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Soliton solutions to constrained nonlocal integrable nonlinear Schrödinger hierarchies of type $(-\lambda, \lambda)$

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The paper aims to generate nonlocal integrable nonlinear Schrödinger hierarchies of type $(-\lambda, \lambda)$ by imposing two nonlocal matrix restrictions of the AKNS matrix characteristic-value problems of arbitrary order. Based on the explored outspreading of characteristic-values and adjoint characteristic-values, exact soliton solutions are formulated by applying the associated reflectionless generalized Riemann–Hilbert problems, in which characteristic-values and adjoint characteristic-values could have a nonempty intersection. Illustrative models of the resultant mixed-type nonlocal integrable nonlinear Schrödinger equations are presented.

Keywords: Integrable hierarchy; Riemann–Hilbert problem; nonlocal integrable equation; nonlinear Schrödinger equations; soliton solution.

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1. Introduction

Zero-curvature conditions play a key role in studying nonlinear integrable equations. Based on matrix characteristic-value problems, with which zero-curvature conditions are associated, the inverse scattering theory can be established to solve Cauchy problems of integrable equations. Infinitely, many Lie symmetries and their corresponding conservation laws can also be generated from matrix characteristic-value problems. More interestingly, carrying out a matrix restriction for matrix characteristic-value problems can yield simplified local and nonlocal integrable equations provided that the restriction keeps the zero-curvature conditions invariant.

Nonlocal integrable equations have already attracted much attention from the community of nonlinear mathematical physics and have close connections with non-Hermitian quantum mechanics. By taking the Ablowitz–Kaup–Newell–Segur (AKNS) matrix characteristic-value problems as an example, two types of nonlocal modified Korteweg–de Vries equations and three types of nonlocal nonlinear Schrödinger equations can be generated from imposing one nonlocal matrix restriction [1,2]. Soliton solutions to such nonlocal integrable equations can be computed by applying the inverse scattering technique indeed (see, e.g. [3–6]).

Nonlocal integrable equations have also been attempted by many other efficient approaches, including the Darboux transformation [7], the Zakharov–Shabat dressing method [8], the Hirota direct method [9] and the Riemann–Hilbert technique [10–12], through which their exact soliton solutions can be explicitly worked out [7–12]. Riemann–Hilbert problems are one of such effective methods, and it can be used to deal with nonlocal integrable modified Korteweg–de Vries and nonlinear Schrödinger equations [2, 13–15]. In this paper, we are going to propose a sort of mixed-type constrained nonlocal integrable nonlinear Schrödinger equations of even order by carrying out two simultaneous nonlocal matrix restrictions and formulate their exact soliton solutions via reflectionless generalized Riemann–Hilbert problems, which amends applications of Riemann–Hilbert problems to nonlocal integrable equations generated by taking one nonlocal matrix restriction [2, 4].

The remainder of this paper is organized as follows. In Sec. 2.1, we will recall the AKNS matrix hierarchies of integrable equations and their corresponding matrix characteristic-value problems to facilitate subsequent analyses. In Sec. 2.2, we will carry out two simultaneous nonlocal matrix restrictions and construct constrained nonlocal integrable nonlinear Schrödinger hierarchies of type $(-\lambda, \lambda)$, where λ is the spectral parameter. Two scalar prototype examples read

$$u_{1,t} = -\frac{\beta}{\alpha^2} i[u_{1,xx} - 2\gamma u_1(u_1 u_1(x,-t) + u_1(-x,-t)u_1(-x,t))]$$

and

$$u_{1,t} = -\frac{\beta}{\alpha^2} i[u_{1,xx} + 2\theta u_1(u_1 u_1(-x,t) + u_1(-x,-t)u_1(x,-t))],$$

where i is the imaginary unit, $\gamma = \pm 1$, $\theta = \pm 1$, and α and β are two arbitrarily given nonzero real constants that, as we will see, are involved in the imposed matrix spectral problems. Those equations provide supplements to nonlocal integrable nonlinear Schrödinger equations involving one of three coordinates $(-x, t)$, $(x, -t)$ or $(-x, -t)$ (see, e.g. [1,8,16]). It seems not easy to establish the well-posedness of such nonlocal integrable equations simply through traditional techniques in the theory of partial differential equations. In Sec. 3, by virtue of the explored distribution of characteristic-values and adjoint characteristic-values, through the associated reflectionless generalized Riemann–Hilbert problems, where characteristic-values and adjoint characteristic-values could have a nonempty intersection, we will be able to formulate soliton solutions for the resultant constrained integrable nonlocal nonlinear Schrödinger equations of even order. In Sec. 4, we will give closing remarks.

2. Constrained Nonlocal Integrable Nonlinear Schrödinger Hierarchies of Type $(-\lambda, \lambda)$

2.1. *The local AKNS matrix hierarchies of integrable equations reviewed*

Let $m, n \in \mathbb{N}$ be arbitrarily prescribed, $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$, where α_1, α_2 and β_1, β_2 are two couples of given different real numbers.

We consider an AKNS matrix hierarchy of integrable equations (see, for example, [17, 18] for more details):

$$u_t = i\alpha f^{[r+1]}, \quad v_t = -i\alpha g^{[r+1]}, \quad r \geq 0. \tag{2.1}$$

where u and v are the two matrix potentials defined by

$$u = u(x,t) = (u_{jk})_{m\times n}, \quad v = v(x,t) = (v_{kj})_{n\times m}, \tag{2.2}$$

and $f^{[r+1]}$ and $g^{[r+1]}$ are matrices of differential polynomials of u and v to be determined later. We point out that the first (i.e. $r = 2$) nonlinear member gives the AKNS local matrix integrable nonlinear Schrödinger equations:

$$u_t = -\frac{\beta}{\alpha^2} i(u_{xx} + 2uvu), \quad v_t = \frac{\beta}{\alpha^2} i(v_{xx} + 2vuv). \tag{2.3}$$

Each hierarchy above consists of the compatibility conditions, namely, the associated zero-curvature conditions:

$$\mathcal{U}_t - \mathcal{V}_x^{[r]} - i[\mathcal{V}^{[r]}, \mathcal{U}] = 0, \quad r \geq 0, \tag{2.4}$$

of the two matrix AKNS characteristic-value problems

$$\begin{cases} \varphi_x = i\mathcal{U}\varphi = i\mathcal{U}(u,v;\lambda)\varphi, \quad \mathcal{U} = \lambda\Lambda + \mathcal{P}, \\ \varphi_t = i\mathcal{V}^{[r]}\varphi = i\mathcal{V}^{[r]}(u,v;\lambda)\varphi, \quad \mathcal{V}^{[r]} = \lambda^r\Omega + \mathcal{Q}^{[r]}, \quad r \geq 0. \end{cases} \tag{2.5}$$

In the above formulas, the pair of constant square matrices of $(m+n)$th order, Λ and Ω, is given as

$$\Lambda = \mathrm{diag}(\alpha_1 \mathcal{I}_m, \alpha_2 \mathcal{I}_n), \quad \Omega = \mathrm{diag}(\beta_1 \mathcal{I}_m, \beta_2 \mathcal{I}_n), \tag{2.6}$$

where $\mathcal{I}_s$ stands for the unit matrix of order s. The other pair of varied square matrices of $(m+n)$th order, $\mathcal{P}$ and $\mathcal{Q}^{[r]}$, is determined via

$$\mathcal{P} = \mathcal{P}(u,v) = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}, \tag{2.7}$$

called the potential matrix in the characteristic-value problem, and

$$\mathcal{Q}^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} e^{[r-s]} & f^{[r-s]} \\ g^{[r-s]} & h^{[r-s]} \end{bmatrix}, \tag{2.8}$$

where $e^{[s]}, f^{[s]}, g^{[s]}$ and $h^{[s]}$ are defined recursively via

$$f^{[0]} = 0, \quad g^{[0]} = 0, \quad e^{[0]} = \beta_1 \mathcal{I}_m, \quad h^{[0]} = \beta_2 \mathcal{I}_n, \tag{2.9a}$$

$$f^{[s+1]} = \frac{1}{\alpha}(-if_x^{[s]} - uh^{[s]} + e^{[s]}u), \quad s \geq 0, \tag{2.9b}$$

$$g^{[s+1]} = \frac{1}{\alpha}(ig_x^{[s]} + ve^{[s]} - h^{[s]}v), \quad s \geq 0, \tag{2.9c}$$

$$e_x^{[s]} = i(ug^{[s]} - f^{[s]}v), \quad h_x^{[s]} = i(vf^{[s]} - g^{[s]}u), \quad s \geq 1, \tag{2.9d}$$

where we take all constants of integration to be zero. By virtue of the recursive relations in (2.9), we can particularly find that the Laurent series

$$\mathcal{W} = \sum_{s \geq 0} \begin{bmatrix} e^{[s]} & f^{[s]} \\ g^{[s]} & h^{[s]} \end{bmatrix} \lambda^{-s} \tag{2.10}$$

presents a formal solution of the corresponding stationary zero-curvature equation

$$\mathcal{W}_x = i[\mathcal{U}, \mathcal{W}]. \tag{2.11}$$

Making use of the trace identity for Hamiltonian structures [19] or alternatively Lax operator algebra formulations [20, 21], one can directly show that (2.1) determines an integrable hierarchy of commuting flows, each of which has a bi-Hamiltonian formulation and so — a hierarchy of commuting conserved densities exhibiting the Liouville integrability.

2.2. *Constrained nonlocal integrable nonlinear Schrödinger hierarchies*

Assume that Γ_1, Θ_1 and Γ_2, Θ_2 are two couples of invertible symmetric constant matrices of orders m and n, respectively. We would like to introduce and analyze a pair of nonlocal matrix restrictions imposed on the prescribed spatial spectral matrix $\mathcal{U}$:

$$\begin{aligned} \mathcal{U}^T(u(x,-t), v(x,-t); -\lambda) &= (\mathcal{U}(u(x,-t), v(x,-t); -\lambda))^T \\ &= -\Gamma \mathcal{U}(u(x,t), v(x,t); \lambda)\Gamma^{-1} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \mathcal{U}^T(u(-x,-t), v(-x,-t); \lambda) &= (\mathcal{U}(u(-x,-t), v(-x,-t); \lambda))^T \\ &= \Theta \mathcal{U}(u(x,t), v(x,t); \lambda)\Theta^{-1} \end{aligned} \tag{2.13}$$

in which Γ and Θ are the following two square matrices

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}, \quad \Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix} \tag{2.14}$$

which are constant and invertible. Clearly, these matrix restrictions are equivalent to the two potential matrix restrictions that we need

$$\mathcal{P}^T(u(x,-t), v(x,-t)) = -\Gamma \mathcal{P}(u(x,t), v(x,t)t)\Gamma^{-1} \tag{2.15}$$

and

$$\mathcal{P}^T(u(-x,-t), v(-x,-t)) = \Theta \mathcal{P}(u(x,t), v(x,t))\Theta^{-1}. \tag{2.16}$$

More precisely, we need to satisfy the following restrictions on the two potentials u and v:

$$v(x,-t) = -\Gamma_2^{-1} u^T(x,t)\Gamma_1 \tag{2.17}$$

and

$$v(-x,-t) = \Theta_2^{-1} u^T(x,t)\Theta_1. \tag{2.18}$$

It thus follows immediately that our matrix of variables u needs to fulfill a restriction:

$$-\Gamma_2^{-1} u^T(x,t)\Gamma_1 = \Theta_2^{-1} u^T(-x,t)\Theta_1, \tag{2.19}$$

or another matrix of variables v needs to fulfill a restriction:

$$-\Gamma_1^{-1} v^T(x,t)\Gamma_2 = \Theta_1^{-1} v^T(-x,t)\Theta_2, \tag{2.20}$$

in order to ensure that both matrix restrictions in (2.12) and (2.13) are met.

Furthermore, a particularly important result can be explored. Using the matrix restrictions determined by (2.12) and (2.13), one can deduce that

$$\begin{cases} \mathcal{W}^T(u(x,-t), v(x,-t); -\lambda) \\ \quad = (\mathcal{W}(u(x,-t), v(x,-t); -\lambda))^T = \Gamma \mathcal{W}(u(x,t), v(x,t); \lambda)\Gamma^{-1}, \\ \mathcal{W}^T(u(-x,-t), v(-x,-t); \lambda) \\ \quad = (\mathcal{W}(u(-x,-t), v(-x,-t); \lambda))^T = \Theta \mathcal{W}(u(x,t), v(x,t); \lambda)\Theta^{-1}, \end{cases} \tag{2.21}$$

for the solution $\mathcal{W}$ to the stationary zero-curvature equation, defined by (2.10). These further imply that

$$\begin{cases} \mathcal{V}^{[2s]T}(u(x,-t), v(x,-t); -\lambda) \\ \quad = (\mathcal{V}^{[2s]}(u(x,-t), v(x,-t); -\lambda))^T = \Gamma \mathcal{V}^{[2s]}(u(x,t), v(x,t); \lambda)\Gamma^{-1}, \\ \mathcal{V}^{[2s]T}(u(-x,-t), v(-x,-t); \lambda) \\ \quad = (\mathcal{V}^{[2s]}(u(-x,-t), v(-x,-t); \lambda))^T = \Theta \mathcal{V}^{[2s]}(u(x,t), v(x,t); \lambda)\Theta^{-1} \end{cases} \tag{2.22}$$

and

$$\begin{cases} \mathcal{Q}^{[2s]T}(u(x,-t),v(x,-t);-\lambda) \\ \quad = (\mathcal{Q}^{[2s]}(u(x,-t),v(x,-t);-\lambda))^T = \Gamma\mathcal{Q}^{[2s]}(u(x,t),v(x,t);\lambda)\Gamma^{-1}, \\ \mathcal{Q}^{[2s]T}(u(-x,-t),v(-x,-t);\lambda) \\ \quad = (\mathcal{Q}^{[2s]}(u(-x,-t),v(-x,-t);\lambda))^T = \Theta\mathcal{Q}^{[2s]}(u(x,t),v(x,t);\lambda)\Theta^{-1}, \end{cases} \tag{2.23}$$

where $s \geq 0$.

Accordingly, by use of the potential restrictions (2.17) and (2.18), the local AKNS matrix integrable equations in (2.1), when r takes on even numbers $2s,\ s \geq 0$, are constrained to nonlocal matrix integrable nonlinear Schrödinger type equations:

$$u_t = i\alpha f^{[2s+1]}|_{v=-\Gamma_2^{-1}u^T(x,-t)\Gamma_1=\Theta_2^{-1}u^T(-x,-t)\Theta_1}, \quad s \geq 0, \tag{2.24}$$

in which u is an $m \times n$ constrained matrix potential which satisfies the constraint (2.19) and Γ_1, Γ_2 and Θ_1, Θ_2 are two couples of invertible constant symmetric matrices of orders m and n, respectively. Being outcomes of the two matrix restrictions, every member in the constrained hierarchy (2.24) possesses its own Lax pair which is formed out of the constrained counterparts of the spatial and temporal matrix characteristic-value problems in (2.5) with $r = 2s,\ s \geq 0$, and countably many constrained counterparts of the symmetries and their corresponding conserved densities that the local matrix AKNS integrable equations in (2.1) with $r = 2s, s \geq 0$, possess.

When we take $s = 1$, namely, $r = 2$, the constrained nonlocal integrable nonlinear Schrödinger type equations (2.24) put forward a sort of type $(-\lambda, \lambda)$ constrained nonlocal integrable nonlinear Schrödinger counterparts:

$$\begin{aligned} u_t &= -\frac{\beta}{\alpha^2} i(u_{xx} - 2u\Gamma_2^{-1}u^T(x,-t)\Gamma_1 u) \\ &= -\frac{\beta}{\alpha^2} i(u_{xx} + 2u\Theta_2^{-1}u^T(-x,-t)\Theta_1 u), \end{aligned} \tag{2.25}$$

in which an $m \times n$ matrix of variables u needs to satisfy the restriction (2.19).

In what follows, we illustrate the general procedure by a few examples of these type $(-\lambda, \lambda)$ constrained nonlocal integrable nonlinear Schrödinger equations by considering different values for m, n and taking appropriate choices for Γ, Θ. In our subsequent construction, we will adopt the following notation for two basic matrices:

$$\mathcal{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.26}$$

for the sake of brevity.

First, we compute two examples in the case where $m = 1$ and $n = 2$. First, we choose

$$\Gamma_1 = 1, \quad \Gamma_2^{-1} = \gamma \mathcal{I}_2, \quad \Theta_1 = 1, \quad \Theta_2^{-1} = \theta \Pi_2, \tag{2.27}$$

where γ and θ are real constants which satisfy $\gamma^2 = \theta^2 = 1$. Such a choice aims to meet the imposed matrix restrictions. Then, equivalently, we see that the potential restriction (2.19) requires

$$u_2 = -\gamma\theta u_1(-x, -t),$$

where $u = (u_1, u_2)$, and at this moment, the resultant potential matrix $\mathcal{P}$ is

$$\mathcal{P} = \begin{bmatrix} 0 & u_1 & -\gamma\theta u_1(-x,-t) \\ -\gamma u_1(x,-t) & 0 & 0 \\ \theta u_1(-x,t) & 0 & 0 \end{bmatrix}. \tag{2.28}$$

Furthermore, the corresponding type $(-\lambda, \lambda)$ constrained nonlocal integrable nonlinear Schrödinger equations read

$$u_{1,t} = -\frac{\beta}{\alpha^2} i[u_{1,xx} - 2\gamma u_1(u_1 u_1(x,-t) + u_1(-x,-t)u_1(-x,t))], \tag{2.29}$$

where $\gamma = \pm 1$.

Similarly, let us choose

$$\Gamma_1 = 1, \quad \Gamma_2^{-1} = \sigma \Pi_2, \quad \Theta_1 = 1, \quad \Theta_2^{-1} = \theta \mathcal{I}_2, \tag{2.30}$$

where γ and θ are real constants which satisfy $\gamma^2 = \theta^2 = 1$. Taking such a choice also aims to meet the imposed matrix restrictions. This choice leads to the following reduced potential matrix $\mathcal{P}$:

$$\mathcal{P} = \begin{bmatrix} 0 & u_1 & -\gamma\theta u_1(-x,-t) \\ \theta u_1(-x,t) & 0 & 0 \\ -\gamma u_1(x,-t) & 0 & 0 \end{bmatrix}, \tag{2.31}$$

and the novel constrained mixed-type nonlocal integrable nonlinear Schrödinger equations:

$$u_{1,t} = -\frac{\beta}{\alpha^2} i[u_{1,xx} + 2\theta u_1(u_1 u_1(-x,t) + u_1(-x,-t)u_1(x,-t))], \tag{2.32}$$

where $\theta = \pm 1$. The mixed-type nonlocality pattern in this pair of equations is different from the one in (2.29).

Second, we compute two examples in the case where $m = 1$ and $n = 4$. Let us take

$$\Gamma_1 = 1, \quad \Gamma_2^{-1} = \mathrm{diag}(\gamma_1 \mathcal{I}_2, \gamma_2 \mathcal{I}_2), \quad \Theta_1 = 1, \quad \Theta_2^{-1} = \mathrm{diag}(\theta_1 \Pi_2, \theta_2 \Pi_2) \tag{2.33}$$

and

$$\Gamma_1 = 1, \quad \Gamma_2^{-1} = \mathrm{diag}(\gamma_1 \Pi_2, \gamma_2 \Pi_2), \quad \Theta_1 = 1, \quad \Theta_2^{-1} = \mathrm{diag}(\theta_1 \mathcal{I}_2, \theta_2 \mathcal{I}_2), \tag{2.34}$$

where γ_j and θ_j are real constants fulfilling $\gamma_j^2 = \theta_j^2 = 1,\ j = 1, 2$. These choices just reflect a requirement that we must satisfy the imposed matrix restrictions, and they put forward, respectively, the reduced potential matrices:

$$\mathcal{P} = \begin{bmatrix} 0 & u_1 & -\gamma_1\theta_1 u_1(-x,-t) & u_3 & -\gamma_2\theta_2 u_3(-x,-t) \\ -\gamma_1 u_1(x,-t) & 0 & 0 & 0 & 0 \\ \theta_1 u_1(-x,t) & 0 & 0 & 0 & 0 \\ -\gamma_2 u_3(x,-t) & 0 & 0 & 0 & 0 \\ \theta_2 u_3(-x,t) & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.35}$$

and

$$\mathcal{P} = \begin{bmatrix} 0 & u_1 & -\gamma_1\theta_1 u_1(-x,-t) & u_3 & -\gamma_2\theta_2 u_3(-x,-t) \\ \theta_1 u_1(-x,t) & 0 & 0 & 0 & 0 \\ -\gamma_1 u_1(x,-t) & 0 & 0 & 0 & 0 \\ \theta_2 u_3(-x,t) & 0 & 0 & 0 & 0 \\ -\gamma_2 u_3(x,-t) & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.36}$$

Consequently, the associated two sorts of two-component mixed-type nonlocal integrable nonlinear Schrödinger counterparts take the form

$$\begin{cases} u_{1,t} = -\dfrac{\beta}{\alpha^2} i[u_{1,xx} - 2\gamma_1 u_1(u_1 u_1(x,-t) + u_1(-x,-t)u_1(-x,t)) \\ \qquad - 2\gamma_2 u_1(u_3 u_3(x,-t) + u_3(-x,-t)u_3(-x,t))], \\ u_{3,t} = -\dfrac{\beta}{\alpha^2} i[u_{3,xx} - 2\gamma_1 u_3(u_1 u_1(x,-t) + u_1(-x,-t)u_1(-x,t)) \\ \qquad - 2\gamma_2 u_3(u_3 u_3(x,-t) + u_3(-x,-t)u_3(-x,t))] \end{cases} \tag{2.37}$$

and

$$\begin{cases} u_{1,t} = -\dfrac{\beta}{\alpha^2} i[u_{1,xx} + 2\theta_1 u_1(u_1 u_1(-x,t) + u_1(-x,-t)u_1(x,-t)) \\ \qquad + 2\theta_2 u_1(u_3 u_3(-x,t) + u_3(-x,-t)u_3(x,-t))], \\ u_{3,t} = -\dfrac{\beta}{\alpha^2} i[u_{3,xx} + 2\theta_1 u_3(u_1 u_1(-x,t) + u_1(-x,-t)u_1(x,-t)) \\ \qquad + 2\theta_2 u_3(u_3 u_3(-x,t) + u_3(-x,-t)u_3(x,-t))], \end{cases} \tag{2.38}$$

respectively, where γ_j and θ_j are real constants satisfying $\gamma_j^2 = \theta_j^2 = 1,\ j = 1, 2$.

Third, we compute four examples in the case where $m = 2$ and $n = 2$. We take

$$\Gamma_1 = \gamma_1 \Pi_2, \quad \Gamma_2^{-1} = \gamma_2 \mathcal{I}_2, \quad \Theta_1 = \theta_1 \Pi_2, \quad \Theta_2^{-1} = \theta_2 \Pi_2, \tag{2.39}$$

$$\Gamma_1 = \gamma_1 \Pi_2, \quad \Gamma_2^{-1} = \gamma_2 \Pi_2, \quad \Theta_1 = \theta_1 \mathcal{I}_2, \quad \Theta_2^{-1} = \theta_2 \Pi_2, \tag{2.40}$$

$$\Gamma_1 = \gamma_1 \mathcal{I}_2, \quad \Gamma_2^{-1} = \gamma_2 \Pi_2, \quad \Theta_1 = \theta_1 \Pi_2, \quad \Theta_2^{-1} = \theta_2 \Pi_2 \tag{2.41}$$

and

$$\Gamma_1 = \gamma_1 \Pi_2, \quad \Gamma_2^{-1} = \gamma_2 \Pi_2, \quad \Theta_1 = \theta_1 \Pi_2, \quad \Theta_2^{-1} = \theta_2 \mathcal{I}_2, \tag{2.42}$$

where γ_j and θ_j are, similarly, real constants fulfilling $\gamma_j^2 = \theta_j^2 = 1,\ j = 1, 2$. As in the examples presented above, making these selections also aims to meet the imposed matrix restrictions. Clearly, such choices bring us the corresponding constrained matrix potentials:

$$u = \begin{bmatrix} u_{11} & -\gamma\theta u_{11}(-x,t) \\ u_{21} & -\gamma\theta u_{21}(-x,t) \end{bmatrix}, \quad v = \begin{bmatrix} -\gamma u_{21}(x,-t) & -\gamma u_{11}(x,-t) \\ \theta u_{21}(-x,-t) & \theta u_{11}(-x,-t) \end{bmatrix}, \tag{2.43}$$

$$u = \begin{bmatrix} u_{11} & u_{12} \\ -\gamma\theta u_{11}(-x,t) & -\gamma\theta u_{12}(-x,t) \end{bmatrix}, \quad v = \begin{bmatrix} \theta u_{12}(-x,-t) & -\gamma u_{12}(x,-t) \\ \theta u_{11}(-x,-t) & -\gamma u_{11}(x,-t) \end{bmatrix}, \tag{2.44}$$

$$u = \begin{bmatrix} u_{11} & u_{12} \\ -\gamma\theta u_{11}(-x,t) & -\gamma\theta u_{12}(-x,t) \end{bmatrix}, \quad v = \begin{bmatrix} -\gamma u_{12}(x,-t) & \theta u_{12}(-x,-t) \\ -\gamma u_{11}(x,-t) & \theta u_{11}(-x,-t) \end{bmatrix} \tag{2.45}$$

and

$$u = \begin{bmatrix} u_{11} & -\gamma\theta u_{11}(-x,t) \\ u_{21} & -\gamma\theta u_{21}(-x,t) \end{bmatrix}, \quad v = \begin{bmatrix} \theta u_{21}(-x,-t) & \theta u_{11}(-x,-t) \\ -\gamma u_{21}(x,-t) & -\gamma u_{11}(x,-t) \end{bmatrix}, \tag{2.46}$$

respectively, where $\gamma = \gamma_1\gamma_2$ and $\theta = \theta_1\theta_2$. Such formulations on the potential matrices allow us to present the corressponding four sorts of two-component mixed-type nonlocal integrable nonlinear Schrödinger counterparts

$$\begin{cases} u_{11,t} = -\dfrac{\beta}{\alpha^2} i[u_{11,xx} - 2\gamma u_{11}(u_{21}(-x,-t)u_{11}(-x,t) + u_{11}u_{21}(x,-t)) \\ \qquad - 2\gamma u_{21}(u_{11}(-x,-t)u_{11}(-x,t) + u_{11}u_{11}(x,-t))], \\ u_{21,t} = -\dfrac{\beta}{\alpha^2} i[u_{21,xx} - 2\gamma u_{11}(u_{21}(-x,-t)u_{21}(-x,t) + u_{21}u_{21}(x,-t)) \\ \qquad - 2\gamma u_{21}(u_{11}(-x,-t)u_{21}(-x,t) + u_{21}u_{11}(x,-t))], \end{cases} \tag{2.47}$$

$$\begin{cases} u_{11,t} = -\dfrac{\beta}{\alpha^2} i[u_{11,xx} + 2\theta u_{11}(u_{11}u_{12}(-x,-t) + u_{12}u_{11}(-x,-t)) \\ \qquad + 2\theta u_{11}(-x,t)(u_{11}u_{12}(x,-t) + u_{12}u_{11}(x,-t))], \\ u_{12,t} = -\dfrac{\beta}{\alpha^2} i[u_{12,xx} + 2\theta u_{12}(u_{11}u_{12}(-x,-t) + u_{12}u_{11}(-x,-t)) \\ \qquad + 2\theta u_{12}(-x,t)(u_{11}u_{12}(x,-t) + u_{12}u_{11}(x,-t))], \end{cases} \tag{2.48}$$

$$\begin{cases} u_{11,t} = -\dfrac{\beta}{\alpha^2} i[u_{11,xx} - 2\gamma u_{11}(u_{11}u_{12}(x,-t) + u_{12}u_{11}(x,-t)) \\ \qquad - 2\gamma u_{11}(-x,t)(u_{11}u_{12}(-x,-t) + u_{12}u_{11}(-x,-t))], \\ u_{12,t} = -\dfrac{\beta}{\alpha^2} i[u_{12,xx} - 2\gamma u_{12}(u_{11}u_{12}(x,-t) + u_{12}u_{11}(x,-t)) \\ \qquad - 2\gamma u_{12}(-x,t)(u_{11}u_{12}(-x,-t) + u_{12}u_{11}(-x,-t))] \end{cases} \tag{2.49}$$

and

$$
\begin{cases}
u_{11,t} = -\dfrac{\beta}{\alpha^2} i[u_{11,xx} + 2\theta u_{11}(u_{11}u_{21}(-x,-t) + u_{21}(x,-t)u_{11}(-x,t)) \\
\qquad\qquad + 2\theta u_{21}(u_{11}u_{11}(-x,-t) + u_{11}(x,-t)u_{11}(-x,t))], \\
u_{21,t} = -\dfrac{\beta}{\alpha^2} i[u_{21,xx} + 2\theta u_{11}(u_{21}u_{21}(-x,-t) + u_{21}(x,-t)u_{21}(-x,t)) \\
\qquad\qquad + 2\theta u_{21}(u_{21}u_{11}(-x,-t) + u_{11}(x,-t)u_{21}(-x,t))],
\end{cases}
\tag{2.50}
$$

respectively, where $\gamma = \gamma_1\gamma_2 = \pm 1$ and $\theta = \theta_1\theta_2 = \pm 1$.

3. Soliton Solutions

3.1. *Outspreading of characteristic-values and adjoint characteristic-values*

Making use of the matrix restriction of (2.12) (or (2.13)), one can prove that if λ is a characteristic-value of the matrix characteristic-value problems (2.5), then $\hat{\lambda} = -\lambda$ (or $\hat{\lambda} = \lambda$) is an adjoint characteristic-value, namely, it fulfills the adjoint matrix characteristic-value problems:

$$
\tilde{\varphi}_x = -i\tilde{\varphi}\mathcal{U} = -i\tilde{\varphi}\mathcal{U}(u,v;\hat{\lambda}), \quad \tilde{\varphi}_t = -i\tilde{\varphi}\mathcal{V}^{[2s]} = -i\tilde{\varphi}\mathcal{V}^{[2s]}(u,v;\hat{\lambda}), \tag{3.1}
$$

where $s \geq 0$ and vice versa. Based on this fact, it is reasonable to assume that one can have characteristic-values consisting of $\mu,\ -\mu$, and adjoint characteristic-values consisting of $-\mu,\ \mu$, in which $\mu \in \mathbb{C}$.

What's more, upon using the matrix restrictions, (2.12) and (2.13), one can observe that both of

$$
\varphi^T(x,-t,-\lambda)\Gamma \quad \text{and} \quad \varphi^T(-x,-t,\lambda)\Theta, \tag{3.2}
$$

solve the above adjoint matrix characteristic-value problems with the originally prescribed characteristic-value λ, as long as $\varphi(\lambda)$ solves the matrix characteristic-value problems (2.5) with a given characteristic-value λ.

3.2. *Solitons by generalized Riemann–Hilbert problems*

This section aims to present a general procedure for computing soliton solutions of the resultant mixed-type nonlocal integrable nonlinear Schrödinger counterparts via determining solutions to the associated reflectionless generalized Riemann–Hilbert problems. Assume that $N_1, N_2 \geq 0$ are two arbitrary integers such that $N = 2N_1 + N_2 \geq 1$.

First, we take a set of characteristic-values λ_k, $1 \leq k \leq N$, and a set of adjoint characteristic-values $\hat{\lambda}_l,\ 1 \leq l \leq N$, as follows:

$$
\lambda_k, 1 \leq k \leq N : \mu_1, \ldots, \mu_{N_1}, -\mu_1, \ldots, -\mu_{N_1}, \nu_1, \ldots, \nu_{N_2} \tag{3.3}
$$

and

$$\hat{\lambda}_l, 1 \leq l \leq N: -\mu_1, \ldots, -\mu_{N_1}, \mu_1, \ldots, \mu_{N_1}, -\nu_1, \ldots, -\nu_{N_2}, \tag{3.4}$$

in which $\mu_k, \nu_l \in \mathbb{C},\ 1 \leq k \leq N_1,\ 1 \leq l \leq N_2$, Also suppose that the corresponding eigenfunctions and adjoint eigenfunctions are denoted, respectively, by

$$\upsilon_k, 1 \leq k \leq N \quad \text{and} \quad \hat{\upsilon}_l, \quad 1 \leq l \leq N. \tag{3.5}$$

And in this nonlocal situation, the following empty intersection condition:

$$\{\lambda_k \,|\, 1 \leq k \leq N\} \cap \{\hat{\lambda}_l \,|\, 1 \leq l \leq N\} = \emptyset, \tag{3.6}$$

is not satisfied.

Next, let us introduce two square matrices of $(m+n)$th order:

$$\begin{cases} \mathcal{T}^+(\lambda) = \mathcal{I}_{m+n} - \displaystyle\sum_{k,l=1}^{N} \frac{1}{\lambda - \hat{\lambda}_l} \upsilon_k (\mathcal{M}^{-1})_{kl} \hat{\upsilon}_l, \\ (\mathcal{T}^-)^{-1}(\lambda) = \mathcal{I}_{m+n} + \displaystyle\sum_{k,l=1}^{N} \frac{1}{\lambda - \lambda_k} \upsilon_k (\mathcal{M}^{-1})_{kl} \hat{\upsilon}_l, \end{cases} \tag{3.7}$$

in which $\mathcal{M}$ is an Nth-order square matrix $\mathcal{M} = (m_{kl})_{N \times N}$, whose elements are determined through

$$m_{lk} = \begin{cases} \dfrac{\hat{\upsilon}_l \upsilon_k}{\lambda_k - \hat{\lambda}_l} & \text{if } \lambda_k \neq \hat{\lambda}_l, \\ 0 & \text{if } \lambda_k = \hat{\lambda}_l, \end{cases} \quad \text{where } 1 \leq l, \quad k \leq N. \tag{3.8}$$

A theorem in [14] tells that the above two square matrices, $\mathcal{T}^+(\lambda)$ and $\mathcal{T}^-(\lambda)$, actually provide a solution to the associated reflectionless generalized Riemann–Hilbert problem. Namely, they are the inverse matrices of each other:

$$(\mathcal{T}^-)^{-1}(\lambda)\mathcal{T}^+(\lambda) = \mathcal{I}_{m+n}, \quad \lambda \in \mathbb{R}, \tag{3.9}$$

as long as an orthogonality condition:

$$\hat{\upsilon}_l \upsilon_k = 0 \quad \text{if } \lambda_k = \hat{\lambda}_l, \quad \text{where } 1 \leq l, \quad k \leq N, \tag{3.10}$$

is fulfilled.

Now, we make a Taylor expansion

$$\mathcal{T}^+(\lambda) = \mathcal{I}_{m+n} + \frac{1}{\lambda}\mathcal{T}_1^+ + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \tag{3.11}$$

at $\lambda = \infty$, to yield

$$\mathcal{T}_1^+ = -\sum_{k,l=1}^{N} \upsilon_k (\mathcal{M}^{-1})_{kl} \hat{\upsilon}_l, \tag{3.12}$$

and then based on the spatial matrix characteristic-value problems in (2.5), we get

$$\mathcal{P} = [\mathcal{T}_1^+, \Lambda] = -\lim_{\lambda\to\infty}[\Lambda, \lambda\mathcal{T}^+(\lambda)]. \tag{3.13}$$

Further, we can conclude that (3.12) and (3.13) produce soliton solutions of the matrix AKNS integrable equations (2.1):

$$u = \alpha\sum_{k,l=1}^{N} v_k^1(\mathcal{M}^{-1})_{kl}\hat{v}_l^2, \quad v = -\alpha\sum_{k,l=1}^{N} v_k^2(\mathcal{M}^{-1})_{kl}\hat{v}_l^1, \tag{3.14}$$

where for each pair of k, l between 1 and N, the two vector functions v_k and $\hat{v}_l$ are split into $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_l = (\hat{v}_l^1, \hat{v}_l^2)$, of which v_k^1 and $\hat{v}_l^1$ are, respectively, column and row vectors of dim m, and v_k^2 and $\hat{v}_l^2$ are, respectively, column and row vectors of dim n.

When we begin with zero matrix potentials, $u = 0$ and $v = 0$, the associated matrix characteristic-value problems in (2.5) tell

$$v_k = v_k(x, t, \lambda_k) = \exp(i\lambda_k\Lambda x + i\lambda_k^{2s}\Omega t)\omega_k, \quad 1 \le k \le N, \tag{3.15}$$

in which all ω's, are column constant vectors. Following the discussion made previously in Sec. 3.1, the associated adjoint eigenfunctions, corresponding to the $\hat{\lambda}$'s, can be taken in the following way:

$$\hat{v}_l = \hat{v}_l(x, t, \hat{\lambda}_l) = v_l^\dagger(-x, t, \lambda_)\Gamma = \hat{\omega}_l e^{-i\hat{\lambda}_l\Lambda x - i\hat{\lambda}_l^{2s}\Omega t}, \quad 1 \le l \le N, \tag{3.16}$$

in which

$$\hat{\omega}_l = \omega_l^T\Gamma, \quad 1 \le l \le N. \tag{3.17}$$

Consequently, the orthogonality condition (3.10) reads

$$\omega_l^T\Gamma\omega_k = 0 \quad \text{if } \lambda_k = \hat{\lambda}_l, \quad \text{where } 1 \le l, \quad k \le N. \tag{3.18}$$

Lastly, in order to put forward soliton solutions of the resultant nonlocal integrable matrix nonlinear Schrödinger counterparts (2.24), one needs to know if $\mathcal{T}_1^+$ defined by (3.12) satisfies the two involutory relations:

$$(\mathcal{T}_1^+)^\dagger(x, t) = \Gamma\mathcal{T}_1^+(x, -t)\Gamma^{-1}, \quad (\mathcal{T}_1^+)^T(x, t) = -\Theta\mathcal{T}_1^+(-x, -t)\Theta^{-1}. \tag{3.19}$$

If the answer is yes, the resultant contrained matrix $\mathcal{P}$ defined by (3.13) will fulfill the couple of nonlocal matrix restriction requirements given by'(2.15) and (2.16) simultaneously. Then, it follows from those two conditions that one can have the following class of soliton solutions:

$$u = \alpha\sum_{k,l=1}^{N} v_k^1(\mathcal{M}^{-1})_{kl}\hat{v}_l^2, \tag{3.20}$$

of the resulting mixed-type nonlocal integrable matrix nonlinear Schrödinger counterparts (2.24). These solutions are just the ones reduced from (3.14) for the matrix AKNS equations (2.1).

3.3. *Realization of the involutory relations*

Let us here build a theoretical framework for satisfying the required involutory relations in (3.19).

First, based on the analyzes made in Sec. 3.1, the associated adjoint eigenfunctions $\hat{\upsilon}_l,\ 1 \le l \le 2N_1$, can be deduced as follows:

$$\hat{\upsilon}_l = \hat{\upsilon}_l(x,t,\hat{\lambda}_l) = \upsilon_l^T(x,-t,\lambda_l)\Gamma = \upsilon_{N_1+l}^T(-x,-t,\lambda_{N_1+l})\Theta, \quad 1 \le l \le N_1 \tag{3.21}$$

and

$$\begin{aligned}\hat{\upsilon}_{N_1+l} &= \hat{\upsilon}_{N_1+l}(x,t,\hat{\lambda}_{N_1+l}) = \upsilon_{N_1+l}^T(x,-t,\lambda_{N_1+l})\Gamma \\ &= \upsilon_l^T(-x,-t,\lambda_l)\Theta, \quad 1 \le l \le N_1.\end{aligned} \tag{3.22}$$

It is direct to see that these determinations by (3.21) and (3.22) create the desired criteria for $\omega_l,\ 1 \le l \le 2N_1$:

$$\begin{cases}\omega_l^T(\Gamma\Theta^{-1} - \Theta\Gamma^{-1}) = 0, & 1 \le l \le N_1, \\ \omega_l = \Theta^{-1}\Gamma\omega_{l-N_1}, & N_1+1 \le l \le 2N_1,\end{cases} \tag{3.23}$$

whose aim is to fulfill the two required restriction conditions in (2.15) and (2.16).

Now, notice that once we solve the reflectionless generalized Riemann–Hilbert problems through (3.7) and (3.8), and fulfill the required involutory relations

$$(\mathcal{T}^+)^\dagger(-\lambda^*) = \Gamma(\mathcal{T}^-)^{-1}(\lambda)\Gamma^{-1}, \quad (\mathcal{T}^+)^T(\lambda) = \Theta(\mathcal{T}^-)^{-1}(\lambda)\Theta^{-1}, \tag{3.24}$$

the corresponding resultant matrix $\mathcal{T}_1^+$ will achieve the involutory relationship goal in (3.19), which comes from the matrix restrictions in (2.12) and (2.13). In this way, the expression (3.20), along with (3.7), (3.8), (3.15) and (3.16) produces a class of soliton solutions of the constrained mixed-type nonlocal integrable matrix nonlinear Schrödinger counterparts (2.24), provided that the ω's fulfill the essential criteria (3.23) and the orthogonality condition (3.18).

At last, in the case where $m = n/2 = N = 1$, we put forward an illustrative example of one-soliton solutions of the mixed-type nonlocal integrable nonlinear Schrödinger equation (2.29). We take $\lambda_1 = \xi i + \eta,\ \hat{\lambda}_1 = -\xi i - \eta,\ \xi, \eta \in \mathbb{R}$, and set $\omega_1 = (\omega_{1,1}, \omega_{1,2}, \omega_{1,3})^T$, where $\omega_{1,1}, \omega_{1,2}, \omega_{1,3} \in \mathbb{C}$ are arbitrary. This choice brings us a class of one-soliton solutions of the mixed-type nonlocal integrable nonlinear Schrödinger equation (2.29):

$$u_1 = \frac{2\alpha\gamma(\xi i + \eta)\omega_{1,1}\omega_{1,2}\mathrm{e}^{-i\beta\xi^2 t - \xi(2\beta\eta t + \alpha x) + i\eta(\beta\eta t + \alpha x)}}{\gamma(\omega_{1,2}^2 + \omega_{1,3}^2) + \omega_{1,1}^2\mathrm{e}^{-2\alpha(\xi i + \eta)x}}, \tag{3.25}$$

where $\xi, \eta \in \mathbb{R}$ are arbitrarily given constants, and $\omega_{1,1}, \omega_{1,2}, \omega_{1,3} \in \mathbb{C}$ are also arbitrarily given constants but required to fulfill the conditions

$$\omega_{1,2}^2 = \omega_{1,3}^2, \quad \omega_{1,1}^4 = 4\omega_{1,3}^4 \tag{3.26}$$

which comes from the involutory relations in (3.19). For the equation with $\gamma = 1$, all solutions with real constants, $\omega_{1,1}, \omega_{1,2}$ and $\omega_{1,3}$, are analytic, when $\xi = 0$. For the equation with $\gamma = -1$, all solutions satisfying $|\omega_{1,1}|^2 \ne 2|\omega_{1,2}|^2$ are analytic, too, when $\eta = 0$.

4. Concluding Remarks

Constrained nonlocal integrable nonlinear Schrödinger counterparts of type $(-\lambda, \lambda)$ have been proposed and their exact soliton solutions have been constructed through solving the associated reflectionless generalized Riemann–Hilbert problems, where characteristic-values and adjoint characteristic-values could have a nonempty intersection. The crucial step is to impose two nonlocal matrix restrictions for the matrix AKNS characteristic-value problems simultaneously. The involved nonlocalities in the resultant nonlocal integrable nonlinear Schrödinger equations of even order are mixed-type, including reverse-space, reverse-time and reverse-spacetime functions. All the presented nonlocal integrable equations are, thus, novel.

We remark that soliton solutions are an important class of exact solutions which can be explicitly worked out for nonlinear integrable equations. In the nonlocal case, such a task is more challenging, since nonlocal equations are significantly different from local equations. In our preceding analysis, we have just applied the Riemann–Hilbert technique. It will be of extreme importance to construct soliton solutions by other traditional methods, such as the Hirota direct method, the Wronskian technique and the Darboux transformation. Moreover, it will be very helpful to study dynamical properties of diverse solutions for nonlocal integrable equations via Riemann–Hilbert problems. Such solutions can be lump and breather wave solutions [22–24], algebro–geometric solutions [25] and solitonless solutions [26].

Note that the mixed-type nonlocalities bring difficulty for establishing global existence of solutions or more generally, the well-posedness theory. It should also be an extremely rewarding experience to look for more nonlocal, both scalar and matrix, integrable equations which are associated with other matrix Lie algebras (see, e.g. [27, 28] for examples associated so(3, $\mathbb{R}$)), including nonlocal integrabe couplings [29], based on non-semisimple Lie algebras [30]. Definitely, we need more new ideas, new research and new tools that will enable us to address problems on nonlocal partial differential equations.

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