International Journal of Geometric Methods in Modern Physics
Vol. 22, No. 3 (2025) 2450293 (13 pages)

DOI: 10.1142/S0219887824502931

A matrix integrable Hamiltonian hierarchy and its two integrable reductions

Wen-Xiu Ma

Department of Mathematics
Zhejiang Normal University
Jinhua, 321004 Zhejiang, P. R. China

Department of Mathematics
King Abdulaziz University
Jeddah 21589, Saudi Arabia

Department of Mathematics and Statistics
University of South Florida
Tampa, FL 33620-5700, USA

Material Science Innovation and Modelling
North-West University Mafikeng Campus
Private Bag X2046, Mmabatho 2735, South Africa
mawx@cas.usf.edu

Received 11 July 2023
Accepted 21 July 2024
Published 24 August 2024

A higher-order spectral problem is introduced, based on a special Lie subalgebra of the general linear algebra. An associated matrix Liouville integrable hierarchy, each of which consists of four submatrix equations, is generated by means of the zero curvature formulation. The corresponding Hamiltonian structure is established by the trace variational identity and two integrable reductions, of which one is real and the other is complex, are constructed under similarity transformations.

Keywords: Lax pair; zero curvature equation; integrable hierarchy; nonlinear Schrödinger equations; integrable reduction.

Mathematics Subject Classification 2020: 37K15, 35Q55, 37K40

1. Introduction

Matrix Lie algebras are used to formulate matrix spectral problems, which yield integrable equations and form the basis for the inverse scattering transform [1–3].

There are many examples of applying the special linear algebra [4–6]. Recently, special orthogonal algebras have also been applied to formulating counterparts of matrix spectral problems associated with the special linear algebra, which lead to integrable hierarchies with Hamiltonian structures (see, e.g. [7, 8]).

It is crucial to choose a pseudoregular element e_0 in the corresponding loop algebra of a matrix Lie algebra in determining a matrix spectral problem of the following form:

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_l e_l(\lambda), \tag{1.1}$$

where $e_0, e_1, \ldots, e_l$ are linearly independent in the loop algebra and $u = (u_1, \ldots, u_l)^T$ is the potential.

Once an appropriate matrix spectral problem is formed, associated zero curvature equations will generate an integrable hierarchy (see, e.g. [9–13] for combined integrable hierarchies). It can be shown that the hierarchy consists of commuting flows. A Hamiltonian structure of the hierarchy can be derived using the trace variational identity, applicable when the Lie algebra is semisimple [14], and the generalized variational identity for non-semisimple cases [15]. This establishes that every member of the hierarchy is Liouville integrable.

The invariance of matrix spectral problems under similarity transformations engenders reduced integrable equations, and such invariance produces local and nonlocal potential reductions, generating reduced integrable equations. Various local and nonlocal reduced integrable equations have been constructed from matrix spectral problems associated with the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problems, indeed (see, e.g. [16–20] for local and nonlocal reductions, respectively). Taking pairs of local and nonlocal reductions, we can generate specific nonlocal reduced integrable equations, and their soliton solutions can be generated by the reflectionless Riemann–Hilbert problems [21, 22] and binary Darboux transformations [23, 24], in which eigenvalues could be equal to adjoint eigenvalues.

This paper aims to present an application of the zero curvature formulation in generating local integrable equations. In Sec. 2, we introduce a higher-order spectral problem with four submatrix potentials, based on a special matrix Lie subalgebra of the general linear algebra. Within the zero curvature formulation, we construct an associated matrix integrable hierarchy, and applying the trace variational identity, we furnish a Hamiltonian structure for the integrable hierarchy. In Sec. 3, we propose and analyze two local reductions of the adopted matrix spectral problem, and present two reduced matrix integrable hierarchies. In Sec. 4, we present a conclusion, together with some concluding remarks.

2. A Matrix Spectral Problem and its Associated Integrable Hierarchy

2.1. *A Lie subalgebra of the general linear algebra*

Let m, n be two given natural numbers, and α, β be two given symmetric and orthogonal matrices of orders n and m:

$$\alpha^T = \alpha, \quad \alpha^T\alpha = I_n, \quad \beta^T = \beta, \quad \beta^T\beta = I_m, \tag{2.1}$$

where T denotes the matrix transpose and I_k is the identity matrix of order k.

We consider a class of square matrices of the following form:

$$A = \begin{bmatrix} -a & b & e \\ c & d & -\alpha b^T \beta \\ f & -\beta c^T \alpha & \beta a^T \beta \end{bmatrix}, \tag{2.2}$$

where a, e, f are $m \times m$ matrices, b and c^T are $m \times n$ matrices, and d is an $n \times n$ matrix, which satisfy

$$(\beta e)^T = -\beta e, \quad (\alpha d)^T = -\alpha d, \quad (\beta f)^T = -\beta f. \tag{2.3}$$

It is easy to see that such matrices form a matrix Lie algebra, under the matrix commutator: $[A, B] = AB - BA$. Actually, we can determine all matrices in (2.2) by the property:

$$(SA)^T = -SA, \quad S = \begin{bmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \\ \beta & 0 & 0 \end{bmatrix}. \tag{2.4}$$

This characteristic feature tells a matrix Lie structure clearly, because noting $S^T = S$, we have

$$\begin{aligned} (S[A,B])^T &= (SAB)^T - (SBA)^T \\ &= B^T(SA)^T - A^T(SB)^T \\ &= B^T(-SA) - A^T(-SB) \\ &= -(SB)^T A + (SA)^T B \\ &= SBA - SAB = -S[A,B], \end{aligned}$$

where SA and SB are assumed to be skew-symmetric.

2.2. *A spectral problem and its integrable hierarchy*

Let λ denote the spectral parameter. We introduce a spatial higher-order spectral problem of the form

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = \begin{bmatrix} -\lambda I_m & p & v \\ q & 0 & -\alpha p^T \beta \\ w & -\beta q^T \alpha & \lambda I_m \end{bmatrix}, \tag{2.5}$$

where the potential $u = u(p, q, v, w)$ consists of four potential submatrices:

$$\begin{cases} p = p(x,t) = (p_{ij})_{m \times n}, & q = q(x,t) = (q_{ij})_{n \times m}, \\ v = v(x,t) = (v_{ij})_{m \times m}, & w = w(x,t) = (w_{ij})_{m \times m}. \end{cases} \tag{2.6}$$

This provides a counterpart of the AKNS spectral problem [1, 4].

To generate an integrable hierarchy, we first solve the stationary zero curvature equationg

$$W_x = i[U, W], \tag{2.7}$$

by looking for a Laurent series solution with the same partitioned form as U:

$$W = \begin{bmatrix} -a & b & e \\ c & d & -\alpha b^T \beta \\ f & -\beta c^T \alpha & \beta a^T \beta \end{bmatrix} = \sum_{s\geq 0} \lambda^{-s} W^{[s]}, \tag{2.8}$$

where the coefficient matrix $W^{[s]}$ is similarly partitioned into

$$W^{[s]} = \begin{bmatrix} -a^{[s]} & b^{[s]} & e^{[s]} \\ c^{[s]} & d^{[s]} & -\alpha b^{[s]T} \beta \\ f^{[s]} & -\beta c^{[s]T} \alpha & \beta a^{[s]} \beta \end{bmatrix}. \tag{2.9}$$

Note that we have

$$[U, W] = ([U, W]_{ij})_{3\times 3},$$

with the matrix blocks being given by

$$\begin{aligned}
[U, W]_{11} &= pc - bq + vf - ew, \\
[U, W]_{12} &= -\lambda b + pd + ap + e\beta q^T \alpha - v\beta c^T \alpha, \\
[U, W]_{13} &= -2\lambda e - p\alpha b^T \beta + b\alpha p^T \beta + v\beta a^T \beta + av, \\
[U, W]_{21} &= \lambda c - qa - dq - \alpha p^T \beta f + \alpha b^T \beta w, \\
[U, W]_{22} &= qb - cp + \alpha p^T c^T \alpha - \alpha b^T q^T \alpha, \\
[U, W]_{23} &= \lambda \alpha b^T \beta + d\alpha p^T \beta + qe - \alpha p^T a^T \beta + cv, \\
[U, W]_{31} &= 2\lambda f - \beta q^T \alpha c + \beta c^T \alpha q - wa - \beta a^T \beta w, \\
[U, W]_{32} &= -\lambda \beta c^T \alpha - \beta q^T \alpha d - fp + \beta a^T q^T \alpha + wb, \\
[U, W]_{33} &= \beta q^T b^T \beta - \beta c^T p^T \beta + we - fv.
\end{aligned}$$

Thus, the stationary zero curvature equation determines the initial values

$$a_x^{[0]} = 0, \quad b^{[0]} = 0, \quad c^{[0]} = 0, \quad d_x^{[0]} = 0, \quad e^{[0]} = 0, \quad f^{[0]} = 0, \tag{2.10}$$

and the recursion relation:

$$\begin{cases}
b^{[s+1]} = ib_x^{[s]} + pd^{[s]} + a^{[s]}p + e^{[s]}\beta q^T \alpha - v\beta c^{[s]T} \alpha, \\
c^{[s+1]} = -ic_x^{[s]} + qa^{[s]} + d^{[s]}q + \alpha p^T \beta f^{[s]} - \alpha b^{[s]T} \beta w, \\
e^{[s+1]} = \dfrac{1}{2}(ie_x^{[s]} - p\alpha b^{[s]T} \beta + b^{[s]} \alpha p^T \beta + v\beta a^{[s]T} \beta + a^{[s]} v),
\end{cases}$$

$$\begin{cases} f^{[s+1]} = \dfrac{1}{2}(-if_x^{[s]} + \beta q^T \alpha c^{[s]} - \beta c^{[s]T} \alpha q + w a^{[s]T} + \beta a^{[s]T} \beta w), \\ a_x^{[s+1]} = i(b^{[s+1]} q - p c^{[s+1]} + e^{[s+1]} w - v f^{[s+1]}), \\ d_x^{[s+1]} = i(q b^{[s+1]} - c^{[s+1]} p + \alpha p^T c^{[s+1]T} \alpha - \alpha b^{[s+1]T} q^T \alpha), \end{cases} \tag{2.11}$$

where $s \geq 0$. Upon selecting

$$a^{[0]} = I_m, \quad d^{[0]} = 0, \tag{2.12}$$

and taking the constant of integration to be zero,

$$a^{[s]}|_{u=0} = 0, \quad d^{[s]}|_{u=0} = 0, \quad s \geq 1, \tag{2.13}$$

we can obtain that

$$\begin{cases} b^{[1]} = p, \quad c^{[1]} = q, \quad e^{[1]} = v, \\ f^{[1]} = w, \quad a^{[1]} = 0, \quad d^{[1]} = 0; \end{cases} \tag{2.14}$$

$$\begin{cases} b^{[2]} = ip_x, \quad c^{[2]} = -iq_x, \quad e^{[2]} = \dfrac{1}{2} i v_x, \quad f^{[2]} = -\dfrac{1}{2} i w_x, \\ a^{[2]} = -pq - \dfrac{1}{2} vw, \quad d^{[2]} = -qp + \alpha p^T q^T \alpha; \end{cases} \tag{2.15}$$

and

$$\begin{cases} b^{[3]} = -p_{xx} - 2pqp + p\alpha p^T q^T \alpha - \dfrac{1}{2} vwp + \dfrac{1}{2} i v_x \beta q^T \alpha + iv\beta q_x^T \alpha, \\ c^{[3]} = -q_{xx} - 2qpq + \alpha p^T q^T \alpha q - \dfrac{1}{2} qvw - \dfrac{1}{2} i\alpha p^T \beta w_x - i\alpha p_x^T \beta w, \\ e^{[3]} = -\dfrac{1}{2}\left(\dfrac{1}{2} v_{xx} + ip\alpha p_x^T \beta - ip_x \alpha p^T \beta + v\beta q^T p^T \beta + pqv \right. \\ \qquad \left. + \dfrac{1}{2} v\beta w^T v^T \beta + \dfrac{1}{2} vwv \right), \\ f^{[3]} = -\dfrac{1}{2}\left(\dfrac{1}{2} w_{xx} + i\beta q^T \alpha q_x - i\beta q_x^T \alpha q + wpq + \beta q^T p^T \beta w \right. \\ \qquad \left. + \dfrac{1}{2} wvw + \dfrac{1}{2} \beta w^T v^T \beta w \right), \\ a^{[3]} = i\left[(pq_x - p_x q) + \dfrac{1}{4}(vw_{xx} - v_x w) + \dfrac{1}{2} iv\beta q^T \alpha q + \dfrac{1}{2} ip\alpha p^T \beta w \right], \\ d^{[3]} = i\left[(q_x p - qp_x) + \alpha(p_x^T q^T - p^T q_x^T)\alpha + iqv\beta q^T \alpha + i\alpha p^T \beta wp \right]. \end{cases} \tag{2.16}$$

At this moment, we readily see that if we define the temporal matrix spectral problems:

$$-i\phi_{t_r} = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi, \quad V^{[r]} = (\lambda^r W)_+ = \sum_{s=0}^{r} \lambda^s W^{[r-s]}, \quad r \geq 0, \tag{2.17}$$

then the compatibility conditions of the two matrix spectral problems in (2.5) and (2.17), namely, the zero curvature equations:

$$U_{t_r} - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r \geq 0, \tag{2.18}$$

yield a matrix integrable hierarchy:

$$u_{t_r} = K^{[r]}, \quad r \geq 0, \tag{2.19}$$

each of which consists of four submatrix equations:

$$\begin{cases} p_{t_r} = ib^{[r+1]}, & q_{t_r} = -ic^{[r+1]}, \\ v_{t_r} = 2ie^{[r+1]}, & w_{t_r} = -2if^{[r+1]}. \end{cases} \tag{2.20}$$

The first example in this matrix integrable hierarchy gives the following coupled integrable nonlinear Schrödinger equations:

$$\begin{cases} ip_{t_2} = p_{xx} + 2pqp - p\alpha p^T q^T \alpha + \frac{1}{2}vwp - \frac{1}{2}iv_x\beta q^T\alpha - iv\beta q_x^T\alpha, \\ iq_{t_2} = -q_{xx} - 2qpq + \alpha p^T q^T \alpha q - \frac{1}{2}qvw - \frac{1}{2}i\alpha p^T\beta w_x - i\alpha p_x^T\beta w, \\ iv_{t_2} = \frac{1}{2}v_{xx} + ip\alpha p_x^T\beta - ip_x\alpha p^T\beta + v\beta q^T p^T\beta + pqv + \frac{1}{2}v\beta w^T v^T\beta + \frac{1}{2}vwv, \\ iw_{t_2} = -\frac{1}{2}w_{xx} - i\beta q^T\alpha q_x + i\beta q_x^T\alpha q - wpq - \beta q^T p^T\beta w \\ \qquad - \frac{1}{2}wvw - \frac{1}{2}\beta w^T v^T\beta w, \end{cases} \tag{2.21}$$

where α and β are two arbitrary symmetric and orthogonal matrices. They extend the range of non-commutative integrable NLS equations (see, e.g. [25]) and coupled integrable NLS equations (see, e.g. [7, 8, 26]), which possess intriguing integrable properties.

2.3. *Hamiltonian structure*

In order to formulate a Hamiltonian structure for the matrix integrable hierarchy (2.20), we take advantage of the trace variational identity

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(W\frac{\partial U}{\partial\lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right), \tag{2.22}$$

where γ is a constant. In our situation, it is easy to see that

$$\begin{cases} \mathrm{tr}\left(W\frac{\partial U}{\partial\lambda}\right) = 2\,\mathrm{tr}\,a, \\ \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = 2c^T, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = 2b^T, \\ \mathrm{tr}\left(W\frac{\partial U}{\partial v}\right) = f^T, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial w}\right) = e^T, \end{cases}$$

and thus, we obtain

$$\begin{cases} \dfrac{\delta}{\delta p}\displaystyle\int a^{[s+1]}\,dx = \lambda^{-\gamma}\dfrac{\partial}{\partial\lambda}\lambda^{\gamma}c^{[s]T}, & \dfrac{\delta}{\delta q}\displaystyle\int a^{[s+1]}\,dx = \lambda^{-\gamma}\dfrac{\partial}{\partial\lambda}\lambda^{\gamma}b^{[s]T}, \\ \dfrac{\delta}{\delta v}\displaystyle\int a^{[s+1]}\,dx = \dfrac{1}{2}\lambda^{-\gamma}\dfrac{\partial}{\partial\lambda}\lambda^{\gamma}f^{[s]T}, & \dfrac{\delta}{\delta w}\displaystyle\int a^{[s+1]}\,dx = \dfrac{1}{2}\lambda^{-\gamma}\dfrac{\partial}{\partial\lambda}\lambda^{\gamma}e^{[s]T}, \end{cases} \quad s \geq 0.$$

Taking a special value $s = 2$, we can determine $\gamma = 0$, and thus, we have

$$\begin{cases} \dfrac{\delta}{\delta p}\displaystyle\int H^{[s]}\,dx = c^{[s+1]T}, & \dfrac{\delta}{\delta q}\displaystyle\int H^{[s]}\,dx = b^{[s+1]T}, \\ \dfrac{\delta}{\delta v}\displaystyle\int H^{[s]}\,dx = \dfrac{1}{2}f^{[s+1]T}, & \dfrac{\delta}{\delta w}\displaystyle\int H^{[s]}\,dx = \dfrac{1}{2}e^{[s+1]T}, \end{cases} \quad s \geq 0, \tag{2.23}$$

where the Hamiltonian functions are defined by

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+2]}}{s+1}\,dx, \quad s \geq 0. \tag{2.24}$$

This allows us to furnish the Hamiltonian structure for the matrix integrable hierarchy (2.20):

$$u_{t_r} = K_r = J\frac{\delta\mathcal{H}^{[r]}}{\delta u}, \text{ i.e. } \begin{cases} p_{t_r} = i\dfrac{\delta\mathcal{H}^{[r]}}{\delta q^T}, & q_{t_r} = -i\dfrac{\delta\mathcal{H}^{[r]}}{\delta p^T}, \\ v_{t_r} = 4i\dfrac{\delta\mathcal{H}^{[r]}}{\delta w^T}, & w_{t_r} = -4i\dfrac{\delta\mathcal{H}^{[r]}}{\delta v^T}, \end{cases} \quad r \geq 0, \tag{2.25}$$

where the Hamiltonian functional $\mathcal{H}^{[r]}$ is determined by (2.24).

The established Hamiltonian structure gives a relation between symmetries and conserved quantities. We can directly show the associated Lax operator algebra:

$$[\![V^{[r]}, V^{[s]}]\!] = V^{[r]\prime}(u)[K^{[s]}] - V^{[s]\prime}(u)[K^{[r]}] + [V^{[r]}, V^{[s]}] = 0, \quad r, s \geq 0, \tag{2.26}$$

and it then follows that infinitely many symmetries commute [27]:

$$[\![K^{[r]}, K^{[s]}]\!] = K^{[r]\prime}(u)[K^{[s]}] - K^{[s]\prime}(u)[K^{[r]}] = 0, \quad r, s \geq 0. \tag{2.27}$$

Further from the Hamiltonian structure (2.25) and the recursion relation (2.11), one can obtain

$$\{\mathcal{H}^{[r]}, \mathcal{H}^{[s]}\}_J = \int \left(\frac{\delta\mathcal{H}^{[r]}}{\delta u}\right)^T J\frac{\delta\mathcal{H}^{[s]}}{\delta u}\,dx = 0, \quad r, s \geq 0. \tag{2.28}$$

By combing J with a hereditary recursion operator Φ, determined from $K^{[s+1]} = \Phi K^{[s]}$ [28], a bi-Hamiltonian structure

$$u_{t_r} = K^{[r]} = J\frac{\delta\mathcal{H}^{[r]}}{\delta u} = M\frac{\delta\mathcal{H}^{[r-1]}}{\delta u}, \quad M = J\Phi,\ r \geq 1, \tag{2.29}$$

can also be formulated for the integrable hierarchy (2.20), and thus, every member in the hierarchy is Liouville integrable.

3. Reduced Integrable Hierarchies

Let δ, η be another pair of constant square matrices of orders n and m, respectively, which satisfy

$$\delta^T = \delta, \quad \delta^T \delta = I_n, \quad \eta^T = \eta, \quad \eta^T \eta = I_m, \tag{3.1}$$

and we assume that

$$\alpha\delta = \delta\alpha, \quad \beta\eta = \eta\beta, \tag{3.2}$$

where α, β are the previous pairs of square matrices involved in the spectral problem (2.5).

3.1. *Real integrable reduction*

First, let us consider a real reduction for the spectral matrix U:

$$CU(\lambda)C^{-1} = U(-\lambda), \quad C = \begin{bmatrix} 0 & 0 & \eta \\ 0 & \delta & 0 \\ \eta & 0 & 0 \end{bmatrix}. \tag{3.3}$$

Noting that

$$CU(\lambda)C^{-1} = \begin{bmatrix} \lambda I_m & -\eta\beta q^T \alpha\delta & \eta w \eta \\ -\delta\alpha p^T \beta\eta & 0 & \delta q \eta \\ \eta v \eta & \eta p \delta & -\lambda I_m \end{bmatrix},$$

the above reduction equivalently requires

$$q = -\delta\alpha p^T \beta\eta, \quad w = \eta v \eta \quad \text{or} \quad p = -\eta\beta q^T \alpha\delta, \quad v = \eta w \eta. \tag{3.4}$$

Due to the commutative conditions in (3.2) between α and δ and between β and η, we see that this matrix still belongs to the previously proposed matrix Lie algebra. It is now direct to know that

$$CW(\lambda)C^{-1} = -W(-\lambda), \tag{3.5}$$

since both Laurent series $CW(\lambda)C^{-1}$ and $W(-\lambda)$ of λ solve the stationary zero curvature equation (2.7) but they possess the opposite initial values at $\lambda = \infty$. Thus, noting that the definition of $V^{[r]} = (\lambda^r W)_+$, we have

$$CV^{[r]}(\lambda)C^{-1} = (-1)^{r+1} V^{[r]}(-\lambda), \quad r \geq 0. \tag{3.6}$$

It then follows that

$$\begin{aligned} &C\big(U_{t_{2s+1}}(\lambda) - V_x^{[2s+1]}(\lambda) + i[U(\lambda), V^{[2s+1]}(\lambda)]\big)C^{-1} \\ &\quad = U_{t_{2s+1}}(-\lambda) - V_x^{[2s+1]}(-\lambda) + i[U(-\lambda), V^{[2s+1]}(-\lambda)], \quad s \geq 0. \end{aligned} \tag{3.7}$$

This yields a reduced matrix integrable hierarchy

$$\begin{aligned} p_{t_{2s+1}} &= ib^{[2s+2]}|_{q=-\delta\alpha p^T\beta\eta,\ w=\eta v\eta}, \\ v_{t_{2s+1}} &= 2ie^{[2s+2]}|_{q=-\delta\alpha p^T\beta\eta,\ w=\eta v\eta}, \quad s \geq 0, \end{aligned} \tag{3.8}$$

each of which also possesses infinitely many symmetries and conserved densities inherited from the original ones under the potential reductions in (3.4). The first nonlinear reduced example consists of the two generalized nonlinear Schrödinger equations:

$$\begin{cases} ip_{t_2} = p_{xx} - 2p\delta\alpha p^T\beta\eta p + p\alpha p^T\beta\eta p\delta \\ \qquad\quad + \dfrac{1}{2}v\eta v\eta p + \dfrac{1}{2}iv_x\eta p\delta + iv\eta p_x\delta, \\ iv_{t_2} = \dfrac{1}{2}v_{xx} + ip\alpha p_x^T\beta - ip_x\alpha p^T\beta - v\eta p\alpha\delta p^T\beta \\ \qquad\quad - p\alpha\delta p^T\beta\eta v + \dfrac{1}{2}v\beta\eta v^T\eta v^T\beta + \dfrac{1}{2}v\eta v\eta v. \end{cases} \tag{3.9}$$

3.2. *Complex integrable reduction*

In what follows, we assume that α, β, δ and η are all real.

Let us second consider a complex reduction for the spectral matrix U:

$$CU(\lambda)C^{-1} = U^{\dagger}(\lambda^*), \quad C = \begin{bmatrix} \eta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \eta \end{bmatrix}, \tag{3.10}$$

where δ, η are again the pair of constant square matrices of orders n and m which satisfy (3.2), and $\dagger$ denotes the Hermitian transpose, and $*$ stands for the complex conjugate.

Upon observing that

$$CU(\lambda)C^{-1} = \begin{bmatrix} -\lambda I_m & \eta p\delta & \eta v\eta \\ \delta q\eta & 0 & -\delta\alpha p^T\beta\eta \\ \eta w\eta & -\eta\beta q^T\alpha\delta & \lambda I_m \end{bmatrix},$$

the above reduction on the spectral matrix exactly requires

$$q^{\dagger} = \eta p\delta, \quad w^{\dagger} = \eta v\eta \quad \text{or} \quad p^{\dagger} = \delta q\eta, \quad v^{\dagger} = \eta w\eta. \tag{3.11}$$

Note that to keep $U^{\dagger}(\lambda^*)$ to be in the chosen algebra, α and β must be real, and to guarantee that the above two sets of potential reductions are in agreement, δ and η must be real. Now taking (3.11) into consideration, we can show that

$$CW(\lambda)C^{-1} = W^{\dagger}(\lambda^*), \tag{3.12}$$

which leads to

$$CV^{[r]}(\lambda)C^{-1} = V^{[r]\dagger}(\lambda^*), \quad r \geq 0. \tag{3.13}$$

This tells that

$$\begin{aligned} &C\big(U_{t_r}(\lambda) - V_x^{[r]}(\lambda) + i[U(\lambda), V^{[r]}(\lambda)]\big)C^{-1} \\ &\quad = U_{t_r}^\dagger(\lambda^*) - V_x^{[r]\dagger}(\lambda^*) + i[U^\dagger(\lambda^*), V^{[r]\dagger}(\lambda^*)], \quad r \geq 0, \end{aligned} \tag{3.14}$$

and thus, we obtain a reduced matrix integrable hierarchy

$$p_{t_r} = ib^{[r+1]}|_{q=\delta p^\dagger \eta,\ w=\eta v^\dagger \eta}, \quad v_{t_r} = 2ie^{[r+1]}|_{q=\delta p^\dagger \eta,\ w=\eta v^\dagger \eta}, \quad r \geq 0, \tag{3.15}$$

whose infinitely many symmetries and conservation laws are similarly inherited from the original ones under the set of potential reductions in (3.11). The first nonlinear reduced integrable system consists of the following generalized nonlinear Schrödinger equations:

$$\begin{cases} ip_{t_2} = p_{xx} + 2p\delta p^\dagger \eta p - p\alpha p^T \eta p^* \delta\alpha + \dfrac{1}{2} v\eta v^\dagger \eta p \\ \qquad\quad - \dfrac{1}{2} iv_x \beta\eta p^* \delta\alpha - iv\beta\eta p_x^* \delta\alpha, \\ iv_{t_2} = \dfrac{1}{2} v_{xx} + ip\alpha p_x^T \beta - ip_x \alpha p^T \beta + v\beta\eta p^* \delta p^T \beta \\ \qquad\quad + p\delta p^\dagger \eta v + \dfrac{1}{2} v\beta\eta v^* \eta v^T \beta + \dfrac{1}{2} v\eta v^\dagger \eta v, \end{cases} \tag{3.16}$$

where † and ∗ again denote the Hermitian transpose and the complex conjugate, respectively.

4. Concluding Remarks

A higher-order spectral problem has been introduced, based on a special Lie subalgebra of the general linear algebra. An associated matrix integrable hierarchy has been generated and its Hamiltonian structure has been established by applying the trace variational identity. Two integrable reductions, of which one is real and the other is complex, yield two reduced matrix integrable hierarchies. Three novel coupled integrable nonlinear Schrödinger systems are presented explicitly.

Recently, reduced nonlocal integrable equations have attracted much attention and such equations exhibit diverse nonlinear wave phenomena different from the ones in the local case. It would be interesting to take the newly introduced matrix spectral problem as an example to construct nonlocal integrable equations (see, e.g. [29] for the case so(3,ℝ)). It should be significantly important to determine special function solutions [31,32] and various soliton-type solutions, including lump solutions [30,33], rogue wave solutions [34,35], multi-pole soliton solutions [36,37], and interaction solutions [38,39]. These investigations encompass both local and nonlocal integrable equations stemming from our newly proposed matrix spectral problems.

We can also combine different group reductions, particularly local and nonlocal ones, and new resultant integrable equations can carry interesting solution structures but complicated characteristics of interaction solutions.

Declaration of Competing Interest

The author declares that there is no known competing interest that could have appeared to influence this work.

Acknowledgments

The work was supported in part by NSFC under the grants 12271488 and 11975145, the Ministry of Science and Technology of China (G2021016032L and G2023016011L), and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020).

ORCID

Wen-Xiu Ma https://orcid.org/0000-0001-5309-1493

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[2] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: the Inverse Scattering Method* (Consultants Bureau, New York, 1984).

[3] V. Drinfeld and V. V. Sokolov, Lie algebras and equations of Korteweg–de Vries type, *J. Sov. Math.* **30** (1985) 1975–2036.

[4] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249–315.

[5] M. Antonowicz and A. P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, *Phys. D* **28** (1987) 345–357.

[6] S. Manukure, Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints, *Commun. Nonlinear Sci. Numer. Simul.* **57** (2018) 125–135.

[7] W. X. Ma, Multi-component bi-Hamiltonian Dirac integrable equations, *Chaos Solitons Fractals* **39** (2009) 282–287.

[8] J. Y. Yang and W. X. Ma, Four-component Liouville integrable models and their bi-Hamiltonian formulations, *Romanian J. Phys.* **69** (2024) 101.

[9] W. X. Ma, A four-component hierarchy of combined integrable equations with bi-Hamiltonian formulations, *Appl. Math. Lett.* **153** (2024) 109025.

[10] W. X. Ma, A combined Liouville integrable hierarchy associated with a fourth-order matrix spectral problem, *Commun. Theor. Phys.* **76** (2024) 075001.

[11] W. X. Ma, Four-component combined integrable equations possessing bi-Hamiltonian formulations, *Modern Phys. Lett. B* **38** (2024) 2450319, doi:10.1142/S0217984924503196.

[12] W. X. Ma, A generalized hierarchy of combined integrable bi-Hamiltonian equations from a specific fourth-order matrix spectral problem, *Mathematics* **12** (2024) 927.

[13] W. X. Ma, A combined Kaup–Newell type integrable Hamiltonian hierarchy with four potentials and a hereditary recursion operator, *Discrete Contin. Dyn. Syst. S* **17** (2024), doi:10.3934/dcdss.2024117.

[14] G. Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy, *J. Phys. A* **22** (1989) 2375–2392.

[15] W. X. Ma and M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semidirect sums of Lie algebras, *J. Phys. A* **39** (2006) 10787–10801.

[16] A. V. Mikhailov, The reduction problem and the inverse scattering method, *Phys. D* **3** (1981) 73–117.

[17] V. S. Gerdjikov, G. G. Grahovski and N. A. Kostov, Reductions of N-wave interactions related to low-rank simple Lie algebras: I. $\mathbb{Z}_2$-reductions, *J. Phys. A* **34** (2001) 9425–9461.

[18] W. X. Ma, A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations, *Modern Phys. Lett. B* **36** (2022) 2250094.

[19] M. J. Ablowitz and Z. H. Musslimani, Integrable nonlocal nonlinear equations, *Stud. Appl. Math.* **139** (2017) 7–59.

[20] W. X. Ma, Nonlocal PT-symmetric integrable equations and related Riemann–Hilbert problems, *Partial Differ. Equ. Appl. Math.* **4** (2021) 100190.

[21] W. X. Ma, Reduced nonlocal integrable mKdV equations of type $(-\lambda, \lambda)$ and their exact soliton solutions, *Commun. Theor. Phys.* **74** (2022) 065002.

[22] W. X. Ma, Type (λ^*, λ) reduced nonlocal integrable AKNS equations and their soliton solutions, *Appl. Numer. Math.* **199** (2024) 105–113.

[23] W. X. Ma, Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations, *Chaos Solitos Fractals* **180** (2024) 114539.

[24] W. X. Ma, Y. H. Huang, F. D. Wang, Y. Zhang and L. Y. Ding, Binary Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations, *Int. J. Geom. Methods Mod. Phys.* **21** (2024) 2450182, doi:10.1142/S0219887824501822.

[25] V. Sokolov, *Algebraic Structures in Integrability* (World Scientific, Singapore, 2020).

[26] W. X. Ma, Integrable couplings and two-dimensional unital algebras, *Axioms* **13** (2024) 481.

[27] W. X. Ma, The algebraic structure of zero curvature representations and application to coupled KdV systems, *J. Phys. A* **26** (1993) 2573–2582.

[28] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D* **4** (1981) 47–66.

[29] W. X. Ma, Integrable nonlocal nonlinear Schrödinger equations associated with so(3,$\mathbb{R}$), *Proc. Amer. Math. Soc. Ser. B* **9** (2022) 1–11.

[30] W. X. Ma and Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, *J. Differential Equations* **264** (2018) 2633–2659.

[31] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-geometric Solutions: $(1+1)$-Dimensional Continuous Models* (Cambridge University Press, Cambridge, 2003).

[32] A. Bashir, A. R. Seadawy, S. Ahmed and S. T. R. Rizvi, The Weierstrass and Jacobi elliptic solutions along with multiwave, homoclinic breather, kink-periodic-cross rational and other solitary wave solutions to Fornberg Whitham equation, *Chaos Solitons Fractals* **163** (2022) 112538.

[33] T. A. Sulaiman, A. Yusuf, A. Abdeljabbar and M. Alquran, Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation, *J. Geom. Phys.* **169** (2021) 104347.

[34] Y. Sun, B. Tian, X. Y. Xie, J. Chai and H. M. Yin, Rogue waves and lump solitons for a (3+1)-dimensional B-type Kadomtsev–Petviashvili equation in fluid dynamics, *Waves Random Complex Media* **28** (2018) 544–552.
[35] Y. L. She and R. X. Yao, Novel particular solutions, breathers, and rogue waves for an integrable nonlocal derivative nonlinear Schrödinger equation, *Adv. Math. Phys.* **2022** (2022) 7670773.
[36] J. J. Yang, S. F. Tian and Z. Q. Li, Riemann–Hilbert method and multi-soliton solutions of an extended modified Korteweg–de Vries equation with N distinct arbitrary-order poles, *J. Math. Anal. Appl.* **511** (2022) 126103.
[37] Y. C. Wei and H. Q. Zhang, Vector multi-pole solutions in the r-coupled Hirota equation, *Wave Motion* **112** (2022) 102959.
[38] S. Manukure, A. Chowdhury and Y. Zhou, Complexiton solutions to the asymmetric Nizhnik–Novikov–Veselov equation, *Internat. J. Modern Phys. B* **33** (2019) 950098.
[39] H. C. Ma, S. P. Yue and A. P. Deng, Lump and interaction solutions for a (2+1)-dimensional combined pKP–BKP equation in fluids, *Modern Phys. Lett. B* **36** (2022) 2250069.