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A matrix second-order negative Ablowitz–Kaup–Newell–Segur flow and its Darboux transformation

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Negative-order integrable flows have significant applications in areas such as fluid dynamics, nonlinear optics, and quantum field theory, particularly in the study of soliton behavior and wave interactions. In this paper, we study a matrix second-order negative flow within the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, formulated through a Lax pair whose temporal part involves a second-order pole in the spectral parameter. This framework facilitates the analysis of shallow water wave dynamics, including the emergence of lump and soliton solutions. The corresponding Darboux transformation is constructed within the general AKNS-type framework. Starting from a seed solution, a class of explicit solutions is generated through a single application of the Darboux transformation.

Keywords: Lax pair; matrix AKNS flows; Darboux transformation.

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1. Introduction

Negative-order flows constitute a fascinating and increasingly significant area within the theory of integrable models. These flows are deeply connected to rich mathematical structures, such as infinite-dimensional symmetries, recursion operators, and algebraic curves, as well as to a variety of physical applications, particularly in fluid dynamics and nonlinear optics (see, e.g. [1–3]). A prominent example is the

Camassa–Holm equation [4], an integrable model with a bi-Hamiltonian structure that describes unidirectional shallow water waves and supports peakon solutions. The study of negative-order flows enhances our understanding of nonlinear wave phenomena, inverse dynamics, and soliton interactions, extending beyond the scope of classical integrable equations.

The Darboux transformation is a powerful method used to generate new solutions from known ones for integrable partial differential equations, especially in soliton theory. It is typically applied to linear spectral problems, known as a Lax pair, associated with nonlinear integrable equations.

Originally introduced by Gaston Darboux in the 19th century in the context of second-order linear differential equations, the transformation was later adapted and extensively developed in the 20th century for integrable systems. In particular, its application to the Korteweg–de Vries hierarchy and other soliton equations has played a pivotal role in modern soliton theory. A comprehensive treatment of the Darboux transformation in integrable systems is provided in the monograph by Matveev and Salle [5]. Further developments, including generalizations to matrix, noncommutative, super-symmetric, and higher-dimensional settings, can be found in [6–8].

At its core, the Darboux transformation constructs a new potential (or solution) by applying an algebraic transformation to the Lax pair of an integrable system. This process preserves the integrability of the original system and allows the recursive generation of multi-soliton, rational, and rogue wave solutions (see, e.g. [9, 10]).

Let us now recall the zero-curvature representation and the formulation of Darboux transformations within this framework. A system of partial differential equations

$$u_t = K(u) = K(x, t, u, u_x, \dots) \tag{1.1}$$

is said to admit a zero-curvature equation representation, if it arises from the compatibility condition of a pair of linear equations:

$$U_t - V_x + [U, V] = 0, \tag{1.2}$$

where U and V are square matrices, known as a Lax pair, depending on the solution u, its derivatives, and the spectral parameter λ. These matrices typically lie in a matrix loop algebra [11]. The zero-curvature equation is equivalent to the compatibility of the linear system

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad \phi_t = V\phi = V(u, \lambda)\phi, \tag{1.3}$$

where ϕ is the eigenfunction vector and λ is the spectral parameter.

A transformation of $\phi' = D\phi$ and $u' = u'(u)$, where $D = D(u, \lambda)$ is a square matrix, is called a Darboux transformation of the spectral problems (1.3), if ϕ' satisfies the same type spectral problems:

$$\phi'_x = U'\phi' = U(u', \lambda)\phi', \quad \phi'_t = V'\phi' = V(u', \lambda)\phi'. \tag{1.4}$$

The matrix D is referred to as the Darboux matrix. The transformed Lax pair (U', V') must satisfy the gauge-type conditions:

$$U'D = DU + D_x, \quad V'D = DV + D_t. \tag{1.5}$$

When U and V are $N \times N$ matrices, a common ansatz for a first-order Darboux matrix is

$$D(\lambda) = \lambda I_N - S, \tag{1.6}$$

where I_N is the identity matrix of order N and S is an $N \times N$ matrix independent of λ. To construct S, one chooses N eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ and their corresponding eigenfunctions solving

$$\phi^{[j]}_x = U(u, \lambda_j)\phi^{[j]}, \quad \phi^{[j]}_t = V(u, \lambda_j)\phi^{[j]}, \quad 1 \leq j \leq N, \tag{1.7}$$

with u being a fixed solution to (1.1). Then, the matrix S is given by [6, 12]:

$$S = HAH^{-1}, \quad H = (\phi^{[1]}, \ldots, \phi^{[N]}), \quad A = \mathrm{diag}(\lambda_1, \ldots, \lambda_N). \tag{1.8}$$

We summarize the Darboux transformation as follows:

	Darboux transformation: $D = \lambda I_N - H\Lambda H^{-1}$	
Lax pair: (U, V)	$\Longrightarrow$	New Lax pair: (U', V')
Solution: $u_t = K(u)$	$\Longrightarrow$	New solution: $u'_t = K(u')$

In this work, we begin with the matrix Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem and derive a matrix-valued second-order negative AKNS flow by introducing a Lax pair involving a second-order pole in the spectral parameter. This flow is a natural extension of the Camassa–Holm type equations associated with the AKNS spectral problem. Based on this Lax representation, we construct the corresponding Darboux transformation within the AKNS framework. By applying this transformation to a seed solution, we generate a class of explicit solutions for the proposed matrix second-order negative AKNS system.

2. Lax Pair and An Associated Integrable System

Let m, n be two natural numbers. We consider the AKNS matrix spectral matrix of the form [13]:

$$U = i\lambda\Lambda + Q, \quad \Lambda = \begin{bmatrix} -I_m & 0 \\ 0 & I_n \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}, \tag{2.1}$$

where I_k denotes the identity matrix of order k, and q and r are the two potential matrices given by

$$q = (q_{jk})_{m\times n}, \quad r = (r_{kj})_{n\times m}. \tag{2.2}$$

We formulate the Lax matrix as follows:

$$V = \frac{1}{1-4\lambda^2} W, \quad W = \lambda W_1 + W_2 \tag{2.3}$$

with

$$W_1 = i\begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}\Lambda - 2i\Lambda Q_t, \quad W_2 = -\frac{1}{2}\begin{bmatrix} v_1 & f \\ g & v_2 \end{bmatrix}\Lambda - Q_{tx}, \tag{2.4}$$

where u_1, v_1 and u_2, v_2 are square potential matrices of orders m and n, respectively, and the off-diagonal blocks f and g are defined as

$$f = -u_1 q - q u_2, \quad g = r u_1 + u_2 r. \tag{2.5}$$

Note that

$$\begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix} = [\Delta\Lambda, Q], \quad \Delta = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}.$$

We present the Lax matrix V in this form to simplify the resulting model equations.

The zero-curvature equation

$$U_t - V_x + [U, V] = 0 \tag{2.6}$$

is equivalent to

$$(1-4\lambda^2)U_t - W_x + [U, W] = 0.$$

By matching powers of λ, we obtain the system

$$\begin{cases} -4Q_t + i[\Lambda, W_1] = 0, \\ -W_{1,x} + i[\Lambda, W_2] + [Q, W_1] = 0, \\ Q_t - W_{2,x} + [Q, W_2] = 0. \end{cases} \tag{2.7}$$

The first equation is automatically satisfied. From the second and third equations, we derive the following system:

$$\begin{cases} q_t + q_{txx} - \frac{1}{2}(u_1 q + q u_2)_x - \frac{1}{2}(v_1 q + q v_2) = 0, \\ r_t + r_{txx} - \frac{1}{2}(r u_1 + u_2 r)_x + \frac{1}{2}(r v_1 + v_2 r) = 0, \\ v_{1,x} + 2q r_{tx} - 2 q_{tx} r = 0, \\ v_{2,x} - 2 r q_{tx} + 2 r_{tx} q = 0, \\ u_{1,x} - 2(qr)_t = 0, \\ u_{2,x} - 2(rq)_t = 0. \end{cases} \tag{2.8}$$

The first four equations come from the third equation in (2.7), and the last two come from the second equation. Becuase the spectal parameter λ appears with a second-order term in the denominator of V, this system corresponds to a second-order negative AKNS flow. It is actually a special case of the system presented in [12], with

$$\begin{cases} n = 2, \quad p_1 = x, \quad x_2 = t, \quad A_1 = -i\lambda\Lambda - Q, \\ a_2 = 1 - 4\lambda^2, \quad A_2 = \lambda W_1 + W_2. \end{cases} \tag{2.9}$$

Note that all functions involved in U and V above, such as q and r, are functions of x and t only, and therefore do not depend on x_1 and p_2.

When we take the following in our construction:

$$m = n = 1, \quad u_1 = u_2 = w, \quad v_1 = v_2 = v, \tag{2.10}$$

the Lax operator V reduces to the scalar form given in [14], and the system (2.8) simplifies to

$$\begin{cases} q_t + q_{txx} - (wq)_x - vq = 0, \\ r_t + r_{txx} - (wr)_x + vr = 0, \\ v_x - 2(r q_{tx} - r_{tx} q) = 0, \\ w_x - 2(qr)_t = 0. \end{cases} \tag{2.11}$$

This is exactly the Nurshuak–Tolkynay–Myrzakulov-II equation as introduced in [14]. There are other negative integrable flows associated with the AKNS spectral problem (see, e.g. [15–19]).

3. Darboux Transformation and Explicit Solutions

3.1. *Compatibility conditions*

We assume the Darboux matrix takes the form

$$D(\lambda) = \lambda I_{m+n} - S. \tag{3.1}$$

Then, D must satisfy the spatial compatibility condition:

$$U'D = DU + D_x, \tag{3.2}$$

where U is given by (2.1) and

$$U' = i\lambda\Lambda + Q'. \tag{3.3}$$

This leads to

$$(i\lambda\Lambda + Q')(\lambda I_{m+n} - S) = (\lambda I_{m+n} - S)(i\lambda\Lambda + Q) - S_x$$

from which we deduce

$$Q' = Q + i[\Lambda, S] \tag{3.4}$$

and

$$S_x = Q'S - SQ = [Q + i\Lambda S, S]. \tag{3.5}$$

Next, we consider the temporal compatibility condition:

$$V'D = DV + D_t, \tag{3.6}$$

where V is defined by (2.3) and

$$V' = \frac{1}{1-4\lambda^2}W', \quad W' = \lambda W_1' + W_2'. \tag{3.7}$$

This implies

$$W'(\lambda I_{m+n} - S) = (\lambda I_{m+n} - S)W - (1 - 4\lambda^2)S_t.$$

Comparing coefficients of powers of λ, we obtain

$$\begin{cases} W_1' = W_1 + 4S_t, \\ W_2' = W_2 + W_1'S - SW_1, \end{cases} \tag{3.8}$$

and the evolution of S satisfies:

$$S_t = W_2'S - SW_2. \tag{3.9}$$

3.2. *Formulation of the Darboux matrix*

Following the general framework (cf. [12]), we define the matrix S as

$$S = HAH^{-1}, \quad H = (\phi^{[1]}, \dots, \phi^{[m+n]}), \quad A = \mathrm{diag}(\lambda_1, \dots, \lambda_{m+n}), \tag{3.10}$$

where each column vector $\phi^{[j]}$ satisfies the eigenvalue problems:

$$\phi_x^{[j]} = U(u,\lambda_j)\phi^{[j]}, \quad \phi_t^{[j]} = V(u,\lambda_j)\phi^{[j]}, \quad 1 \le j \le m+n. \tag{3.11}$$

From this, we compute

$$H_x = i\Lambda HA + QH, \quad H_t = W_1HB_1 + W_2HB_2, \tag{3.12}$$

where the diagonal matrices B_1 and B_2 are given by

$$\begin{aligned} B_1 &= \operatorname{diag}\left(\frac{\lambda_1}{1-4\lambda_1^2}, \dots, \frac{\lambda_{m+n}}{1-4\lambda_{m+n}^2}\right), \\ B_2 &= \operatorname{diag}\left(\frac{1}{1-4\lambda_1^2}, \dots, \frac{1}{1-4\lambda_{m+n}^2}\right). \end{aligned} \tag{3.13}$$

Using (3.12), we compute the derivatives S_x and S_t, and verify that both the spatial and temporal compatibility conditions (3.5) and (3.9) are satisfied.

The spatial derivative yields:

$$\begin{aligned} S_x &= H_xAH^{-1} - HA(H^{-1}H_xH^{-1}) \\ &= i\Lambda HA^2H^{-1} + QHAH^{-1} - HAH^{-1}(i\Lambda HA + QH)H^{-1} \\ &= i\Lambda HA^2H^{-1} + QHAH^{-1} - iHAH^{-1}\Lambda HAH^{-1} - HAH^{-1}Q \\ &= QS - SQ + i\Lambda S^2 - iS\Lambda S, \end{aligned}$$

which confirms that the spatial compatibility condition (3.5) holds.

To verify the temporal condition, we compute

$$\begin{aligned} S_t &= H_tAH^{-1} - HA(H^{-1}H_tH^{-1}) \\ &= W_1HB_1AH^{-1} + W_2HB_2AH^{-1} - HAH^{-1}W_1HB_1H^{-1} \\ &\quad - HAH^{-1}W_2HB_2H^{-1}. \end{aligned}$$

Based on this, we then define

$$\begin{aligned} h_1(W_1) + h_2(W_2) := W_2'S - SW_2 - S_t = \big[W_2 + (W_1 + 4S_t)S - SW_1\big]S \\ - SW_2 - S_t, \end{aligned} \tag{3.14}$$

where h_1 collects all terms containing W_1, and h_2 those containing W_2. Using the identities

$$4B_1A^2 = B_1 - A, \quad 4B_2A^2 = B_2 - I_{m+n}, \tag{3.15}$$

we can compute that

$$\begin{aligned} h_1(W_1) &\equiv W_1HA^2H^{-1} + 4W_1HB_1A^3H^{-1} - 4HAH^{-1}W_1HB_1A^2H^{-1} \\ &\quad - HAH^{-1}W_1HAH^{-1} - W_1HB_1AH^{-1} + HAH^{-1}W_1HB_1H^{-1} \\ &= W_1H(A + 4B_1A^2 - B_1)AH^{-1} - HAH^{-1}W_1H(4B_1A^2 + A - B_1)H^{-1} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} h_2(W_2) &\equiv W_2HAH^{-1} + 4W_2HB_2A^3H^{-1} - 4HAH^{-1}W_2HB_2A^2H^{-1} \\ &\quad - HAH^{-1}W_2 - W_2HB_2AH^{-1} + HAH^{-1}W_2HB_2H^{-1} \\ &= W_2H(I_{m+n} + 4B_2A^2 - B_2)AH^{-1} \\ &\quad - HAH^{-1}W_2H(4B_2A^2 + I_{m+n} - B_2)H^{-1} \\ &= 0. \end{aligned}$$

These computations confirm the validity of the temporal compatibility condition (3.9).

3.3. *The Darboux transformation*

Therefore, the Darboux transformation is given by

$$\phi' = (\lambda I_{m+n} - S)\phi, \quad U' = i\lambda\Lambda + Q', \quad V' = \frac{1}{1-4\lambda^2}(\lambda W_1' + W_2') \tag{3.16}$$

with

$$Q' = Q + i[\Lambda, S], \quad W_1' = W_1 + 4S_t, \quad W_2' = W_2 + W_1'S - SW_1, \tag{3.17}$$

where the matrix S is defined by (3.10). Observing the structures of U, W_1, and W_2 in (2.1) and (2.4), this transformation yields the following expressions for the transformed quantities:

$$\begin{cases} q' = q + i[\Lambda, S]_{12}, \quad r' = r + i[\Lambda, S]_{21}, \\ u_1' = u_1 + 4iS_{11,t}, \quad u_2' = u_2 - 4iS_{22,t}, \\ v_1' = v_1 + 2[W_1, S]_{11} + 8(S_tS)_{11}, \quad v_2' = v_2 - 2[W_1, S]_{22} - 8(S_tS)_{22}. \end{cases} \tag{3.18}$$

Here, a matrix A of order $m+n$ is partitioned into block form as

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] = \left[\begin{array}{c|c} (M_{11})_{m\times m} & (M_{12})_{m\times n} \\ \hline (M_{21})_{n\times m} & (M_{22})_{n\times n} \end{array}\right].$$

3.4. *Application: Explicit solutions*

Consider a seed solution

$$q = r = 0, \quad u_1 = u_{1,0}, \quad u_2 = u_{2,0}, \quad v_1 = v_{1,0}, \quad v_2 = v_{2,0}, \tag{3.19}$$

where $u_{1,0}$, $u_{2,0}$, $v_{1,0}$, and $v_{2,0}$, are constant matrices. The fundamental eigenfunctions

$$\phi^{[j]} = (\phi_1^{[j]T}, \phi_2^{[j]T})^T \tag{3.20}$$

take the form

$$\begin{cases} \phi_1^{[j]} = \exp\left(i\lambda_j I_m x - \frac{1}{1-4\lambda_j^2}\left(i\lambda_j u_{1,0} - \frac{1}{2}v_{1,0}\right)t\right)\mu_1^{[j]}, \\ \phi_2^{[j]} = \exp\left(-i\lambda_j I_n x + \frac{1}{1-4\lambda_j^2}\left(i\lambda_j u_{2,0} - \frac{1}{2}v_{2,0}\right)t\right)\mu_2^{[j]}, \end{cases} \quad 1 \le j \le m+n, \tag{3.21}$$

where $\mu_1^{[j]} \in \mathbb{C}^m$ and $\mu_2^{[j]} \in \mathbb{C}^n$ are arbitrary constant column vectors. Due to $W_1 = \mathrm{diag}(-iu_{1,0}, iu_{2,0})$, we have $[W_1, S]_{11} = 0$ and $[W_1, S]_{22} = 0$, and the resulting explicit solution is then given by

$$\begin{cases} q' = i[\Lambda, S]_{12}, & r' = i[\Lambda, S]_{21}, \\ u_1' = u_{1,0} + 4iS_{11,t}, & u_2' = u_{2,0} - 4iS_{22,t}, \\ v_1' = v_{1,0} + 8(S_tS)_{11}, & v_2' = v_{2,0} - 8(S_tS)_{22}. \end{cases} \tag{3.22}$$

where $S = HAH^{-1}$ with $H = (\phi^{[1]}, \ldots, \phi^{[m+n]})$ and $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_{m+n})$.

4. Conclusions

Starting from a specific matrix Lax pair, we have constructed a matrix second-order negative AKNS flow and its associated Darboux transformation, along with a class of explicit solutions. In the scalar case, this construction recovers the Nurshuak–Tolkynay–Myrzakulov-II equation introduced in [14]. Additional developments on Darboux transformations for negative integrable flows can be found in the literature (see, e.g. [20–23]).

Constrained Lax pairs are of particular interest, especially when reductions are imposed on the general potential matrix U (see, e.g. [24, 25]). A natural question then arises: how can one construct Darboux transformations for such constrained matrix systems, given their reduced Lax pairs? This question becomes even more intriguing in the broader context where the Lax operator V contains higher-order poles in its dependence on the spectral parameter λ. A compelling open problem is thus to determine the structure of Darboux transformations in this more general negative-order setting, where both constraints and nontrivial spectral singularities come into play.

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