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ORIGINAL PAPER

A Higher-Order Spectral Problem and Associated Matrix Integrable Hierarchies

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Abstract
Based on a special Lie subalgebra of the general linear algebra, a higher-order matrix spectral problem is proposed. An associated matrix integrable hierarchy, each of which consists of four submatrix equations, is constructed from the associated zero curvature equations. The corresponding Hamiltonian formulation is furnished by utilizing the trace identity, and two integrable reductions over the real and complex fields are presented by means of similarity transformations.

Keywords Matrix spectral problem · Zero curvature equation · Integrable hierarchy · NLS equations · mKdV equations

Mathematics Subject Classification 37K15 · 35Q55 · 37K40

Introduction

Matrix spectral problems over matrix Lie algebras are the staring points to generate integrable equations, and provide the basis for the inverse scattering transform [1–3]. There are various examples of applying the special linear algebra [4–6]. Very recently, special orthogonal algebras have also been used to construct counterparts of matrix spectral problems associated with the special linear algebra, which yield integrable hierarchies with Hamiltonian formulations (see, e.g., [7]).

It is a key to select a pseudoregular element e_0 in the corresponding loop algebra of a matrix Lie algebra in determining a matrix spectral problem of the form

$$i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_l e_l(\lambda),$$

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where $e_0, e_1, \ldots, e_l$ are linearly independent matrices in the loop algebra and $u = (u_1, \ldots, u_l)^T$ is the potential vector. Once a proper matrix spectral problem is determined, associated zero curvature equations will engender integrable equations, which represent commuting flows, and their Hamiltonian formulations could be furnished via the trace identity if the underlying Lie algebra is semisimple [8], and the variational identity if the underlying Lie algebra is non-seimsimple [9].

Similarity transformations keep matrix spectral problems invariant and so engender reduced integrable equations. The resulting invariance requires local and nonlocal potential reductions, producing reduced integrable equations. Various local and nonlocal reduced integrable equations have been presented from matrix spectral problems associated with the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problems, indeed (see, e.g., [10–12] and [13, 14] for local and nonlocal reductions, respectively). Taking pairs of local and nonlocal reductions, we can construct specific nonlocal reduced integrable equations, and their soliton solutions can be formulated by the reflectionless Riemann–Hilbert problems, in which eigenvalues could be equal to adjoint eigenvalues [15].

The aim of this paper is to provide an application of the zero curvature formulation in generating local integrable equations. In Sect. 2, we propose a special matrix Lie sub-algebra of the general linear algebra, and construct a higher-order matrix spectral problem with four submatrix potentials. Based on the zero curvature formulation, we work out an associated matrix integrable hierarchy, and by applying the trace identity, we present a Hamiltonian formulation for the resulting integrable hierarchy. In Sect. 3, we present and discuss two local reductions of the adopted matrix spectral problem, and compute two reduced matrix integrable hierarchies, over the real and complex fields. In the last section, we give rise to a conclusion and a few concluding remarks.

A Matrix Spectral Problem and Associated Integrable Equations

A Lie Sub-algebra of Matrices

Let m, n be two given natural numbers, T stand for the matrix transpose, and I_k denote the identity matrix of order k.

We consider a set of square matrices of the form

$$A = \begin{bmatrix} -a & b & e \\ c & d & -b^T \\ f & -c^T & a^T \end{bmatrix}, \tag{1}$$

where a, e, f are $m \times m$ matrices, b and c^T are $m \times n$ matrices, and d is an $n \times n$ matrix, and e, f, d are assumed to be skew-symmetric. It is direct to show that such matrices constitute a matrix Lie algebra, under the matrix commutator: $[A, B] = AB - BA$. Actually, we can determine all matrices in (1) by the property:

$$(SA)^T = -SA, \quad S = \begin{bmatrix} 0 & 0 & I_m \\ 0 & I_n & 0 \\ I_m & 0 & 0 \end{bmatrix}. \tag{2}$$

Such a characteristic feature determines a matrix Lie structure indeed, because due to $S^T = S$, we can compute

$$(S[A, B])^T = (SAB)^T - (SBA)^T$$

$$
\begin{aligned}
&= B^T(SA)^T - A^T(SB)^T \\
&= B^T(-SA) - A^T(-SB) \\
&= -(SB)^T A + (SA)^T B \\
&= SBA - SAB = -S[A, B],
\end{aligned}
$$

where SA and SB are assumed to be skew-symmetric.

A Spectral Problem and Its Integrable Equations

Let λ stand for the spectral parameter. We introduce a spatial higher-order spectral problem of the form

$$
-i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = \begin{bmatrix} -\lambda I_m & p & v \\ q & 0 & -p^T \\ w & -q^T & \lambda I_m \end{bmatrix}, \tag{3}
$$

where the potential $u = u(p, q, v, w)$ consists of four potential submatrices:

$$
\begin{cases} p = p(x,t) = (p_{ij})_{m\times n}, & q = q(x,t) = (q_{ij})_{n\times m}, \\ v = v(x,t) = (v_{ij})_{m\times m}, & w = w(x,t) = (w_{ij})_{m\times m}. \end{cases} \tag{4}
$$

This tells a counterpart spectral problem of the AKNS spectral problem [1, 4].

To construct an integrable hierarchy, we usually first solve the stationary zero curvature equation

$$
W_x = i[U, W], \tag{5}
$$

by searching for a Laurent series solution with the same partitioned form as U:

$$
W = \begin{bmatrix} -a & b & e \\ c & d & -b^T \\ f & -c^T & a^T \end{bmatrix} = \sum_{s\ge 0} \lambda^{-s} W^{[s]}, \tag{6}
$$

where the coefficient matrix $W^{[s]}$ is similarly partitioned into

$$
W^{[s]} = \begin{bmatrix} -a^{[s]} & b^{[s]} & e^{[s]} \\ c^{[s]} & d^{[s]} & -b^{[s]T} \\ f^{[s]} & -c^{[s]T} & a^{[s]} \end{bmatrix}. \tag{7}
$$

Assuming that

$$
[U, W] = \big([U, W]_{ij}\big)_{3\times 3},
$$

we can work out those matrix blocks:

$$
\begin{aligned}
&[U, W]_{11} = pc - bq + vf - ew, \\
&[U, W]_{12} = -\lambda b + pd + ap + eq^T - vc^T, \\
&[U, W]_{13} = -2\lambda e - pb^T + bp^T + va^T + av, \\
&[U, W]_{21} = \lambda c - qa - dq - p^T f + b^T w, \\
&[U, W]_{22} = qb - cp + p^T c^T - b^T q^T, \\
&[U, W]_{23} = \lambda b^T + dp^T + qe - p^T a^T + cv, \\
&[U, W]_{31} = 2\lambda f - q^T c + c^T q - wa - a^T w, \\
&[U, W]_{32} = -\lambda c^T - q^T d - fp + a^T q^T + wb, \\
&[U, W]_{33} = q^T b^T - c^T p^T + we - fv.
\end{aligned}
$$

Thus, the corresponding stationary zero curvature equation determines the initial values

$$a_x^{[0]} = 0, \quad b^{[0]} = 0, \quad c^{[0]} = 0, \quad d_x^{[0]} = 0, \quad e^{[0]} = 0, \quad f^{[0]} = 0, \tag{8}$$

and the recursion relation:

$$\begin{cases} b^{[s+1]} = ib_x^{[s]} + pd^{[s]} + a^{[s]}p + e^{[s]}q^T - vc^{[s]T}, \\ c^{[s+1]} = -ic_x^{[s]} + qa^{[s]} + d^{[s]}q + p^T f^{[s]} - b^{[s]T}w, \\ e^{[s+1]} = \frac{1}{2}\big(ie_x^{[s]} - pb^{[s]T} + b^{[s]}p^T + va^{[s]T} + a^{[s]}v\big), \\ f^{[s+1]} = \frac{1}{2}\big(-if_x^{[s]} + q^T c^{[s]} - c^{[s]T}q + wa^{[s]T} + a^{[s]T}w\big), \\ a_x^{[s+1]} = i\left(b^{[s+1]}q - pc^{[s+1]} + e^{[s+1]}w - vf^{[s+1]}\right), \\ d_x^{[s+1]} = i\left(qb^{[s+1]} - c^{[s+1]}p + p^T c^{[s+1]T} - b^{[s+1]T}q^T\right), \end{cases} \tag{9}$$

where $s \geq 0$.

To determine a unique sequence of $W^{[s]}$, $s \geq 0$, we take

$$a^{[0]} = I_m, \quad d^{[0]} = 0, \tag{10}$$

and select the constant of integration as zero,

$$a^{[s]}|_{u=0} = 0, \quad d^{[s]}|_{u=0} = 0, \quad s \geq 1. \tag{11}$$

In this way, we can work out that

$$\begin{cases} b^{[1]} = p, \quad c^{[1]} = q, \quad e^{[1]} = v, \\ f^{[1]} = w, \quad a^{[1]} = 0, \quad d^{[1]} = 0; \end{cases} \tag{12}$$

$$\begin{cases} b^{[2]} = ip_x, \quad c^{[2]} = -iq_x, \quad e^{[2]} = \frac{1}{2}iv_x, \quad f^{[2]} = -\frac{1}{2}iw_x, \\ a^{[2]} = -pq - \frac{1}{2}vw, \quad d^{[2]} = -qp + p^T q^T; \end{cases} \tag{13}$$

and

$$\begin{cases} b^{[3]} = -p_{xx} - 2pqp + pp^T q^T - \frac{1}{2}vwp + \frac{1}{2}iv_x q^T + ivq_x^T, \\ c^{[3]} = -q_{xx} - 2qpq + p^T q^T q - \frac{1}{2}qvw - \frac{1}{2}ip^T w_x - ip_x^T w, \\ e^{[3]} = -\frac{1}{2}\left(\frac{1}{2}v_{xx} + ipp_x^T - ip_x p^T + vq^T p^T + pqv + \frac{1}{2}vw^T v^T + \frac{1}{2}vwv\right), \\ f^{[3]} = -\frac{1}{2}\left(\frac{1}{2}w_{xx} + iq^T q_x - iq_x^T q + wpq + q^T p^T w + \frac{1}{2}wvw + \frac{1}{2}w^T v^T w\right), \\ a^{[3]} = i\left[(pq_x - p_x q) + \frac{1}{4}(vw_{xx} - v_x w) + \frac{1}{2}ivq^T q + \frac{1}{2}ipp^T w\right], \\ d^{[3]} = i\left[(q_x p - qp_x) + \left(p_x^T q^T - p^T q_x^T\right) + iqvq^T + ip^T wp\right]. \end{cases} \tag{14}$$

At this moment, we can easily see that if we take the temporal matrix spectral problems as

$$-i\phi_t = V^{[r]}\phi = V^{[r]}(u, \lambda)\phi, \quad V^{[r]} = (\lambda^r W)_+ = \sum_{s=0}^{r} \lambda^s W^{[r-s]}, \quad r \geq 0, \tag{15}$$

then the compatibility conditions of the two matrix spectral problems in (3) and (15), i.e., the associated zero curvature equations:

$$U_{t_r} - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r \geq 0, \tag{16}$$

generate a hierarchy of matrix integrable equations:

$$u_{t_r} = K^{[r]},\ r \geq 0, \tag{17}$$

each of which consists of four submatrix equations:

$$p_{t_r} = ib^{[r+1]}, \quad q_{t_r} = -ic^{[r+1]}, \quad v_{t_r} = 2ie^{[r+1]}, \quad w_{t_r} = -2if^{[r+1]}. \tag{18}$$

The first nonlinear example in this matrix integrable hierarchy presents the following generalized nonlinear Schrödinger equations:

$$\begin{cases} ip_{t_2} = p_{xx} + 2pqp - pp^T q^T + \frac{1}{2}vwp - \frac{1}{2}iv_x q^T - ivq_x^T, \\ iq_{t_2} = -q_{xx} - 2qpq + p^T q^T q - \frac{1}{2}qvw - \frac{1}{2}ip^T w_x - ip_x^T w, \\ iv_{t_2} = \frac{1}{2}v_{xx} + ipp_x^T - ip_x p^T + vq^T p^T + pqv + \frac{1}{2}vw^T v^T + \frac{1}{2}vwv, \\ iw_{t_2} = -\frac{1}{2}w_{xx} - iq^T q_x + iq_x^T q - wpq - q^T p^T w - \frac{1}{2}wvw - \frac{1}{2}w^T v^T w. \end{cases} \tag{19}$$

Hamiltonian Formulation

To present a Hamiltonian formulation for the matrix integrable hierarchy (18), we usually take advantage of the trace indenty

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(W\,\tfrac{\partial U}{\partial \lambda}\big)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}\big(W\,\tfrac{\partial U}{\partial u}\big), \tag{20}$$

where γ is a constant related to W. In our situation, it is easy to observe that

$$\begin{cases} \mathrm{tr}\big(W\,\frac{\partial U}{\partial \lambda}\big) = 2\,\mathrm{tr}\,a, \\ \mathrm{tr}\big(W\,\frac{\partial U}{\partial p}\big) = 2c^T, \qquad \mathrm{tr}\big(W\,\frac{\partial U}{\partial q}\big) = 2b^T, \\ \mathrm{tr}\big(W\,\frac{\partial U}{\partial v}\big) = f^T, \qquad \mathrm{tr}\big(W\,\frac{\partial U}{\partial w}\big) = e^T, \end{cases} \tag{21}$$

and thus, we arrive at

$$\begin{cases} \frac{\delta}{\delta p}\int a^{[s+1]}\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}c^{[s]T}, \qquad \frac{\delta}{\delta q}\int a^{[s+1]}\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}b^{[s]T}, \\ \frac{\delta}{\delta v}\int a^{[s+1]}\,dx = \frac{1}{2}\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}f^{[s]T}, \qquad \frac{\delta}{\delta w}\int a^{[s+1]}\,dx = \frac{1}{2}\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}e^{[s]T}, \end{cases} \quad s \geq 0. \tag{22}$$

Considering a special case of $s = 2$, we can find $\gamma = 0$, and thus, we obtain

$$\begin{cases} \frac{\delta}{\delta p}\int H^{[s]}\,dx = c^{[s]T}, \qquad \frac{\delta}{\delta q}\int H^{[s]}\,dx = b^{[s]T}, \\ \frac{\delta}{\delta v}\int H^{[s]}\,dx = \frac{1}{2}f^{[s]T}, \qquad \frac{\delta}{\delta w}\int H^{[s]}\,dx = \frac{1}{2}e^{[s]T}, \end{cases} \quad s \geq 1, \tag{23}$$

where the Hamiltonian functions are given by

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+1]}}{s}\,dx, \quad s \geq 1. \tag{24}$$

This allows us to present the Hamiltonian formulation for the matrix integrable hierarchy (18):

$$\begin{cases} p_{t_r} = i\dfrac{\delta \mathcal{H}^{[r]}}{\delta q^T}, & q_{t_r} = -i\frac{\delta \mathcal{H}^{[r]}}{\delta p^T}, \\ v_{t_r} = 4i\dfrac{\delta \mathcal{H}^{[r]}}{\delta w^T}, & w_{t_r} = -4i\frac{\delta \mathcal{H}^{[r]}}{\delta v^T}, \end{cases} \quad r \geq 0, \tag{25}$$

where the Hamiltonian functional $\mathcal{H}^{[r]}$ is defined by (24).

The established Hamiltonian formulation provides a relation between symmetries and conserved quantities. We can directly explore the associated Lax operator algebra:

$$[\![V^{[r]}, V^{[s]}]\!] = V^{[r]\prime}(u)[K^{[s]}] - V^{[s]\prime}(u)[K^{[r]}] + [V^{[r]}, V^{[s]}] = 0, \quad r, s \geq 0, \tag{26}$$

and it further follows that infinitely many symmetries $\{K^{[s]}\}_{s=0}^{\infty}$ commute [16]:

$$[\![K^{[r]}, K^{[s]}]\!] = K^{[r]\prime}(u)[K^{[s]}] - K^{[s]\prime}(u)[K^{[r]}] = 0, \quad r, s \geq 0. \tag{27}$$

A bi-Hamiltonian formulation can also be established by combing J with a recursion relation for $K^{[s]}$ generated from (9) [17].

Reduced Local Integrable Hierarchies

Integrable Reduction Over the Real Field $\mathbb{R}$

First, let us consider a reduction for the spectral matrix U over the real field $\mathbb{R}$:

$$CU(\lambda)C^{-1} = U(-\lambda), \quad C = \begin{bmatrix} 0 & 0 & I_m \\ 0 & I_n & 0 \\ I_m & 0 & 0 \end{bmatrix}, \tag{28}$$

Noting that

$$CU(\lambda)C^{-1} = \begin{bmatrix} \lambda I_m & -q^T & w \\ -p^T & 0 & q \\ v & p & -\lambda I_m \end{bmatrix}, \tag{29}$$

the above reduction equivalently requires

$$q = -p^T, \quad w = v \quad \text{or} \quad p = -q^T, \quad v = w. \tag{30}$$

Obviously, we can see that this matrix still belongs to the previously proposed matrix Lie sub-algebra. It is direct to check that

$$CW(\lambda)C^{-1} = -W(-\lambda), \tag{31}$$

since both Laurent series $CW(\lambda)C^{-1}$ and $W(-\lambda)$ of λ solve the stationary zero cuvature Eq. (5) but they possess the opposite initial values at $\lambda = \infty$. Thus, by the definition of $V^{[r]} = (\lambda^r W)_+$, we know

$$CV^{[r]}(\lambda)C^{-1} = (-1)^{r+1}V^{[r]}(-\lambda), \ r \geq 0. \tag{32}$$

Further, it follows that

$$\begin{aligned} &C\big(U_{t_{2s+1}}(\lambda) - V_x^{[2s+1]}(\lambda) + i[U(\lambda), V^{[2s+1]}(\lambda)]\big)C^{-1} \\ &= U_{t_{2s+1}}(-\lambda) - V_x^{[2s+1]}(-\lambda) + i[U(-\lambda), V^{[2s+1]}(-\lambda)], \ s \geq 0. \end{aligned} \tag{33}$$

This engenders a reduced matrix integrable hierarchy

$$p_{t_{2s+1}} = ib^{[2s+2]}|_{q=-p^T,\ w=v}, \quad v_{t_{2s+1}} = 2ie^{[2s+2]}|_{q=-p^T,\ w=v}, \quad s \geq 0, \tag{34}$$

each of which also has infinitely many symmetries and conserved densities inherited from the original ones under the potential reductions in (30).

Integrable Reduction Over the Complex Field $\mathbb{C}$

Let us second consider a reduction for the spectral matrix U over the complex field $\mathbb{C}$:

$$CU(\lambda)C^{-1} = U^\dagger(\lambda^*), \quad C = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}, \tag{35}$$

where $\dagger$ stands for the Hermitian transpose and $*$ denotes the complex conjugate.

Upon noticing that

$$CU(\lambda)C^{-1} = \begin{bmatrix} -\lambda I_m & p & v \\ q & 0 & -p^T \\ w & -q^T & \lambda I_m \end{bmatrix}, \tag{36}$$

the above reduction on the spectral matrix exactly requires

$$q^\dagger = p, \quad w^\dagger = v \quad \text{or} \quad p^\dagger = q, \quad v^\dagger = w. \tag{37}$$

Note that this keeps $U^\dagger(\lambda^*)$ to be in the chosen matrix Lie sub-algebra, Now taking (37) into consideration, we can verify that

$$CW(\lambda)C^{-1} = W^\dagger(\lambda^*), \tag{38}$$

which yields

$$CV^{[r]}(\lambda)C^{-1} = V^{[r]\dagger}(\lambda^*), \ r \geq 0. \tag{39}$$

This ensures that

$$\begin{aligned} &C\big(U_{t_r}(\lambda) - V_x^{[r]}(\lambda) + i[U(\lambda), V^{[r]}(\lambda)]\big)C^{-1} \\ &\quad = U_{t_r}^\dagger(\lambda^*) - V_x^{[r]\dagger}(\lambda^*) + i[U^\dagger(\lambda^*), V^{[r]\dagger}(\lambda^*)], \ r \geq 0, \end{aligned} \tag{40}$$

and consequently, we obtain a reduced matrix integrable hieierachy

$$p_{t_r} = ib^{[r+1]}|_{q=p^\dagger,\ w=v^\dagger}, \quad v_{t_r} = 2ie^{[r+1]}|_{q=p^\dagger,\ w=v^\dagger}, \quad r \geq 0, \tag{41}$$

whose infinitely many symmetries and conservation laws are similarly inherited from the original ones under the set of potential reductions in (37). The first nonlinear reduced integrable system presents the following generalized nonlonear Schrödinger equations:

$$\begin{cases} ip_{t_2} = p_{xx} + 2pp^\dagger p - pp^T p^* + \frac{1}{2}vv^\dagger p - \frac{1}{2}iv_x p^* - ivp_x^*, \\ iv_{t_2} = \frac{1}{2}v_{xx} + ipp_x^T - ip_x p^T + vp^* p^T + pp^\dagger v + \frac{1}{2}vv^* v^T + \frac{1}{2}vv^\dagger v, \end{cases} \tag{42}$$

where $\dagger$ and $*$ again stand for the Hermitian transpose and the complex conjugate, respectively.

Concluding Remarks

Based on a special Lie sub-algebra of the general linear algebra, a higher-order matrix spectral problem has been proposed. An associated matrix integrable hierarchy has been constructed from the corresponding zero curvature equations, and its Hamiltonian formulation has been furnished by utilizing the trace identity. Two integrable reductions over the real and complex fields were made, which yield two reduced matrix integrable hierarchies. Two generalized nonlinear Schrödinger integrable systems were presented explicitly.

Recently, reduced nonlocal integrable equations have attracted much attention and such equations exhibit diverse nonlinear wave phenomena different from the ones in the local case. It would be interesting to take the newly introduced matrix spectral problem as an example to construct nonlocal integrable equations (see, e.g., [18] for the case so(3, $\mathbb{R}$)). it should be significantly important to look for special function solutions [19, 20] and soliton type solutions, including lump solutions [21, 22], rogue wave solutions [23, 24], multi-pole soliton solutions [25, 26], and interaction solutions [27, 28], in both cases of local and nonlocal integrable equations associated with our new matrix spectral problems. Moreover, we can combine two different group reductions, particularly local and nonlocal ones, and new resultant integrable equations can carry interesting soliton structures with complicated characteristics of soliton interactions.

Author Contributions W-XM: Writing—original draft, writing—review and editing, methodology, formal analysis, conceptualization.

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Data Availability All data generated or analyzed during this study are included in this published article.

Declarations

Competing interests The author declares that there is no known competing interests that could have appeared to influence this work.

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