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N -soliton solutions and the Hirota conditions in $(1 + 1)$ -dimensions

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Abstract: We analyze N -soliton solutions and explore the Hirota N -soliton conditions for scalar $(1 + 1)$ -dimensional equations, within the Hirota bilinear formulation. An algorithm to verify the Hirota conditions is proposed by factoring out common factors out of the Hirota function in N wave vectors and comparing degrees of the involved polynomials containing the common factors. Applications to a class of generalized KdV equations and a class of generalized higher-order KdV equations are made, together with all proofs of the existence of N -soliton solutions to all equations in two classes.

Keywords: $(1 + 1)$ -dimensional integrable equations; Hirota N -soliton condition; N -soliton solution.

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1 Introduction

Soliton solutions are a class of self-reinforcing wave packets which maintain the balance between dispersive and nonlinear effects, and they possess important applications in physical and engineering sciences [1–3]. Breather, lump and rogue wave solutions, which have been extensively studied in recent years, are all special reductions of soliton solutions. The Hirota bilinear formulation is a powerful approach for constructing soliton solutions [4]. The concept of bilinear derivatives is the key tool in the method, and Hirota bilinear forms are crucial in presenting soliton solutions.

Hirota bilinear derivatives are defined by [5]:

$$D_x f \cdot g = f_x g - f g_x, \quad D_x^2 f \cdot g = f_{xx} g - 2f_x g_x + f g_{xx}, \dots,$$

$$D_x^m f \cdot g = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} (\partial_x^i f)(\partial_x^{m-i} g), \quad m \geq 1,$$

and more generally, bilinear partial derivatives with multiple variables are similarly defined as follows:

$$(D_x^m D_t^n f \cdot g)(x, t) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t) g(x', t')|_{x'=x, t'=t}, \quad m, n \geq 1. \quad (1.1)$$

When $f = g$, we get Hirota bilinear expressions:

$$D_x f \cdot f = 0, \quad D_x^2 f \cdot f = 2(f_{xx} f - f_x^2), \dots,$$

$$D_x^{2m-1} f \cdot f = 0, \quad D_x^{2m} f \cdot f = \sum_{i=0}^{2m} (-1)^{2m-i} \binom{2m}{i} (\partial_x^i f)(\partial_x^{2m-i} f), \quad m \geq 1,$$

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and similarly, bilinear partial derivative expressions:

$$D_x^m D_t^n f \cdot f = \sum_{i=0}^m \sum_{j=0}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (\partial_x^i \partial_t^j f) (\partial_x^{m-i} \partial_t^{n-j} f), \quad m, n \geq 1. \quad (1.2)$$

In terms of Hirota bilinear expressions, we can define Hirota bilinear equations. Take an even polynomial $P(x_1, x_2, \dots, x_M)$ in M variables with no constant term i.e., $P(\mathbf{0}) = P(0, 0, \dots, 0) = 0$. The associated Hirota bilinear equation is defined by

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) f \cdot f = 0, \quad (1.3)$$

where each term of which is a Hirota bilinear expression. If a partial differential equation (PDE) can be transformed into a Hirota bilinear equation, then we say that it possesses a Hirota bilinear form. The basic question of when and how bilinear forms could be obtained for PDEs is closely connected with Bell polynomial theories [6, 7].

The KdV equation

$$N(u) := u_t + 6uu_x + u_{xxx} = 0 \quad (1.4)$$

possesses the Hirota bilinear form:

$$B(f) := (D_x^4 + D_x D_t) f \cdot f = 2(f_{4x} f - 4f_{3x} f_x + 3f_{xx}^2 + f_{xt} f - f_x f_t) = 0, \quad (1.5)$$

associated with $P(x, t) = x^4 + xt$, under the logarithmic derivative transformation $u = 2(\ln f)_{xx}$. The link is $N(u) = (B(f)/f^2)_x$ [4]. The Boussinesq equations

$$N(u) := u_{tt} + (u^2)_{xx} \pm u_{4x} = 0 \quad (1.6)$$

possess the Hirota bilinear forms:

$$B(f) := (D_t^2 \pm D_x^4) f \cdot f = 2[f_{tt} f - f_t^2 \pm (f_{4x} f - 4f_{3x} f_x + 3f_{xx}^2)] = 0, \quad (1.7)$$

associated with $P(x, t) = t^2 \pm x^4$, under the same logarithmic derivative transformation $u = \pm 6(\ln f)_{xx}$, and the links are $N(u) = \pm 3(B(f)/f^2)_{xx}$ [8].

We would like to analyze N -soliton solutions and derive the corresponding Hirota N -soliton conditions for scalar $(1 + 1)$ -dimensional equations. An algorithm will be proposed for verifying the Hirota conditions by figuring out common factors out of the Hirota function in N wave vectors and comparing degrees of the involved polynomials containing common factors. Applications will be made to a class of generalized KdV equations associated with

$$P(x, t) = ax^4 + bx^3t + cx^2 + dxt, \quad (1.8)$$

where a, b, c, d are arbitrary constants satisfying $b^2 + d^2 \neq 0$, and a class of generalized higher-order KdV equations associated with

$$P(x, t) = ax^6 + bx^4 + cx^2 + xt, \quad (1.9)$$

where a, b, c are arbitrary constants. Our analysis implies that all equations in the two classes possess N -soliton solutions, which contain the KdV equation associated with $P = x^4 + xt$, and the Sawada–Kotera equation [11] or the Caudrey–Dodd–Gibbon equation [12] associated with $P = x^6 + xt$.

2 The Hirota N -soliton conditions

Let us denote N wave vectors by

$$\mathbf{k}_i = (k_{1,i}, k_{2,i}, \dots, k_{M,i}), \quad 1 \leq i \leq N, \quad (2.1)$$

where $k_{1,i}, k_{2,i}, \dots, k_{M,i}, 1 \leq i \leq N$, are constants. An N -soliton solution to a Hirota bilinear Eq. (1.3) is given by [9]:

$$f = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \eta_i + \sum_{i<j} a_{ij} \mu_i \mu_j \right), \quad (2.2)$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)$, $\mu = 0, 1$ means that each μ_i takes 0 or 1, and

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M + \eta_{i,0}, \quad 1 \leq i \leq N, \quad (2.3)$$

$$e^{a_{ij}} = A_{ij} := -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)}, \quad 1 \leq i < j \leq N, \quad (2.4)$$

$\eta_{i,0}$'s being arbitrary constant phase shifts. We will show that a Hirota bilinear Eq. (1.3) has an N -soliton solution (2.2) if and only if

$$H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n}) := \sum_{\sigma=\pm 1} P \left(\sum_{r=1}^n \sigma_r \mathbf{k}_{i_r} \right) \prod_{1 \leq r < s \leq n} P(\sigma_r \mathbf{k}_{i_r} - \sigma_s \mathbf{k}_{i_s}) \sigma_r \sigma_s = 0, \quad 1 \leq n \leq N, \quad (2.5)$$

where $1 \leq i_1 < \dots < i_n \leq N$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ and $\sigma = \pm 1$ means that each σ_i takes 1 or -1 . This is called the Hirota condition for an N -soliton solution, or simply, the N -soliton condition [10]. We also call all $H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n})$ the Hirota functions. There are very few studies on this Hirota N -soliton condition, due to its complexity [10, 13].

The Hirota condition in the case of $n = 1$ leads to the dispersion relations

$$P(\mathbf{k}_i) = 0, \quad 1 \leq i \leq N, \quad (2.6)$$

because of the even property of P . The one-soliton condition is just the dispersion relation $P(\mathbf{k}_1) = 0$, which means that $f = 1 + e^{\eta_1}$ is a solution. Besides the dispersion relations, the two-soliton condition requires

$$2(P(\mathbf{k}_1 + \mathbf{k}_2)P(\mathbf{k}_1 - \mathbf{k}_2) - P(\mathbf{k}_1 - \mathbf{k}_2)P(\mathbf{k}_1 + \mathbf{k}_2)) = 0, \quad (2.7)$$

which is an identity. Therefore, there always exists a two-soliton solution:

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \quad (2.8)$$

to a Hirota bilinear equation. Furthermore, taking $N = 3$, we see that the three-soliton condition requires:

$$\sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} P(\sigma_1 \mathbf{k}_1 + \sigma_2 \mathbf{k}_2 + \sigma_3 \mathbf{k}_3) P(\sigma_1 \mathbf{k}_1 - \sigma_2 \mathbf{k}_2) \\ \times P(\sigma_2 \mathbf{k}_2 - \sigma_3 \mathbf{k}_3) P(\sigma_1 \mathbf{k}_1 - \sigma_3 \mathbf{k}_3) = 0,$$

in addition to the dispersion relations. Again due to the even property of P , this is equivalent to

$$\sum_{(\sigma_1, \sigma_2, \sigma_3) \in S} P(\sigma_1 \mathbf{k}_1 + \sigma_2 \mathbf{k}_2 + \sigma_3 \mathbf{k}_3) P(\sigma_1 \mathbf{k}_1 - \sigma_2 \mathbf{k}_2) \\ \times P(\sigma_2 \mathbf{k}_2 - \sigma_3 \mathbf{k}_3) P(\sigma_1 \mathbf{k}_1 - \sigma_3 \mathbf{k}_3) = 0, \quad (2.9)$$

where $S = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$. The three-soliton solution is given by

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1 + \eta_2} + A_{13} e^{\eta_1 + \eta_3} \\ + A_{23} e^{\eta_2 + \eta_3} + A_{123} e^{\eta_1 + \eta_2 + \eta_3}, \quad A_{123} = A_{12} A_{13} A_{23}. \quad (2.10)$$

If condition (2.9) is satisfied, we say that an equation passes the three-soliton test [15, 16].

It is now a direct computation, particularly by symbolic computation (see, e.g., [17, 18]), that both the KdV equation and the Boussinesq equations pass the three-soliton test. It is commonly believed that the three-soliton condition implies the N -soliton condition, and no counterexample is found, indeed.

If we require a sufficient Hirota N -soliton condition [19]:

$$P(\mathbf{k}_i - \mathbf{k}_j) = 0, \quad 1 \leq i < j \leq N, \quad (2.11)$$

we obtain a resonant N -soliton solution

$$f = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + \cdots + c_N e^{\eta_N}, \quad (2.12)$$

where c_i 's are arbitrary constants. All wave vectors \mathbf{k}_i 's associated with resonant solutions form an affine space in \mathbb{R}^M [20].

Note that we have

$$P(D_{x_1}, \dots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j) e^{\eta_i + \eta_j}, \quad (2.13)$$

and

$$P(D_{x_1}, \dots, D_{x_M}) e^{\eta_n} f \cdot e^{\eta_n} g = e^{2\eta_n} P(D_{x_1}, \dots, D_{x_M}) f \cdot g, \quad (2.14)$$

where η_i, η_j and η_n are arbitrary linear functions but do not need to satisfy the dispersion relations. The first formula tells how to compute Hirota bilinear expressions of exponential functions, and the second formula tells how to take out a common factor from Hirota bilinear expressions. Based on these two rules, we can derive the following expression.

Theorem 2.1. *Let f be defined by (2.2), and $\hat{\xi}$ mean that no ξ is involved. Then we have*

$$\begin{aligned} & P(D_{x_1}, \dots, D_{x_M}) f \cdot f \\ &= (-1)^{\frac{1}{2}N(N-1)} \frac{H(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_N)}{\prod_{1 \leq i < j \leq N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ &+ \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1 \leq i_1 < \cdots < i_n \leq N} \frac{H(\mathbf{k}_1, \dots, \hat{\mathbf{k}}_{i_1}, \dots, \hat{\mathbf{k}}_{i_n}, \dots, \mathbf{k}_N)}{\prod_{\substack{1 \leq i < j \leq N \\ i, j \notin \{i_1, \dots, i_n\}}} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \cdots + \hat{\eta}_{i_1} + \cdots + \hat{\eta}_{i_n} + \cdots + \eta_N} \\ &+ \sum_{n=1}^{N-1} \sum_{1 \leq i_1 < \cdots < i_n \leq N} e^{2\left(\eta_{i_1} + \cdots + \eta_{i_n} + \sum_{1 \leq r < s \leq n} a_{i_r i_s}\right)} P(D_{x_1}, \dots, D_{x_M}) \tilde{f}_{i_1 \dots i_n} \cdot \tilde{f}_{i_1 \dots i_n} \end{aligned} \quad (2.15)$$

with

$$\tilde{f}_{i_1 \dots i_n} = \sum_{\tilde{\mu}_{i_1 \dots i_n} = 0, 1} \exp \left(\sum_{\substack{1 \leq i \leq N \\ i \notin \{i_1, \dots, i_n\}}} \mu_i \tilde{\eta}_i + \sum_{\substack{1 \leq i < j \leq N \\ i, j \notin \{i_1, \dots, i_n\}}} a_{ij} \mu_i \mu_j \right), \quad \tilde{\eta}_i = \eta_i + \sum_{r=1}^n a_{ir},$$

where $\tilde{\mu}_{i_1 \dots i_n} = (\mu_1, \dots, \hat{\mu}_{i_1}, \dots, \hat{\mu}_{i_n}, \dots, \mu_N)$ and $\tilde{\mu}_{i_1 \dots i_n} = 0, 1$ means that each μ_i in $\tilde{\mu}_{i_1 \dots i_n}$ takes 0 or 1.

Proof. Note that we have the computational rules (2.13) and (2.14), and so, we can expand all terms in $P(D_{x_1}, \dots, D_{x_M}) f \cdot f$.

Let us first consider the case, in which there is no common factor. We compute the terms which involve $e^{\eta_1 + \eta_2 + \cdots + \eta_N}$. For example, we have the following term of such a type:

$$\begin{aligned} & P(D_{x_1}, \dots, D_{x_M}) (A_{12 \dots (N-1)} e^{\eta_1 + \eta_2 + \cdots + \eta_{N-1}} \cdot e^{\eta_N}) \\ &= A_{12 \dots (N-1)} P(D_{x_1}, \dots, D_{x_M}) (e^{\eta_1 + \eta_2 + \cdots + \eta_{N-1}} \cdot e^{\eta_N}) \\ &= A_{12 \dots (N-1)} P(\mathbf{k}_1 + \cdots + \mathbf{k}_{N-1} - \mathbf{k}_N) e^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ &= (-1)^{\frac{1}{2}(N-1)(N-2)} \prod_{1 \leq i < j \leq N-1} \frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)} P(\mathbf{k}_1 + \cdots + \mathbf{k}_{N-1} - \mathbf{k}_N) e^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ &= (-1)^{\frac{1}{2}N(N-1)} \frac{P(\sigma_1 \mathbf{k}_1 + \cdots + \sigma_N \mathbf{k}_N) \prod_{1 \leq i < j \leq N} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j}{\prod_{1 \leq i < j \leq N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \eta_2 + \cdots + \eta_N}, \end{aligned}$$

where $\sigma = (\sigma_1, \dots, \sigma_{N-1}, \sigma_N) = (1, \dots, 1, -1)$ and $A_{12\dots(N-1)} = \prod_{1 \leq i < j \leq N-1} A_{ij}$. Taking all possibilities of $\sigma_i = \pm 1, 1 \leq i \leq N$, we obtain the first sum determined by $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$ in (2.15). The other sums determined by

$$H(\mathbf{k}_1, \dots, \hat{\mathbf{k}}_{i_1}, \dots, \hat{\mathbf{k}}_{i_n}, \dots, \mathbf{k}_N), \quad \text{where } 1 \leq i_1 < \dots < i_n \leq N,$$

can be similarly obtained.

If we have a common factor $e^{\eta_{i_1} + \dots + \eta_{i_n} + \sum_{1 \leq r < s \leq n} a_{i_r i_s}}$, where $1 \leq i_1 < \dots < i_n \leq N$ and $1 \leq n \leq N-1$, then we can use (2.14) to take out this factor to get the terms in the last sum in (2.15). If we have a common factor $e^{\eta_1 + \dots + \eta_N + \sum_{1 \leq i < j \leq N} a_{ij}}$, then the resulting term in $P(D_{x_1}, \dots, D_{x_M})f \cdot f$ is zero. Therefore, the formula (2.15) holds. The proof is finished. \square

Based on this theorem, by a recursive procedure, we can see that the Hirota condition is a necessary and sufficient condition for a Hirota bilinear equation to have an N -soliton solution, which is summarized in the following theorem.

Theorem 2.2. *A Hirota bilinear Eq. (1.3) possesses an N -soliton solution (2.2) if and only if the Hirota condition in (2.5) is satisfied.*

In order to figure out as more common factors out of the Hirota function $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$ as possible, we will use the following result, which is an automatic consequence of the definition of the Hirota functions.

Proposition 2.1. *The Hirota functions defined in (2.5) are symmetric and even functions in the involved wave vectors.*

Taking $\mathbf{k}_{N-1} = \pm \mathbf{k}_N$, we have

$$P(\sigma_i \mathbf{k}_i - \mathbf{k}_{N-1})P(\sigma_i \mathbf{k}_i \pm \mathbf{k}_N) = P(\mathbf{k}_i - \mathbf{k}_N)P(\mathbf{k}_i + \mathbf{k}_N) \quad (2.16)$$

in any case of $\sigma_i = \pm 1$, due to the even property of the polynomial P . Based on the properties in (2.16), we can show the following result.

Theorem 2.3. *If $\mathbf{k}_{N-1} = \pm \mathbf{k}_N$, then we have*

$$H(\mathbf{k}_1, \dots, \mathbf{k}_N) = 2H(\mathbf{k}_1, \dots, \mathbf{k}_{N-2})P(2\mathbf{k}_N) \prod_{i=1}^{N-2} P(\mathbf{k}_i - \mathbf{k}_N)P(\mathbf{k}_i + \mathbf{k}_N). \quad (2.17)$$

Proof. When $\mathbf{k}_{N-1} = \pm \mathbf{k}_N$, we can compute that

$$\begin{aligned} H(\mathbf{k}_1, \dots, \mathbf{k}_N) &= \sum_{\sigma=\pm 1} P(\sigma_1 \mathbf{k}_1 + \dots + \sigma_N \mathbf{k}_N) \prod_{1 \leq i < j \leq N} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j \\ &= \sum_{\sigma=\pm 1} P(\sigma_1 \mathbf{k}_1 + \dots + \sigma_N \mathbf{k}_N) \prod_{1 \leq i < j \leq N-2} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j \\ &\quad \times \prod_{i=1}^{N-2} P(\sigma_i \mathbf{k}_i - \sigma_{N-1} \mathbf{k}_{N-1}) \sigma_i \sigma_{N-1} \prod_{i=1}^{N-1} P(\sigma_i \mathbf{k}_i - \sigma_N \mathbf{k}_N) \sigma_i \sigma_N \\ &= 2 \sum_{\sigma=\pm 1} P(\sigma_1 \mathbf{k}_1 + \dots + \sigma_N \mathbf{k}_{N-2}) \prod_{1 \leq i < j \leq N-2} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j \\ &\quad \times \prod_{i=1}^{N-2} P(\sigma_i \mathbf{k}_i - \mathbf{k}_{N-1}) \prod_{i=1}^{N-2} P(\sigma_i \mathbf{k}_i \pm \mathbf{k}_N) P(2\mathbf{k}_N) \\ &= 2H(\mathbf{k}_1, \dots, \mathbf{k}_{N-2})P(2\mathbf{k}_N) \prod_{i=1}^{N-2} P(\mathbf{k}_i - \mathbf{k}_N)P(\mathbf{k}_i + \mathbf{k}_N), \end{aligned}$$

where the last step is due to (2.16), and the last but one step follows from the fact that the two cases $(1, \mp 1)$ and $(-1, \pm 1)$ of (σ_{N-1}, σ_N) are left and the other two cases lead to a zero factor owing to $P(\mathbf{0}) = 0$. Therefore, the proof of the theorem is finished. \square

This theorem will be used to factor out common factors out of the Hirota function $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$, while verifying the Hirota N -soliton condition.

3 Applications to $(1 + 1)$ -dimensional equations

3.1 A general algorithm

In the $(1 + 1)$ -dimensional case, the wave vectors can be expressed as

$$\mathbf{k}_i = (k_i, -\omega_i), \quad 1 \leq i \leq N. \quad (3.1)$$

We assume that the dispersion relations (2.6) determine all frequencies $\omega_i = \omega(k_i)$, $1 \leq i \leq N$. Therefore, $P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j)$ are functions of k_i and k_j only.

On one hand, we further assume that $P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j)$ and $P(\sigma_1 \mathbf{k}_1 + \dots + \sigma_N \mathbf{k}_N)$ can be simplified into rational functions as follows:

$$P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) = \frac{\sigma_i \sigma_j k_i k_j Q_1(k_i, k_j, \sigma_i, \sigma_j)}{Q_2(k_i, k_j)}, \quad (3.2)$$

where Q_1 and Q_2 are polynomial functions, and

$$P(\sigma_1 \mathbf{k}_1 + \dots + \sigma_N \mathbf{k}_N) = \frac{Q_3(k_1, \dots, k_N, \sigma_1, \dots, \sigma_N)}{Q_4(k_1, \dots, k_N)}, \quad (3.3)$$

where Q_3 and Q_4 are polynomial functions. Let us define a new polynomial

$$\tilde{H} = H(\mathbf{k}_1, \dots, \mathbf{k}_N) Q_4(k_1, \dots, k_N) \prod_{1 \leq i < j \leq N} Q_2(k_i, k_j), \quad (3.4)$$

for convenience of discussion. The stated assumption in (3.2) exhibits a characteristic of multivariate polynomials.

On the other hand, Theorem 2.3 tells that under the induction assumption, the Hirota function $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$ will be zero, if two of the wave vectors satisfy $\mathbf{k}_i = \pm \mathbf{k}_j$ ($1 \leq i < j \leq N$). Based on the even property of H and P , we know that $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$ is still even with respect to the wave numbers k_i , $1 \leq i \leq N$. Therefore, from the symmetric property in Proposition 2.1, we can factor out a factor $(k_i^2 - k_j^2)^2$ out of the polynomial \tilde{H} :

$$\tilde{H} = (k_i^2 - k_j^2)^2 g_{ij}, \quad \text{for any pair } 1 \leq i < j \leq N, \quad (3.5)$$

where g_{ij} is a polynomial of k_n , $1 \leq n \leq N$.

Finally, it follows from the characteristic property of P in (3.2) that the Hirota function $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$ can be written as

$$H(\mathbf{k}_1, \dots, \mathbf{k}_N) = \frac{\prod_{1 \leq i < j \leq N} k_i^2 k_j^2 \prod_{1 \leq i < j \leq N} (k_i^2 - k_j^2)^2 g}{Q_4(k_1, \dots, k_N) \prod_{1 \leq i < j \leq N} Q_2(k_i, k_j)} \quad (3.6)$$

where g is another polynomial of k_n , $1 \leq n \leq N$. Then, we can see that based on

$$\tilde{H} = \prod_{1 \leq i < j \leq N} k_i^2 k_j^2 \prod_{1 \leq i < j \leq N} (k_i^2 - k_j^2)^2 g, \quad (3.7)$$

the degree of the polynomial \tilde{H} is at least $2N(N - 1) + 2N(N - 1) = 4N(N - 1)$, if $H(\mathbf{k}_1, \dots, \mathbf{k}_N) \neq 0$, implying $g \neq 0$.

Now if $H(\mathbf{k}_1, \dots, \mathbf{k}_N) \neq 0$, the degree of the polynomial \tilde{H} defined by (3.4), which also equals

$$\tilde{H} = \sum_{\sigma=\pm 1} Q_3(k_1, \dots, k_N, \sigma_1, \dots, \sigma_N) \prod_{1 \leq i < j \leq N} k_i k_j Q_1(k_i, k_j, \sigma_i, \sigma_j), \quad (3.8)$$

should then not be less than $4N(N-1)$. Otherwise, we will have $H(\mathbf{k}_1, \dots, \mathbf{k}_N) = 0$, which is what we need to prove for the existence of N -soliton solutions. Thus, the problem for verifying the Hirota condition becomes quite simple, and one basically just needs to compute the degree of the polynomial in (3.8) and determine if it is less than $4N(N-1)$.

3.2 Applications

3.2.1 Generalized KdV equations

Let us consider a class of generalized KdV equations, which are associated with

$$P(x, t) = ax^4 + bx^3t + cx^2 + dxt \quad (3.9)$$

where a, b, c, d are arbitrary constants satisfying $b^2 + d^2 \neq 0$, which guarantees we will have a PDE. The corresponding bilinear generalized KdV equations read

$$\begin{aligned} B(f) &:= (aD_x^4 + bD_x^3D_t + cD_x^2 + dD_xD_t)f \cdot f \\ &= 2 \left[a(f_{4x}f - 4f_{3x}f_x + 3f_{xx}^2) + b(f_{3xt}f - 3f_{xxt}f_x + 3f_{xt}f_{xx} - f_t f_{3x}) \right. \\ &\quad \left. + c(f_{xx}f - f_x^2) + d(f_{yt}f - f_y f_t) \right] = 0. \end{aligned} \quad (3.10)$$

They are equivalent to the following generalized KdV equations:

$$N(u) := a(6u_x u_{xx} + u_{4x}) + b[3(u_x u_t)_x + u_{3xt}] + cu_{xx} + du_{xt} = 0, \quad (3.11)$$

under the logarithmic derivative transformation $u = 2(\ln f)_x$. The link is $N(u) = (B(f)/f^2)_x$. If $b = 0$, then we get the KdV equation, and if $a = 0$, we get the Hirota–Satsuma equation [21].

In what follows, we would like to show that each equation in (3.10) possesses an N -soliton solution. Let us set

$$\Delta = ad - bc. \quad (3.12)$$

It is direct to compute that

$$\omega_i = \omega_i(k_i) = \frac{ak_i^3 + ck_i}{bk_i^2 + d}, \quad 1 \leq i \leq N, \quad (3.13)$$

and

$$P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) = -\frac{\sigma_i \sigma_j k_i k_j \Delta (\sigma_i k_i - \sigma_j k_j)^2 [b(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) + 3d]}{(bk_i^2 + d)(bk_j^2 + d)}, \quad 1 \leq i < j \leq N. \quad (3.14)$$

Case 1. $\Delta = 0$:

In this case, we have $P(\mathbf{k}_i \pm \mathbf{k}_j) = 0$, $1 \leq i < j \leq N$, and thus, the Hirota N -soliton condition is automatically satisfied. This implies that we have a set of resonant solutions:

$$f = 1 + c_1 e^{\eta_1} + \dots + c_N e^{\eta_N}, \quad \eta_i = k_i x - \omega_i(k_i)t, \quad 1 \leq i \leq N, \quad (3.15)$$

where c_i 's and k_i 's are arbitrary constants.

Case 2. $\Delta \neq 0$:

Sub-case 2.1. $d = 0$:

In this subcase, we have $c \neq 0$ and directly obtain

$$\begin{cases} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) = \frac{R_1}{R_2}, & R_1 = c\sigma_i\sigma_j(\sigma_i k_i - \sigma_j k_j)^2(k_i^2 - \sigma_i\sigma_j k_i k_j + k_j^2), & R_2 = k_i k_j, \\ P(\sigma_1 \mathbf{k}_1 + \dots + \sigma_N \mathbf{k}_N) = \frac{R_3}{R_4}, & \deg R_3 = N + 2, & R_4 = \prod_{i=1}^N k_i. \end{cases} \quad (3.16)$$

Now if $H(\mathbf{k}_1, \dots, \mathbf{k}_N) \neq 0$, let us check the degree of the polynomial

$$\begin{aligned} & H(\mathbf{k}_1, \dots, \mathbf{k}_N) R_4(k_1, \dots, k_N) \prod_{1 \leq i < j \leq N} R_2(k_i, k_j) \\ &= R_3(k_1, \dots, k_N, \sigma_1, \dots, \sigma_N) \prod_{1 \leq i < j \leq N} R_1(k_i, k_j, \sigma_i, \sigma_j) \sigma_i \sigma_j. \end{aligned}$$

We apply the same idea as in the general algorithm. On one hand, based on the expression on the right hand side, the degree is $(N + 2) + 2N(N - 1) = 2N^2 - N + 2$. But on the other hand, since $HR_4 \sum_{i < j} R_2$ can have a factor $\sum_{i < j} (k_i^2 - k_j^2)^2$ as explained before, based on the expression on the left hand side, the degree is at least $2N(N - 1) + N + N(N - 1) = 3N^2 - 2N$. Those two numbers could not be equal, and actually, we have $3N^2 - 2N > 2N^2 - N + 2$, when $N \geq 3$. Therefore, $H(\mathbf{k}_1, \dots, \mathbf{k}_N) = 0, N \geq 1$.

Sub-case 2.2. $d \neq 0$:

Sub-subcase 2.2.1. $b = 0$: this is the KdV case. It is easy to work out

$$Q_1 = -3a(\sigma_i k_i - \sigma_j k_j)^2, \quad \deg Q_3 = 4, \quad Q_2 = 1, \quad Q_4 = 1. \quad (3.17)$$

Now if $H(\mathbf{k}_1, \dots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial $H(\mathbf{k}_1, \dots, \mathbf{k}_N)$ ($= \tilde{H}$) is $2N(N - 1) + 4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N - 1)$ when $N \geq 3$. Therefore, $H(\mathbf{k}_1, \dots, \mathbf{k}_N) = 0, N \geq 1$, and the KdV equation has N -soliton solutions, as shown in [9].

Sub-case 2.2.2. $b \neq 0$:

It is direct to get

$$\begin{cases} Q_1 = \Delta[b(k_i^2 - \sigma_i\sigma_j k_i k_j + k_j^2) + 3d](\sigma_i k_i - \sigma_j k_j)^2, \\ \deg Q_3 = 2(N + 1), & Q_2 = (bk_i^2 + d)(bk_j^2 + d), & Q_4 = \prod_{i=1}^N (bk_i^2 + d). \end{cases} \quad (3.18)$$

Now if $H(\mathbf{k}_1, \dots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial

$$\begin{aligned} \tilde{H} &= H(\mathbf{k}_1, \dots, \mathbf{k}_N) Q_4(k_1, \dots, k_N) \prod_{1 \leq i < j \leq N} Q_2(k_i, k_j) \\ &= Q_3(k_1, \dots, k_N, \sigma_1, \dots, \sigma_N) \prod_{1 \leq i < j \leq N} k_i k_j Q_1(k_i, k_j, \sigma_i, \sigma_j). \end{aligned}$$

is $2(N + 1) + 3N(N - 1) = 3N^2 - N + 2$ (from the second expression of \tilde{H}), which could not be greater than $4N(N - 1) + 2N + 2N(N - 1) = 6N^2 - 4N$ (from the first expression of \tilde{H} and (3.6)) when $N \geq 2$. Therefore, $H(\mathbf{k}_1, \dots, \mathbf{k}_N) = 0, N \geq 1$.

We remark that the three-soliton condition is also satisfied for all bilinear equations associated with

$$P = ax^4 + bx^3t + cx^2 + dxt + et^2, \quad e \neq 0, \quad (3.19)$$

where a, b, c, d, e are arbitrary constants. This leads to a class of generalized Boussinesq equations, and the case of $b = c = d = 0$ corresponds to the Boussinesq equations. But we need a more general argument to verify the Hirota N -soliton condition, since the frequency functions involve square roots.

3.2.2 Generalized higher-order KdV equations

Let us consider a class of higher-order generalized higher-order KdV equations associated with

$$P(x, t) = ax^6 + bx^4 + cx^2 + xt, \quad (3.20)$$

where a, b, c are arbitrary constants. This class of polynomials generates the following bilinear generalized higher-order KdV equations:

$$\begin{aligned} B(f) &:= (aD_x^6 + bD_x^4 + cD_x^2 + D_x D_t) f \cdot f \\ &= 2 [a(f_{6x} f - 6f_{5x} f_x + 15f_{4x} f_{xx} - 10f_{3x}^3) \\ &\quad + b(f_{4x} f - 4f_{3x} f_x + 3f_x^2) + c(f_{xx} f - f_x^2) + f_{xt} f - f_x f_t] = 0. \end{aligned} \quad (3.21)$$

The corresponding generalized higher-order KdV equations read as follows:

$$N(u) := a(15u_x^3 + 15u_x u_{3x} + u_{5x})_x + b(6u_x u_{xx} + u_{4x})_x + cu_{xx} + du_{xt} = 0. \quad (3.22)$$

The transformation is $u = 2(\ln f)_x$ and the link is $N(u) = (B(f)/f^2)_x$. The case of $b = c = 0$ leads to the Sawada–Kotera equation [11] or the Caudrey–Dodd–Gibbon equation [12].

Using the dispersion relations, we can directly obtain

$$\omega_i = \omega_i(k_i) = ak_i^5 + bk_i^3 + ck_i, \quad 1 \leq i \leq N, \quad (3.23)$$

and

$$P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) = -\sigma_i \sigma_j k_i k_j (\sigma_i k_i - \sigma_j k_j)^2 [5a(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) + 3b], \quad 1 \leq i < j \leq N. \quad (3.24)$$

Therefore, it is easy to find that

$$Q_1 = -(\sigma_i k_i - \sigma_j k_j)^2 [5a(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) + 3b], \quad \deg Q_3 = 6, \quad Q_2 = 1, \quad Q_4 = 1. \quad (3.25)$$

Now if $H(\mathbf{k}_1, \dots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial \tilde{H} ($= H$) is at most $3N(N - 1) + 6 = 3N^2 - 3N + 6$, which could not be greater than $4N(N - 1)$ when $N \geq 4$. Another direct computation can show that the three-soliton condition holds for all generalized higher-order KdV equations in (3.21). Therefore, $H(\mathbf{k}_1, \dots, \mathbf{k}_N) = 0$, $N \geq 1$, and each of the generalized higher-order KdV equations in (3.21) possesses N -soliton solutions.

This class is different from the fifth-order KdV equations studied in the literature [22]. It has also been proved [14] that the higher-order KdV equations associated with

$$P_1(x, t) = x^{2n} + xt, \quad n \geq 4, \quad (3.26)$$

does not pass the three-soliton test. A direct computation can show that all generalized higher-order KdV equations associated with

$$P_2(x, t) = x^6 + ax^4 + bx^2 + cxt + dt^2, \quad d \neq 0, \quad (3.27)$$

do not possess three-soliton solutions, either, but all generalized higher-order KdV equations associated with

$$P_3(x, t) = x^6 + ax^4 + 5x^3 t + bx^2 + cxt - 5t^2, \quad (3.28)$$

pass the three-soliton test. In the above polynomials by (3.27) and (3.28), a, b and c are three arbitrary constants. The class (3.28) with $a = b = c = 0$ gives the Ramani equation [23], which is also a dimensional reduction of the BKP equation. Similarly, because square roots are involved in the frequency functions in the case of (3.28), a more careful algorithm is needed for verifying the Hirota N -soliton condition.

4 Concluding remarks

We have analyzed the Hirota N -soliton conditions for bilinear differential equations and shown the existence of N -soliton solutions to two classes of generalized KdV equations. Our examples are all supplements to the list of bilinear equations which pass the three-soliton test in [15], and they generalize many existing examples in the literature since they can yield a linear combination of different nonlinear terms. For example, the multivariate polynomial in (3.9) contains two monomials x^4 and x^3t , which lead to two kinds of nonlinear terms in (3.11). This presents a novel class of bilinear equations and their corresponding nonlinear equations, which possess N -soliton solutions. Definitely, there should be more bilinear equations, which could possess N -soliton solutions. In the case of even higher-order differential equations and systems of coupled bilinear equations, the involved computations will be much more complicated. New ideas are needed to prove the existence of N -soliton solutions.

There are generalized bilinear derivatives, and particularly, we have the $D_{p,x}$ -operators [24]:

$$D_{p,x}^m D_{p,t}^n f \cdot g = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \alpha_p^{i+j} (\partial_x^{m-i} \partial_t^{n-j} f) (\partial_x^i \partial_t^j g), \quad m, n \geq 0, \quad m + n \geq 1, \tag{4.1}$$

where the powers of α_p are determined by

$$\alpha_p^i = (-1)^{r(i)}, \quad i = r(i) \bmod p, \quad i \geq 0, \tag{4.2}$$

with $0 \leq r(i) < p$. The patterns of those powers for $i = 1, 2, 3, \dots$ read

$$p = 3: \quad -, +, +, -, +, +, \dots;$$

$$p = 5: \quad -, +, -, +, +, -, +, -, +, +, \dots;$$

$$p = 7: \quad -, +, -, +, -, +, +, -, +, -, +, -, +, +, \dots$$

Particularly, we have $D_{3,x}$ and $D_{5,x}$ associated with the two odd prime numbers: $p = 3, 5$. There exist new characteristic properties of the corresponding generalized bilinear derivatives. For example, we have

$$D_{3,x}^3 f \cdot f = 2f_{xxx} f, \quad D_{3,x}^4 f \cdot f = 6f_{xx}^2, \tag{4.3}$$

which is different from the Hirota case (corresponding to $p = 2$). Of course, we can have other generalized bilinear derivatives: $D_{6,x}, D_{9,x}, \dots$

The corresponding generalized bilinear equations [6, 7] or trilinear equations [25] can possess resonant N -solitons. A generalized bilinear equation in $(1 + 1)$ -dimensions:

$$P(D_{p,x}, D_{p,t}) f \cdot f = 0 \tag{4.4}$$

possesses a resonant N -soliton [6, 7]:

$$f = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + \dots + c_N e^{\eta_N} \tag{4.5}$$

where c_i 's are arbitrary constants and $\eta_i = k_i x - \omega_i t, 1 \leq i \leq N$, iff

$$P(\mathbf{k}_i + \alpha_p \mathbf{k}_j) + P(\mathbf{k}_j + \alpha_p \mathbf{k}_i) = 0, \quad 1 \leq i < j \leq N, \tag{4.6}$$

where $\mathbf{k}_i = (k_i, -\omega_i), 1 \leq i \leq N$. However, we do not have any example of generalized bilinear equations which have general N -soliton solutions. There are many interesting questions that we need to answer first. For example, what is the generalized N -soliton condition, i.e., the N -soliton condition for generalized bilinear equations? How to formulate generalized bilinear equations, for example,

$$P(D_{3,x}, D_{3,t}) = 0,$$

in $(1 + 1)$ -dimensions, which possess general N -soliton solutions?

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References

- [1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, Philadelphia, SIAM, 1981.
- [2] F. Calogero and A. Degasperis, *Solitons and Spectral Transform I*, Amsterdam, North-Holland, 1982.
- [3] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, New York, Consultants Bureau, 1984.
- [4] R. Hirota, *Direct Method in Soliton Theory*, Cambridge, Cambridge University Press, 2004.
- [5] R. Hirota, "A new form of Bäcklund transformations and its relation to the inverse scattering problem," *Prog. Theor. Phys.*, vol. 52, pp. 1498–1512, 1974.
- [6] W. X. Ma, "Bilinear equations, Bell polynomials and linear superposition principle," *J. Phys.: Conf. Ser.*, vol. 411, p. 012021, 2013.
- [7] W. X. Ma, "Bilinear equations and resonant solutions characterized by Bell polynomials," *Rep. Math. Phys.*, vol. 72, pp. 41–56, 2013.
- [8] W. X. Ma, C. X. Li, and J. S. He, "A second Wronskian formulation of the Boussinesq equation," *Nonlinear Anal.: TMA*, vol. 70, pp. 4245–4258, 2009.
- [9] R. Hirota, "Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons," *Phys. Rev. Lett.*, vol. 27, pp. 1192–1194, 1971.
- [10] R. Hirota *Direct Methods in Soliton Theory, Solitons*, R. K. Bullough and P. Caudrey, Eds., Berlin, Heidelberg, Springer-Verlag, 1980.
- [11] K. Sawada and T. Kotera, "A method for finding N -soliton solutions of the K.d.V. equation and K.d.V.-like equation," *Prog. Theor. Phys.*, vol. 51, pp. 1355–1367, 1974.
- [12] Caudrey P. J., Dodd R. K., Gibbon J. D., A new hierarchy of Korteweg–de Vries equations, *Proc. R. Soc. Lond. A*, vol. 351, pp. 407–422, 1976.
- [13] A. C. Newell and Y. B. Zeng, "The Hirota conditions," *J. Math. Phys.*, vol. 27, pp. 2016–2021, 1986.
- [14] Guo F. K., A simple approach to negating the Hirota condition, *Acta Math. Appl. Sin.* 14 (1991) 111–114.
http://www.applmath.com.cn/EN/volumn/volumn_1490.shtml.
- [15] J. Hietarinta, "A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations," *J. Math. Phys.*, vol. 28, pp. 1732–1742, 1987.
- [16] J. Hietarinta, "Introduction to the Hirota bilinear method," in *Integrability of Nonlinear Systems, Lecture Notes in Physics*, vol. 495, Y. Kosmann-Schwarzbach, B. Grammaticos, and K. M. Tamizhmani, Eds., Berlin, Springer, 1997, pp. 95–103.
- [17] W. Hereman and W. Zhuang, "Symbolic software for soliton theory," *Acta Appl. Math.*, vol. 39, pp. 361–378, 1995.
- [18] Z. J. Zhou, J. Z. Fu, and Z. B. Li, "Maple packages for computing Hirota's bilinear equation and multisoliton solutions of nonlinear evolution equations," *Appl. Math. Comput.*, vol. 217, pp. 92–104, 2010.
- [19] W. X. Ma and E. G. Fan, "Linear superposition principle applying to Hirota bilinear equations," *Comput. Math. Appl.*, vol. 61, pp. 950–959, 2011.
- [20] W. X. Ma, Y. Zhang, Y. N. Tang, and J. Y. Tu, "Hirota bilinear equations with linear subspaces of solutions," *Appl. Math. Comput.*, vol. 218, pp. 7174–7183, 2012.
- [21] R. Hirota and J. Satsuma, "N-soliton solutions of model equations for shallow water waves," *J. Phys. Soc. Jpn.*, vol. 40, pp. 611–612, 1976.
- [22] A. M. Wazwaz, "A fifth-order Korteweg–de Vries equation for shallow water with surface tension: multiple soliton solutions," *Acta Phys. Pol. A*, vol. 130, pp. 679–682, 2016.
- [23] A. Ramani, "Inverse scattering, ordinary differential equations of Painlevé-type, and Hirota's bilinear formalism," *Ann. NY Acad. Sci.*, vol. 373, pp. 54–67, 1981.
- [24] Ma W. X., Generalized bilinear differential equations, *Stud. Nonlinear Sci.*, vol. 2, pp. 140–144, 2011.
- [25] W. X. Ma, "Trilinear equations, Bell polynomials, and resonant solutions," *Front. Math. China*, vol. 8, pp. 1139–1156, 2013.