynamics, nonlinear optics and quantum mechanics.

Among typical examples of integrable hierarchies are the Ablowitz–Kaup–Newell–Segur hierarchy [4] and its various hierarchies of integrable couplings [6]. Matrix Lie algebras provide a solid foundation for constructing integrable models and building their Lax pairs [5–7]. It is always intriguing to explore what Lax pairs will yield integrable models. In this paper, we would like to present a novel matrix eigenvalue problem and compute an associated integrable hierarchy, based on a specific matrix Lie algebra.

It is known that the zero curvature formulation paves the way for exploring integrable models (see [7, 8] for details). As usual, we denote a $q$-dimensional column potential vector by $u = (u_1, \cdots, u_q)^T$ and the spectral parameter by $\lambda$. Starting from a given loop matrix algebra $\tilde{g}$ with the loop parameter $\lambda$, we take a spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = u_1 F_1(\lambda) + \cdots + u_q F_q(\lambda) + F_0(\lambda), \quad (1)$$

where the elements $F_1, \cdots, F_q$ are linear independent in the vector space $\tilde{g}$. The above element $F_0$ is always assumed to be pseudo-regular:

$$\mathrm{Im}\,\mathrm{ad}_{F_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{F_0} = \tilde{g}, \ \ [\mathrm{Ker}\,\mathrm{ad}_{F_0}, \mathrm{Ker}\,\mathrm{ad}_{F_0}] = 0,$$

where $\mathrm{ad}_{F_0}$ denotes the adjoint action of $F_0$ on $\tilde{g}$. Then we can find a Laurent series solution $Y = \sum_{n \geq 0} \lambda^{-n} Y^{[n]}$ to the stationary zero curvature equation

$$Y_x = [\mathcal{M}, Y] \quad (2)$$

in the underlying loop algebra $\tilde{g}$.

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The second step is to define an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (\lambda^m Y)_+ + \Delta_r = \sum_{n=0}^{m} \lambda^{m-n} Y^{[n]} + \Delta_m,\ m \geq 0, \tag{3}$$

where $\Delta_m \in \tilde{g},\ m \geq 0$, which provide the other parts of Lax pairs. The zero curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}_x^{[m]} + [\mathcal{M}, \mathcal{N}^{[m]}] = 0,\ m \geq 0, \tag{4}$$

produce a hierarchy of integrable models:

$$u_{t_m} = X^{[m]} = X^{[m]}(u),\ m \geq 0, \tag{5}$$

which actually represent the solvability conditions of the spatial and temporal matrix eigenvalue problems:

$$\varphi_x = \mathcal{M}\varphi,\ \varphi_{t_m} = \mathcal{N}^{[m]}\varphi,\ m \geq 0. \tag{6}$$

The last step is to establish a bi-Hamiltonian structure for the hierarchy (5), by finding a recursion operator and applying the so-called trace identity:

$$\frac{\delta}{\delta u} \int \mathrm{tr}\left(Y \frac{\partial \mathcal{M}}{\partial \lambda}\right) dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \mathrm{tr}\left(Y \frac{\partial \mathcal{M}}{\partial u}\right), \tag{7}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to $u$, and $\kappa$ is a constant, independent of the spectral parameter $\lambda$. It then follows from that every member in the hierarchy is Liouville integrable (see, e.g., [7, 9]).

There are abundant hierarchies of Liouville integrable models, presented in the literature [4–21]. The case of two components is popular and the well-known examples include the Ablowitz-Kaup-Newell-Segur integrable hierarchy [4], the Heisenberg hintegrable ierarchy [22], the Kaup–Newell integrable hierarchy [23] and the Wadati–Konno–Ichikawa integrable hierarchy [24]. Their associated spectral matrices are given by

$$\mathcal{M} = \begin{bmatrix} \lambda & u_1 \\ u_2 & -\lambda \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} \lambda u_3 & \lambda u_1 \\ \lambda u_2 & -\lambda u_3 \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} \lambda^2 & \lambda u_1 \\ \lambda u_2 & -\lambda^2 \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} \lambda & \lambda u_1 \\ \lambda u_2 & -\lambda \end{bmatrix} \tag{8}$$

where $u_1 u_2 + u_3^2 = 1$, respectively.

This paper aims to propose a specific $4 \times 4$ spectral matrix and generate a hierarchy of four-component Liouville integrable models within the zero curvature formulation, based on a special matrix Lie algebra. A recurson operator and a bi-Hamiltonian structure will be explored to show the Liouville integrability for the resulting hierarchy. Two illustrative examples, consisting of generalized combined integrable nonlinear Schrödinger and modified Korteweg-de Vries models, are presented. The success lies in presenting a specific $4 \times 4$ spectral matrix which leads to an integrable hierarchy. A conclusion and a few concluding remarks are given in the last section.

## 2. A four-component integrable hierarchy

Let $\delta$ be an arbitrary real number, and $T$ be a square matrix of order $r \in \mathbb{N}$ such that

$$T^2 = I_r, \tag{9}$$

wheer $I_r$ denotes the identity matrix of order $r$. Let us define a set $\tilde{g}$ of block matrices to be

$$\tilde{g} = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r \times 2r} \middle| A_4 = T A_1 T^{-1},\ A_3 = \delta T A_2 T^{-1} \right\} \tag{10}$$

Obviously, this forms a matrix Lie algebra under the matrix commutator $[A, B] = AB - BA$. We will utilize the Lie algebra with $r = 2$ and

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{11}$$

to introduce a specific spectral matrix below.

Let $\alpha_1$ and $\alpha_2$ be two arbitrary real numbers, and $u = u(x,t) = (u_1, u_2, u_3, u_4)^T$, a column vector with four potentials. Assume that

$$\alpha = \alpha_1 - \alpha_2 \neq 0. \tag{12}$$

Based on recent studies on matrix eigenvalue problems involving four potentials (see, e.g., [25–27] and [28, 29] for examples of matrix eigenvalue problems of arbitrary-order and fourth-order, respectively), we would like to propose a matrix eigenvalue problem of the form:

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u,\lambda)\varphi,\ \mathcal{M} = \begin{bmatrix} \alpha_1\lambda & u_1 & u_2 & 0 \\ u_3 & \alpha_2\lambda & 0 & u_4 \\ \delta u_4 & 0 & \alpha_2\lambda & u_3 \\ 0 & \delta u_2 & u_1 & \alpha_1\lambda \end{bmatrix}, \tag{13}$$

where $\lambda$ is again the spectral parameter. This spectral matrix $\mathcal{M}$ is from the previous matrix Lie algebra $\tilde{g}$, mentioned previously. The eigenvalue problem can not be any reduction of the matrix Ablowitz-Kaup-Newell-Segur eigenvalue problem (see, e.g., [30]). Interestingly, an associated integrable hierarchy of bi-Hamiltonian equations can be generated, which shows particular combined structures of integrable models. Obviously, the case of $\delta = 0$ yields integrable couplings, which are not of perturbation type.

To constuct an associated integrable hierarchy, let us first solve the corresponding stationary zero curvature equation (2) by taking

$$Y = \begin{bmatrix} a & b & e & f \\ c & -a & -f & g \\ \delta g & -\delta f & -a & c \\ \delta f & \delta e & b & a \end{bmatrix} = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}, \tag{14}$$

where the basic objects can be stated a sfollows:

$$\begin{cases} a = \sum_{n\geq 0}\lambda^{-n}a^{[n]},\ b = \sum_{n\geq 0}\lambda^{-n}b^{[n]},\ c = \sum_{n\geq 0}\lambda^{-n}c^{[n]}, \\ e = \sum_{n\geq 0}\lambda^{-n}e^{[n]},\ f = \sum_{n\geq 0}\lambda^{-n}f^{[n]},\ g = \sum_{n\geq 0}\lambda^{-n}g^{[n]}. \end{cases} \tag{15}$$

The reason to take this form is that with $\mathcal{M}$ in (13), an arbitrary matrix in $\tilde{g}$ will lead to a commutator matrix of the above mentioned form. Now we can observe that the corresponding stationary zero curvature equation (2) becomes

$$\begin{cases} a_x = cu_1 + \delta g u_2 - bu_3 - \delta e u_4, \\ b_x = \alpha\lambda b - 2au_1 - 2\delta f u_2, \\ c_x = -\alpha\lambda c + 2au_3 + 2\delta f u_4, \end{cases} \tag{16}$$

$$\begin{cases} e_x = \alpha\lambda e - 2au_2 - 2fu_1, \\ g_x = -\alpha\lambda g + 2au_4 + 2fu_3, \\ f_x = gu_1 + cu_2 - eu_3 - bu_4. \end{cases} \tag{17}$$

These equations equivalently yield the initial conditions:

$$a_x^{[0]} = 0,\ b^{[0]} = c^{[0]} = e^{[0]} = g^{[0]} = 0,\ f_x^{[0]} = 0, \tag{18}$$

and the recursion relations which determine the Laurent series solution:

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha}[b_x^{[n]} + 2a^{[n]}u_1 + 2\delta f^{[n]}u_2], \\ c^{[n+1]} = -\frac{1}{\alpha}[c_x^{[n]} - 2a^{[n]}u_3 - 2\delta f^{[n]}u_4], \end{cases} \tag{19}$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha}[e_x^{[n]} + 2f^{[n]}u_1 + 2a^{[n]}u_2], \\ g^{[n+1]} = -\frac{1}{\alpha}[g_x^{[n]} - 2f^{[n]}u_3 - 2a^{[n]}u_4], \end{cases} \tag{20}$$

$$\begin{cases} a_x^{[n+1]} = c^{[n+1]}u_1 + \delta g^{[n+1]}u_2 - b^{[n+1]}u_3 - \delta e^{[n+1]}u_4, \\ f_x^{[n+1]} = g^{[n+1]}u_1 + c^{[n+1]}u_2 - e^{[n+1]}u_3 - b^{[n+1]}u_4, \end{cases} \tag{21}$$

where $n \geq 0$. To have a uniqueness of Laurent series solutions, we just need to take the initial data,

$$a^{[0]} = \frac{1}{2}\beta,\ f^{[0]} = \frac{1}{2}\gamma, \tag{22}$$

where $\beta$ and $\gamma$ are two arbitrary constants, and select the constants of integration to be zero,

$$a^{[n]}|_{u=0} = 0,\ f^{[n]}|_{u=0} = 0,\ n \geq 1. \tag{23}$$

In this way, one can work out that

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}(\beta u_1 + \delta\gamma u_2),\ c^{[1]} = \frac{1}{\alpha}(\beta u_3 + \delta\gamma u_4), \\ e^{[1]} = \frac{1}{\alpha}(\gamma u_1 + \beta u_2),\ g^{[1]} = \frac{1}{\alpha}(\gamma u_3 + \beta u_4), \\ a^{[1]} = f^{[1]} = 0; \end{cases}$$

$$\begin{cases} b^{[2]} = \frac{1}{\alpha^2}(\beta u_{1,x} + \delta\gamma u_{2,x}),\ c^{[2]} = -\frac{1}{\alpha^2}(\beta u_{3,x} + \delta\gamma u_{4,x}), \\ e^{[2]} = \frac{1}{\alpha^2}(\gamma u_{1,x} + \beta u_{2,x}),\ g^{[2]} = -\frac{1}{\alpha^2}(\gamma u_{3,x} + \beta u_{4,x}), \end{cases}$$

$$\begin{cases} a^{[2]} = -\frac{1}{\alpha^2}[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2], \\ f^{[2]} = -\frac{1}{\alpha^2}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]; \end{cases}$$

$$\begin{cases} b^{[3]} = \frac{1}{\alpha^3}[\beta u_{1,xx} + \delta\gamma u_{2,xx} - 2(\beta u_3 + \delta\gamma u_4)u_1^2 - 4\delta(\gamma u_3 + \beta u_4)u_1u_2 - 2\delta(\beta u_3 + \delta\gamma u_4)u_2^2], \\ c^{[3]} = \frac{1}{\alpha^3}[\beta u_{3,xx} + \delta\gamma u_{4,xx} - 2(\beta u_1 + \delta\gamma u_2)u_3^2 - 4\delta(\gamma u_1 + \beta u_2)u_3u_4 - 2\delta(\beta u_1 + \delta\gamma u_2)u_4^2], \end{cases}$$

$$\begin{cases} e^{[3]} = \frac{1}{\alpha^3}[\gamma u_{1,xx} + \beta u_{2,xx} - 2(\gamma u_3 + \beta u_4)u_1^2 - 4(\beta u_3 + \delta\gamma u_4)u_1u_2 - 2\delta(\gamma u_3 + \beta u_4)u_2^2], \\ g^{[3]} = \frac{1}{\alpha^3}[\gamma u_{3,xx} + \beta u_{4,xx} - 2(\gamma u_1 + \beta u_2)u_3^2 - 4(\beta u_1 + \delta\gamma u_2)u_3u_4 - 2\delta(\gamma u_1 + \beta u_2)u_4^2], \end{cases}$$

$$\begin{cases} a^{[3]} = \frac{1}{\alpha^3}[-(\beta u_3 + \delta\gamma u_4)u_{1,x} - \delta(\gamma u_3 + \beta u_4)u_{2,x} + (\beta u_1 + \delta\gamma u_2)u_{3,x} + \delta(\gamma u_1 + \beta u_2)u_{4,x}], \\ f^{[3]} = \frac{1}{\alpha^3}[-(\gamma u_3 + \beta u_4)u_{1,x} - (\beta u_3 + \delta\gamma u_4)u_{2,x} + (\gamma u_1 + \beta u_2)u_{3,x} + (\beta u_1 + \delta\gamma u_2)u_{4,x}]; \end{cases}$$

and

$$\begin{cases} b^{[4]} = \frac{1}{\alpha^4}\{\beta u_{1,xxx} + \delta\gamma u_{2,xxx} - 6[(\beta u_3 \\ \quad +\delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{1,x} \\ \quad -6\delta[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{2,x}\}, \\ c^{[4]} = -\frac{1}{\alpha^4}\{\beta u_{3,xxx} + \delta\gamma u_{4,xxx} - 6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{3,x} \\ \quad -6\delta[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{4,x}\}, \end{cases}$$

$$\begin{cases} e^{[4]} = \frac{1}{\alpha^4}\{\gamma u_{1,xxx} + \beta u_{2,xxx} - 6[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{1,x} \\ \quad -6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{2,x}\}, \\ g^{[4]} = -\frac{1}{\alpha^4}\{\gamma u_{3,xxx} + \beta u_{4,xxx} - 6[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{3,x} \\ \quad -6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{4,x}\}, \end{cases}$$

$$\begin{aligned} a^{[4]} = \frac{1}{\alpha^4}[&-(\beta u_3 + \delta\gamma u_4)u_{1,xx} - \delta(\gamma u_3 + \beta u_4)u_{2,xx} - (\beta u_1 + \delta\gamma u_2)u_{3,xx} - \delta(\gamma u_1 + \beta u_2)u_{4,xx} \\ &+(\beta u_{3,x} + \delta\gamma u_{4,x})u_{1,x} + \delta(\gamma u_{3,x} + \beta u_{4,x)})u_{2,x} + 3(\beta u_3^2 + 2\delta\gamma u_3u_4 + \delta\beta u_4^2)u_1^2 \\ &+6\delta(\gamma u_3^2 + 2\beta u_3u_4 + \delta\gamma u_4^2)u_1u_2 + 3\delta(\beta u_3^2 + 2\delta\gamma u_3u_4 + \delta\beta u_4^2)u_2^2], \\ f^{[4]} = \frac{1}{\alpha^4}[&-(\gamma u_3 + \beta u_4)u_{1,xx} - (\beta u_3 + \delta\gamma u_4)u_{2,xx} - (\gamma u_1 + \beta u_2)u_{3,xx} - (\beta u_1 + \delta\gamma u_2)u_{4,xx} \\ &+(\gamma u_{3,x} + \beta u_{4,x})u_{1,x} + (\beta u_{3,x} + \delta\gamma u_{4,x)})u_{2,x} + 3(\gamma u_3^2 + 2\beta u_3u_4 + \delta\gamma u_4^2)u_1^2 \\ &+6(\beta u_3^2 + 2\delta\gamma u_3u_4 + \delta\beta u_4^2)u_1u_2 + 3\delta(\gamma u_3^2 + 2\beta u_3u_4 + \delta\gamma u_4^2)u_2^2]. \end{aligned}$$

Upon observing the above results, one can impose $\Delta_r = 0,\ m \geq 0$, to formulate

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(u,\lambda)\varphi,\ \mathcal{N}^{[m]} = (\lambda^m Y)_+ = \sum_{n=0}^{m} \lambda^n Y^{[m-n]},\ m \geq 0, \tag{24}$$

as the temporal matrix eigenvalue problems within the zero curvature formulation. The solvability conditions of the spatial and temporal matrix eigenvalue problems in (13) and (24) are the zero curvature equations in (4). These equations yield

a hierarchy of integrable models with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) = (\alpha b^{[m+1]}, \alpha e^{[m+1]}, -\alpha c^{[m+1]}, -\alpha g^{[m+1]})^T,\ m \geq 0, \tag{25}$$

or more concretely,

$$u_{1,t_m} = \alpha b^{[m+1]},\ u_{2,t_m} = \alpha e^{[m+1]},\ u_{3,t_m} = -\alpha c^{[m+1]},\ u_{4,t_m} = -\alpha g^{[m+1]},\ m \geq 0. \tag{26}$$

The first two nonlinear examples in this hierarchy are the model of combined integrable nonlinear Schrödinger equations:

$$\begin{cases} u_{1,t_2} = \frac{1}{\alpha^2}[\beta u_{1,xx} + \delta\gamma u_{2,xx} - 2(\beta u_3 + \delta\gamma u_4)u_1^2 - 4\delta(\gamma u_3 + \beta u_4)u_1u_2 - 2\delta(\beta u_3 + \delta\gamma u_4)u_2^2], \\ u_{2,t_2} = \frac{1}{\alpha^2}[\gamma u_{1,xx} + \beta u_{2,xx} - 2(\gamma u_3 + \beta u_4)u_1^2 - 4\delta u_1u_2(\beta u_3 + \delta\gamma u_4) - 2\delta(\gamma u_3 + \beta u_4)u_2^2], \\ u_{3,t_2} = -\frac{1}{\alpha^2}[\beta u_{3,xx} + \delta\gamma u_{4,xx} - 2(\beta u_1 + \delta\gamma u_2)u_3^2 - 4\delta(\gamma u_1 + \beta u_2)u_3u_4 - 2\delta(\beta u_1 + \delta\gamma u_2)u_4^2], \\ u_{4,t_2} = -\frac{1}{\alpha^2}[\gamma u_{3,xx} + \beta u_{4,xx} - 2(\gamma u_1 + \beta u_2)u_3^2 - 4\delta(\beta u_1 + \delta\gamma u_2)u_3u_4 - 2\delta(\gamma u_1 + \beta u_2)u_4^2], \end{cases} \tag{27}$$

and the model of combined integrable modified Korteweg-de Vries equations:

$$\begin{cases} u_{1,t_3} = \frac{1}{\alpha^3}\{\beta u_{1,xxx} + \delta\gamma u_{2,xxx} - 6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{1,x} \\ \quad -6\delta[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{2,x}\}, \\ u_{2,t_3} = \frac{1}{\alpha^3}\{\gamma u_{1,xxx} + \beta u_{2,xxx} - 6[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{1,x} \\ \quad -6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{2,x}\}, \\ u_{3,t_3} = -\frac{1}{\alpha^3}\{-\beta u_{3,xxx} - \delta\gamma u_{4,xxx} + 6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{3,x} \\ \quad +6\delta[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{4,x}\}, \\ u_{4,t_3} = -\frac{1}{\alpha^3}\{-\gamma u_{3,xxx} - \beta u_{4,xxx} + 6[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \delta\gamma u_4)u_2]u_{3,x} \\ \quad +6[(\beta u_3 + \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2]u_{4,x}\}. \end{cases} \tag{28}$$

These systems provide two coupled integrable models with four components, which enlarge the category of coupled integrable models of nonlinear Schrödinger equations and modified Korteweg-de Vries equations (see, e.g., [31–33]). One characteristic phenomenon is that each equation contains two derivative terms of the highest order, and so, we call them combined models.

Three special cases of $\delta = 0$, $\beta = 0$ and $\gamma = 0$ in the resulting hierarchy are interesting. The first case presents novel integrable couplings of the the AKNS hierarchy, which are not of perturbation type. The other two cases produce reduced hierarchies of uncombined integrable models.

If one takes $\alpha=-\delta=\beta=1$ and $\gamma=0$ in the model (27), one gets a coupled integrable nonlinear Schrödinger type model:

$$\begin{cases} u_{1,t_2}=u_{1,xx}-2u_3(u_1^2+u_2^2)+4u_1u_2u_4, \\ u_{2,t_2}=u_{2,xx}+2u_4(u_1^2+u_2^2)-4u_1u_2u_3, \\ u_{3,t_2}=-u_{3,xx}+2u_1(u_3^2+u_4^2)-4u_2u_3u_4, \\ u_{4,t_2}=-u_{4,xx}-2u_2(u_3^2+u_4^2)+4u_1u_3u_4. \end{cases} \tag{29}$$

If one takes $\alpha=-\delta=\gamma=1$ and $\beta=0$ in the model (27), one obtains another coupled integrable nonlinear Schrödinger type model:

$$\begin{cases} u_{1,t_2}=-u_{2,xx}+2u_4(u_1^2-u_2^2)+4u_1u_2u_3, \\ u_{2,t_2}=u_{1,xx}-2u_3(u_1^2-u_2^2)+4u_1u_2u_4, \\ u_{3,t_2}=u_{4,xx}-2u_2(u_3^2-u_4^2)-4u_1u_3u_4, \\ u_{4,t_2}=-u_{3,xx}+2u_1(u_3^2-u_4^2)-4u_2u_3u_4. \end{cases} \tag{30}$$

The selection of $\alpha=-\delta=\beta=1$ and $\gamma=0$ in the model (28), leads to a coupled integrable modified Korteweg-de Vries type model:

$$\begin{cases} u_{1,t_3}=u_{1,xxx}-6(u_1u_3-u_2u_4)u_{1,x}+6(u_1u_4+u_2u_3)u_{2,x}, \\ u_{2,t_3}=u_{2,xxx}-6(u_1u_4+u_2u_3)u_{1,x}-6(u_1u_3-u_2u_4)u_{2,x}, \\ u_{3,t_3}=u_{3,xxx}-6(u_1u_3-u_2u_4)u_{3,x}+6(u_1u_4+u_2u_3)u_{4,x}, \\ u_{4,t_3}=u_{4,xxx}-6(u_1u_4+u_2u_3)u_{3,x}-6(u_1u_3-u_2u_4)u_{4,x}. \end{cases} \tag{31}$$

The selection of $\alpha=-\delta=\gamma=1$ and $\beta=0$ in the model (28) yields another coupled integrable modified Korteweg-de Vries type model:

$$\begin{cases} u_{1,t_3}=-u_{2,xxx}+6(u_1u_4+u_2u_3)u_{1,x}+6(u_1u_3-u_2u_4)u_{2,x}, \\ u_{2,t_3}=u_{1,xxx}-6(u_1u_3-u_2u_4)u_{1,x}+6(u_1u_4+u_2u_3)u_{2,x}, \\ u_{3,t_3}=-u_{4,xxx}+6(u_1u_4+u_2u_3)u_{3,x}+6(u_1u_3-u_2u_4)u_{4,x}, \\ u_{4,t_3}=u_{3,xxx}-6(u_1u_3-u_2u_4)u_{3,x}+6(u_1u_4+u_2u_3)u_{4,x}. \end{cases} \tag{32}$$

These models are different from the vector AKNS integrable models. The first class of integrable models contain the ones, previously presented in [28, 29]. Moreover, there is an interesting phenomenon that the two models in each pair just exchange the first component with the second component and the third component with the fourth component in the vector fields on the right hand sides.

## 3. Recursion operator and bi-Hamiltonian structure

Let us assume $\delta\neq 0$ now. To furnish Hamiltonian structures to explore the Liouville integrability for the soliton hierarchy (26), we can take advantage of the trace identity (7) in the case of the spatial matrix eigenvalue problem (13). Noting that the Laurent series solution $Y$ is determined by (14), one can then easily work out

$$\operatorname{tr}\Big(Y\frac{\partial\mathcal{M}}{\partial\lambda}\Big)=2\alpha a,\ \operatorname{tr}\Big(Y\frac{\partial\mathcal{M}}{\partial u}\Big)=(2c,2\delta g,2b,2\delta e)^T, \tag{33}$$

and consequently, the trace identity leads to

$$\begin{aligned}&\frac{\delta}{\delta u}\int\lambda^{-(n+1)}\alpha a^{[n+1]}\,dx\\ &=\lambda^{-\kappa}\frac{\partial}{\partial\lambda}\lambda^{\kappa-n}(c^{[n]},\delta g^{[n]},b^{[n]},\delta e^{[n]})^T,\ n\geq 0.\end{aligned} \tag{34}$$

Checking with $n=2$ tells $\kappa=0$, and therefore, one arrives at

$$\frac{\delta}{\delta u}\mathcal{H}^{[n]}=(c^{[n+1]},\delta g^{[n+1]},b^{[n+1]},\delta e^{[n+1]})^T,\ n\geq 0, \tag{35}$$

where the Hamiltonian functionals are determined by

$$\mathcal{H}^{[n]}=-\int\frac{\alpha a^{[n+2]}}{n+1}dx,\ n\geq 0. \tag{36}$$

This enables us to produce a Hamiltonian structure for the hierarchy (26):

$$u_{t_m}=X^{[m]}=J_1\frac{\delta\mathcal{H}^{[m]}}{\delta u},\ m\geq 0, \tag{37}$$

where the Hamiltonian operator $J_1$ is given by

$$J_1=\begin{bmatrix} \begin{matrix}0\end{matrix} & \begin{matrix}\alpha & 0\\ 0 & \frac{\alpha}{\delta}\end{matrix} \\ \begin{matrix}-\alpha & 0\\ 0 & -\frac{\alpha}{\delta}\end{matrix} & \begin{matrix}0\end{matrix}\end{bmatrix} \tag{38}$$

and the functionals $\mathcal{H}^{[m]}$ are defined by (36). As a consequence, we have an interrelation $S=J_1\frac{\delta\mathcal{H}}{\delta u}$ between a symmetry $S$ and a conserved functional $\mathcal{H}$ of each model in the hierarchy.

The characteristic commutative property for the vector fields $X^{[n]}$

$$\begin{aligned}&[\![X^{[n_1]},X^{[n_2]}]\!]=X^{[n_1]\prime}(u)[X^{[n_2]}]\\ &-X^{[n_2]\prime}(u)[X^{[n_1]}]=0,\ n_1,n_2\geq 0,\end{aligned} \tag{39}$$

follows from an algebra of Lax operators:

$$\begin{aligned}&[\![\mathcal{N}^{[n_1]},\mathcal{N}^{[n_2]}]\!]=\mathcal{N}^{[n_1]\prime}(u)[X^{[n_2]}]-\mathcal{N}^{[n_2]\prime}(u)[X^{[n_1]}]\\ &+[\mathcal{N}^{[n_1]},\mathcal{N}^{[n_2]}]=0,\ n_1,n_2\geq 0.\end{aligned} \tag{40}$$

This can directly be verified by analyzing the relation between the isospectral zero curvature equations (see [36] for details).

On the other hand, from the recursion relation $X^{[m+1]}=\Phi X^{[m]}$, we can compute a hereditary recursion operator $\Phi=(\Phi_{jk})_{4\times 4}$ [34] for the hierarchy (26), which reads as follows:

$$\begin{cases} \Phi_{11}=\frac{1}{\alpha}(\partial_x-2u_1\partial^{-1}u_3-2\delta u_2\partial^{-1}u_4),\ \Phi_{12}=\frac{1}{\alpha}(-2\delta u_1\partial^{-1}u_4-2\delta u_2\partial^{-1}u_3),\\ \Phi_{13}=\frac{1}{\alpha}(-2u_1\partial^{-1}u_1-2\delta u_2\partial^{-1}u_2),\ \Phi_{14}=\frac{1}{\alpha}(-2\delta u_1\partial^{-1}u_2-2\delta u_2\partial^{-1}u_1); \end{cases} \tag{41}$$

$$\begin{cases} \Phi_{21}=\frac{1}{\alpha}(-2u_1\partial^{-1}u_4-2u_2\partial^{-1}u_3),\ \Phi_{22}=\frac{1}{\alpha}(\partial_x-2u_1\partial^{-1}u_3-2\delta u_2\partial^{-1}u_4),\\ \Phi_{23}=\frac{1}{\alpha}(-2u_1\partial^{-1}u_2-2u_2\partial^{-1}u_1),\ \Phi_{24}=\frac{1}{\alpha}(-2u_1\partial^{-1}u_1-2\delta u_2\partial^{-1}u_2); \end{cases} \tag{42}$$

$$\begin{cases} \Phi_{31}=\frac{1}{\alpha}(2u_3\partial^{-1}u_3+2\delta u_4\partial^{-1}u_4),\ \Phi_{32}=\frac{1}{\alpha}(2\delta u_3\partial^{-1}u_4+2\delta u_4\partial^{-1}u_3),\\ \Phi_{33}=\frac{1}{\alpha}(-\partial_x+2u_3\partial^{-1}u_1+2\delta u_4\partial^{-1}u_2),\ \Phi_{34}=\frac{1}{\alpha}(2\delta u_3\partial^{-1}u_2+2\delta u_4\partial^{-1}u_1); \end{cases} \tag{43}$$

$$\begin{cases} \Phi_{41}=\frac{1}{\alpha}(2u_3\partial^{-1}u_4+2u_4\partial^{-1}u_3),\ \Phi_{42}=\frac{1}{\alpha}(2u_3\partial^{-1}u_3+2\delta u_4\partial^{-1}u_4),\\ \Phi_{43}=\frac{1}{\alpha}(2u_3\partial^{-1}u_2+2u_4\partial^{-1}u_1),\ \Phi_{44}=\frac{1}{\alpha}(-\partial_x+2u_3\partial^{-1}u_1+2\delta u_4\partial^{-1}u_2). \end{cases} \tag{44}$$

With some analysis, we can see that $J_1$ and $J_2=\Phi J_1$ constitute a Hamiltonian pair. Namely, an arbitrary linear combination of $J_1$ and $J_2$ is again Hamiltonian. Accordingly, the hierarchy (26) possesses a bi-Hamiltonian structure [35]:

$$u_{t_m}=X^{[m]}=J_1\frac{\delta\mathcal{H}^{[m]}}{\delta u}=J_2\frac{\delta\mathcal{H}^{[m-1]}}{\delta u},\quad m\geq 1. \tag{45}$$

It then follows that the associated Hamiltonian functionals commute with each other under the corresponding two Poisson brackets [7]:

$$\{\mathcal{H}^{[n_1]},\mathcal{H}^{[n_2]}\}_{J_1}=\int\Big(\frac{\delta\mathcal{H}^{[n_1]}}{\delta u}\Big)^T J_1\frac{\delta\mathcal{H}^{[n_2]}}{\delta u}\,dx=0,\ n_1,n_2\geq 0, \tag{46}$$

and

$$\{\mathcal{H}^{[n_1]},\mathcal{H}^{[n_2]}\}_{J_2}=\int\Big(\frac{\delta\mathcal{H}^{[n_1]}}{\delta p}\Big)^T J_2\frac{\delta\mathcal{H}^{[n_2]}}{\delta u}\,dx=0,\ n_1,n_2\geq 0. \tag{47}$$

To conclude, each model in the hierarchy (26) is Liouville integrable and possesses infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. Two particular illustrative integrable models are the systems in (27) and (28), which add examples to the existing category of nonlinear combined Liouville integrable Hamiltonian models with four components.

## 4. Conclusions

From a specific special matrix eigenvalue problem, a hierarchy of four-component Liouville integrable models has been presented within the zero curvature formulation. A particular Laurent series solution of the corresponding stationary zero curvature equation plays a crucial role. The resulting integrable models have been shown to be bi-Hamiltonian by determining a recursion operator and applying the trace identity in the case the underlying matrix eigenvalue problem.

We point out that the case of $\delta=0$ corresponds to integrable couplings and the variational identity can be used to establish a Hamiltonian structure (see, e.g., [6] for details). It should be particularly interesting to study structures of soliton solutions (see, e.g., [37]) and long-time behaviours of global solutions to Cauchy problems (see, e.g., [38, 39]) for the presented integrable models. Powerful and effective approaches could be used, which include the Riemann-Hilbert technique [40], the Zakharov-Shabat dressing method [41], the Darboux transformation [42–46], and the determinant approach [47]. In addition to solitons, lump, kink, breather and rogue wave solutions, particularly their interaction solutions (see, e.g., [48–55]), are interesting, and one can generate them from soliton solutions by taking wave number reductions. On the other hand, nonlocal group reductions or similarity transformations of matrix eigenvalue problems can yield nonlocal reduced integrable models and their solitons are significant in mathematics as well as physics (see, e.g., [56]). It is known that nonlocal differential equation models exhibit significantly different solution behaviors [57, 58].

Integrable models are of great interest and have close connections to various areas of mathematics, including algebraic geometry, Lie theory, and the theory of special functions. The study of integrable models offers insights into the dynamic behavior of physical systems and underpins the fundamental understanding of complex nonlinear mathematical and physical phenomena,

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