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ORIGINAL PAPER

Reduced Non-Local Integrable NLS Hierarchies by Pairs of Local and Non-Local Constraints

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Abstract
We conduct three pairs of local and non-local group constraints for the Ablowitz-Kaup-Newell-Segur matrix eigenvalue problems and generate three reduced non-local integrable nonlinear Schrödinger (NLS) hierarchies. All resulting non-local equations possess infinitely many Lie-Bäcklund symmetries and conservation laws expressed in terms of differential functions of potentials.

Keywords Matrix eigenvalue problem · Non-local group constraint · Integrable hierarchy

Mathematics Subject Classification 37K15 · 35Q55 · 37K40

Introduction

Lax pairs of matrix eigenvalue problems form the foundation for the theory of integrable equations. Their group constraints can lead to local and non-local reduced integrable equations. For the Ablowitz-Kaup-Newell-Segur (AKNS) matrix eigenvalue problems, there are two local group constraints which can engender local integrable modified Kortweweg-de Vries (mKdV) equations, but only one group constraint which can engender local integrable nonlinear Schrödinger (NLS) equations [1, 2].

Taking one non-local group constraint, one can present three kinds of non-local integrale NLS equations and two kinds of non-local integrable mKdV equations [3–6]. Recently, conducting two simultaneous group constraints, consisting of one local constraint and one non-local constraint, leads to other kinds of reduced non-local integrable mKdV equations (see, e.g., [7, 8]). Cauchy problems of non-local integrable equations can be solved by the

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inverse scattering transform (see, e.g., [9, 10]) and the existence of infinitely many symmetries can be showed by considering Lax operator algebras [11, 12]. Soliton solutions can be systematically presented through Darboux transformations [13, 14]), the Hirota bilinear method [15–17] and the Riemann-Hilbert technique [4, 18–22]. In this paper, we would like to we take two group constraints into consideration to construct reduced non-local integrable NLS hierarchies.

The other sections of the paper is organized as follows. In the next section, we revisit the general AKNS hierarchies of integrable equations for subsequent analysis. Then in another section, we conduct three pairs of local and non-local group constraints for the AKNS matrix eigenvalue problems and present three reduced non-local integrable NLS hierarchies. Three scalar novel prototypical examples are

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} - 2\delta(r_1 r_1^*(-x,t) + r_1^* r_1(-x,t))r_1], \tag{1}$$

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} - 2\delta(r_1 r_1(x,-t) + r_1^* r_1^*(x,-t))r_1], \tag{2}$$

and

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\delta(r_1 r_1(-x,-t) + r_1^* r_1^*(-x,-t))r_1], \tag{3}$$

where $\delta = \pm 1$, i is the unit imaginary number, α and β are arbitrary real constants and r_1^* denotes the complex conjugate of r_1. In the last section, we present a conclusion and a few concluding remarks.

The Matrix AKNS Integrable Equations Revisited

In order to facilitate the exposition, we revisit the AKNS hierarchies of matrix integrable equations and their associated matrix eigenvalue problems.

First, let λ stand for the eigenvalue parameter, and assume that r and s denote two matrix potentials:

$$r = r(x,t) = (r_{jk})_{p\times q},\ s = s(x,t) = (s_{kj})_{q\times p}, \tag{4}$$

where $p, q \geq 1$ are two arbitrarily given natural numbers.

It is known that the matrix AKNS eigenvalue problems read

$$\begin{cases} -i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda\Lambda + R)\phi, \\ -i\phi_t = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi = (\lambda^m\Omega + S^{[m]})\phi,\ m \geq 0. \end{cases} \tag{5}$$

In the above Lax pair of matrix eigenvalue problems, the $(p+q)$-th order square matrices, Λ and Ω, read

$$\Lambda = \mathrm{diag}(\alpha_1 I_p, \alpha_2 I_q),\ \Omega = \mathrm{diag}(\beta_1 I_p, \beta_2 I_q), \tag{6}$$

where I_n is the identity matrix of size n, and α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct real constants. The other two $(p+q)$-th order square matrices, R and $S^{[r]}$, read

$$R = R(u) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, \tag{7}$$

which is called the potential matrix, and

$$S^{[m]} = \sum_{n=0}^{m-1} \lambda^n \begin{bmatrix} a^{[m-n]} & b^{[m-n]} \\ c^{[m-n]} & d^{[m-n]} \end{bmatrix}, \tag{8}$$

where $a^{[n]}$, $b^{[n]}$, $c^{[n]}$ and $d^{[n]}$ are defined recursively through

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a^{[0]} = \beta_1 I_p,\ d^{[0]} = \beta_2 I_q, \tag{9a}$$

$$b^{[n+1]} = \frac{1}{\alpha}(-ib_x^{[n]} - rd^{[n]} + a^{[n]}r),\ n \geq 0, \tag{9b}$$

$$c^{[n+1]} = \frac{1}{\alpha}(ic_x^{[n]} + sa^{[n]} - d^{[n]}s),\ n \geq 0, \tag{9c}$$

$$a_x^{[n]} = i(rc^{[n]} - b^{[n]}s),\ d_x^{[n]} = i(sb^{[n]} - c^{[n]}r),\ n \geq 1, \tag{9d}$$

with zero constants of integration being taken. Particularly, we can get

$$S^{[1]} = \frac{\beta}{\alpha}R,\ S^{[2]} = \frac{\beta}{\alpha}\lambda R - \frac{\beta}{\alpha^2} I_{p,q}(R^2 + iR_x), \tag{10}$$

and

$$S^{[3]} = \frac{\beta}{\alpha}\lambda^2 R - \frac{\beta}{\alpha^2}\lambda I_{p,q}(R^2 + iR_x) - \frac{\beta}{\alpha^3}(i[R, R_x] + R_{xx} + 2R^3), \tag{11}$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{p,q} = \text{diag}(I_p, -I_q)$. It also follows from the recursive relations in (9) that

$$W = \sum_{n \geq 0} \lambda^{-n} \begin{bmatrix} a^{[n]} & b^{[n]} \\ c^{[n]} & d^{[n]} \end{bmatrix} \tag{12}$$

gives a Laurent series solution to the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{13}$$

Then, it is direct to see that for each pair of $p, q \geq 1$, the compatibility conditions of the two matrix eigenvalue problems in (5), i.e., the zero curvature equations:

$$U_t - V_x^{[m]} + i[U, V^{[m]}] = 0,\ m \geq 0, \tag{14}$$

present one matrix AKNS integrable hierarchy (see, e.g., [22] for more details):

$$r_t = i\alpha b^{[m+1]},\ s_t = -i\alpha c^{[m+1]},\ m \geq 0. \tag{15}$$

Each member in the hierarchy can be showed to possess a bi-Hamiltonian structure and infinitely many symmetries and conservation laws. The first nonlinear (i.e., $m = 2$) integrable system in (15) gives rise to the AKNS matrix NLS equations:

$$r_t = -\frac{\beta}{\alpha^2} i(r_{xx} + 2rsr),\ s_t = \frac{\beta}{\alpha^2} i(s_{xx} + 2srs), \tag{16}$$

where r and s are the two matrix potentials defined by (4).

Reduced Non-Local Integrable NLS Hierarchies

We would like to generate reduced non-local integrable NLS hierarchies by conducting three pairs of local and non-local group constraints for the matrix AKNS eigenvalue problems in (5) (see also [23] for the local case). In each pair of group constraints, one constraint is local while the other is non-local. Moreover, we assume that $A^\dagger$ and A^T denote the Hermitian transpose and the matrix transpose of a matrix A, respectively.

Type $(\lambda^*, -\lambda^*)$ Reduced NLS Hierarchies

Let Σ_1, Σ_2 and Δ_1, Δ_2 be two pairs of constant invertible Hermitian matrices of sizes p and q, respectively.

We propose the first pair of local and non-local group constraints for the spectral matrix U:

$$U^\dagger(x,t,\lambda^*) = (U(x,t,\lambda^*))^\dagger = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{17}$$

and

$$U^\dagger(-x,t,-\lambda^*) = (U(-x,t,-\lambda^*))^\dagger = -\Delta U(x,t,\lambda)\Delta^{-1}, \tag{18}$$

where the two constant invertible matrices, Σ and Δ, are defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},\ \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{19}$$

We will see that these two constraints are crucial in generating reduced integrable NLS equations.

Evidently, to satisfy the above two group constraints, we need to impose the two equivalent constraints on the potential matrix R:

$$R^\dagger(x,t) = \Sigma R(x,t)\Sigma^{-1}, \tag{20}$$

and

$$R^\dagger(-x,t) = -\Delta R(x,t)\Delta^{-1}, \tag{21}$$

respectively. Furthermore, these requirements generate the following constraints on the matrix potentials r and s:

$$s(x,t) = \Sigma_2^{-1} r^\dagger(x,t)\Sigma_1, \tag{22}$$

and

$$s(x,t) = -\Delta_2^{-1} r^\dagger(-x,t)\Delta_1, \tag{23}$$

respectively. Therefore, it follows that the matrix potential r must satisfy an additional constraint:

$$\Sigma_2^{-1} r^\dagger(x,t)\Sigma_1 = -\Delta_2^{-1} r^\dagger(-x,t)\Delta_1, \tag{24}$$

or the matrix potential s must satisfy a similar additional constraint:

$$\Sigma_1^{-1} s^\dagger(x,t)\Sigma_2 = -\Delta_1^{-1} s^\dagger(-x,t)\Delta_2, \tag{25}$$

to ensure that both group constraint requirements in (17) and (18) are met.

Moreover, under the pair of group constraints in (17) and (18), we can show that

$$\begin{cases} W^{\dagger}(x,t,\lambda^*) = (W(x,t,\lambda^*))^{\dagger} = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^{\dagger}(-x,t,-\lambda^*) = (W(-x,t,-\lambda^*))^{\dagger} = \Delta W(x,t,\lambda)\Delta^{-1}. \end{cases} \tag{26}$$

This guarantees that

$$\begin{cases} V^{[2n]\dagger}(x,t,\lambda^*) = (V^{[2n]}(x,t,\lambda^*))^{\dagger} = \Sigma V^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2n]\dagger}(-x,t,-\lambda^*) = (V^{[2n]}(-x,t,-\lambda^*))^{\dagger} = \Delta V^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{27}$$

and

$$\begin{cases} R^{[2n]\dagger}(x,t,\lambda^*) = (R^{[2n]}(x,t,\lambda^*))^{\dagger} = \Sigma R^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ R^{[2n]\dagger}(-x,t,-\lambda^*) = (R^{[2n]}(-x,t,-\lambda^*))^{\dagger} = \Delta R^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{28}$$

where $n \geq 0$.

Now, based on those explored properties on the Lax pairs, we know that under the pair of potential constraints (22) and (23), the integrable matrix AKNS equations in (15) with $m = 2n,\ n \geq 0$, are reduced to a hierarchy of non-local integrable NLS type equations:

$$r_t = i\alpha b^{[2n+1]}|_{s=\Sigma_2^{-1}r^{\dagger}\Sigma_1 = -\Delta_2^{-1}r^{\dagger}(-x,t)\Delta_1},\ n \geq 0, \tag{29}$$

where r is a $p \times q$ reduced matrix potential which satisfies (24), Σ_1, Σ_2 and Δ_1, Δ_2 are two pairs of arbitrary invertible Hermitian matrices of sizes p and q, respectively. Furthermore, each member in the reduced hierarchy (29) possesses a Lax pair consisting of the reduced spatial and temporal matrix eigenvalue problems in (5) with $m = 2n,\ n \geq 0$, and infinitely many symmetries and conservation laws that are reduced from those for the integrable matrix AKNS equations in (15) with $m = 2n,\ n \geq 0$.

If we set $n = 1$, i.e., $m = 2$, then the reduced non-local reverse-space integrable equations in (29) with $n = 1$ become a kind of reduced non-local reverse-space integrable NLS equations:

$$r_t = -\frac{\beta}{\alpha^2} i(r_{xx} + 2r\Sigma_2^{-1}r^{\dagger}\Sigma_1 r) = -\frac{\beta}{\alpha^2} i(r_{xx} - 2r\Delta_2^{-1}r^{\dagger}(-x,t)\Delta_1 r), \tag{30}$$

where r is a $p \times q$ reduced matrix potential satisfying (24).

Let us now compute a few examples corresponding to different values for p, q and appropriate choices for Σ, Δ, to illustrate those novel reduced non-local reverse-space integrable NLS equations. We recall $I_2 = \text{diag}(1, 1)$ and assume below that

$$\Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{31}$$

First, we consider the case of $p = 1$ and $q = 2$. Let us take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \sigma I_2,\ \Delta_1 = 1,\ \Delta_2^{-1} = \delta \Pi_2, \tag{32}$$

where σ and δ are the plus or minus sign, i.e., two real constants satisfying $\sigma^2 = \delta^2 = 1$. Obviously, the potential constraint (24) becomes

$$r_2 = -\sigma\delta r_1(-x,t), \tag{33}$$

where $r = (r_1, r_2)$. In this way, the corresponding reduced potential matrix R reads

$$R = \begin{bmatrix} 0 & r_1 & -\sigma\delta r_1(-x,t). \\ \sigma r_1^* & 0 & 0 \\ -\delta r_1^*(-x,t) & 0 & 0 \end{bmatrix}, \tag{34}$$

and the corresponding reduced non-local reverse-space integrable NLS equations in (30) become

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\sigma(r_1 r_1^* + r_1(-x,t) r_1^*(-x,t)) r_1], \tag{35}$$

where $\sigma = \pm 1$ and r_1^* denotes the complex conjugate of r_1. This pair of equations has been studied in [24]. A similar argument with

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \sigma \Pi_2,\ \Delta_1 = 1,\ \Delta_2^{-1} = \delta I_2, \tag{36}$$

where σ and δ are the plus or minus sign again, i.e., two real constants satisfying $\sigma^2 = \delta^2 = 1$, can lead to the reduced potential matrix R:

$$R = \begin{bmatrix} 0 & r_1 & -\sigma\delta r_1(-x,t) \\ -\delta r_1^*(-x,t) & 0 & 0 \\ \sigma r_1^* & 0 & 0 \end{bmatrix}, \tag{37}$$

and the non-local reverse-space integrable NLS equations:

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} - 2\delta(r_1 r_1^*(-x,t) + r_1^* r_1(-x,t)) r_1], \tag{38}$$

where $\delta = \pm 1$ and r_1^* denotes the complex conjugate of r_1 again. This pair of equations has only one non-local factor in the nonlinear terms and so its non-locality pattern is different from the one in (35).

Second, we consider the case of $p = 1$ and $q = 4$. Let us take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \mathrm{diag}(\sigma_1 I_2, \sigma_2 I_2),\ \Delta_1 = 1,\ \Delta_2^{-1} = \mathrm{diag}(\delta_1 \Pi_2, \delta_2 \Pi_2), \tag{39}$$

where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1,\ j = 1, 2$. The potential constraint (24) gives rise to

$$r_2 = -\sigma_1\delta_1 r_1(-x,t),\ r_4 = -\sigma_2\delta_2 r_3(-x,t), \tag{40}$$

where $r = (r_1, r_2, r_3, r_4)$. Therefore, the corresponding reduced potential matrix R becomes

$$R = \begin{bmatrix} 0 & r_1 & -\sigma_1\delta_1 r_1(-x,t) & r_3 & -\sigma_2\delta_2 r_3(-x,t) \\ \sigma_1 r_1^* & 0 & 0 & 0 & 0 \\ -\delta_1 r_1^*(-x,t) & 0 & 0 & 0 & 0 \\ \sigma_2 r_3^* & 0 & 0 & 0 & 0 \\ -\delta_2 r_3^*(-x,t) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

and we obtain a class of two-component non-local reverse-space integrable NLS equations:

$$\begin{cases} r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\sigma_1(r_1 r_1^* + r_1(-x,t) r_1^*(-x,t)) r_1 \\ \qquad +2\sigma_2(r_3 r_3^* + r_3(-x,t) r_3^*(-x,t)) r_1], \\ r_{3,t} = -\frac{\beta}{\alpha^2} i[r_{3,xx} + 2\sigma_1(r_1 r_1^* + r_1(-x,t) r_1^*(-x,t)) r_3 \\ \qquad +2\sigma_2(r_3 r_3^* + r_3(-x,t) r_3^*(-x,t)) r_3], \end{cases} \tag{42}$$

where σ_j are real constants satisfying $\sigma_j^2 = 1,\ j = 1, 2$. A similar argument with

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \mathrm{diag}(\sigma_1 \Pi_2, \sigma_2 \Pi_2),\ \Delta_1 = 1,\ \Delta_2^{-1} = \mathrm{diag}(\delta_1 I_2, \delta_2 I_2), \tag{43}$$

where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1,\ j = 1, 2$, can yield the reduced potential matrix R:

$$R = \begin{bmatrix} 0 & r_1 & -\sigma_1\delta_1 r_1(-x,t) & r_3 & -\sigma_2\delta_2 r_3(-x,t) \\ -\delta_1 r_1^*(-x,t) & 0 & 0 & 0 & 0 \\ \sigma_1 r_1^* & 0 & 0 & 0 & 0 \\ -\delta_2 r_3^*(-x,t) & 0 & 0 & 0 & 0 \\ \sigma_2 r_3^* & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{44}$$

and a class of two-component non-local reverse-space integrable NLS equations:

$$\begin{cases} r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} - 2\delta_1(r_1 r_1^*(-x,t) + r_1^* r_1(-x,t))r_1 \\ \qquad -2\delta_2(r_3 r_3^*(-x,t) + r_3^* r_3(-x,t))r_1], \\ r_{3,t} = -\frac{\beta}{\alpha^2} i[r_{3,xx} - 2\delta_1(r_1 r_1^*(-x,t) + r_1^* r_1(-x,t))r_3 \\ \qquad -2\delta_2(r_3 r_3^*(-x,t) + r_3^* r_3(-x,t))r_3], \end{cases} \tag{45}$$

where δ_j are real constants satisfying $\delta_j^2 = 1,\ j = 1, 2$. Again, this class of equations has only one non-local factor in the nonlinear terms and so the non-locality pattern in the equations is different from the one in (42).

Third, we consider a more general case of $p = 2$ and $q = 2$. Let us take

$$\Sigma_1 = \sigma_1 \Pi_2,\ \Sigma_2^{-1} = \sigma_2 I_2,\ \Delta_1 = \delta_1 \Pi_2,\ \Delta_2^{-1} = \delta_2 \Pi_2, \tag{46}$$

where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1$. Then, the potential constraint (24) gives

$$r_{12} = -\sigma_1\delta_1\sigma_2\delta_2 r_{11}(-x,t),\ r_{22} = -\sigma_1\delta_1\sigma_2\delta_2 r_{21}(-x,t). \tag{47}$$

In this way, the corresponding reduced matrix potentials become

$$r = \begin{bmatrix} r_{11} & -\sigma\delta r_{11}(-x,t) \\ r_{21} & -\sigma\delta r_{21}(-x,t) \end{bmatrix},\ s = \begin{bmatrix} \sigma r_{21}^* & \sigma r_{11}^* \\ -\delta r_{21}^*(-x,t) & -\delta r_{11}^*(-x,t) \end{bmatrix}, \tag{48}$$

where $\sigma = \sigma_1\sigma_2$ and $\delta = \delta_1\delta_2$, and we obtain a class of two-component non-local reverse-space integrable NLS equations:

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} + 2\sigma(r_{11} r_{21}^* + r_{11}(-x,t) r_{21}^*(-x,t))r_{11} \\ \qquad +2\sigma(r_{11} r_{11}^* + r_{11}(-x,t) r_{11}^*(-x,t))r_{21}], \\ r_{21,t} = -\frac{\beta}{\alpha^2} i[r_{21,xx} + 2\sigma(r_{21} r_{21}^* + r_{21}(-x,t) r_{21}^*(-x,t))r_{11} \\ \qquad +2\sigma(r_{11}^* r_{21} + r_{11}^*(-x,t) r_{21}(-x,t))r_{21}], \end{cases} \tag{49}$$

where $\sigma = \sigma_1\sigma_2 = \pm 1$.

Similarly, we can conduct three other pairs of local and non-local group constraints with

$$\Sigma_1 = \sigma_1 \Pi_2,\ \Sigma_2^{-1} = \sigma_2 \Pi_2,\ \Delta_1 = \delta_1 I_2,\ \Delta_2^{-1} = \delta_2 \Pi_2, \tag{50}$$

$$\Sigma_1 = \sigma_1 I_2,\ \Sigma_2^{-1} = \sigma_2 \Pi_2,\ \Delta_1 = \delta_1 \Pi_2,\ \Delta_2^{-1} = \delta_2 \Pi_2, \tag{51}$$

and

$$\Sigma_1 = \sigma_1 \Pi_2,\ \Sigma_2^{-1} = \sigma_2 \Pi_2,\ \Delta_1 = \delta_1 \Pi_2,\ \Delta_2^{-1} = \delta_2 I_2, \tag{52}$$

where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1$. The corresponding three pairs of reduced matrix potentials read

$$r = \begin{bmatrix} r_{11} & r_{12} \\ -\sigma\delta r_{11}(-x,t) & -\sigma\delta r_{12}(-x,t) \end{bmatrix}, \; s = \begin{bmatrix} -\delta r_{12}^*(-x,t) & \sigma r_{12}^* \\ -\delta r_{11}^*(-x,t) & \sigma r_{11}^* \end{bmatrix}, \tag{53}$$

$$r = \begin{bmatrix} r_{11} & r_{12} \\ -\sigma\delta r_{11}(-x,t) & -\sigma\delta r_{12}(-x,t) \end{bmatrix}, \; s = \begin{bmatrix} \sigma r_{12}^* & -\delta r_{12}^*(-x,t) \\ \sigma r_{11}^* & -\delta r_{11}^*(-x,t) \end{bmatrix}, \tag{54}$$

and

$$r = \begin{bmatrix} r_{11} & -\sigma\delta r_{11}(-x,t) \\ r_{21} & -\sigma\delta r_{21}(-x,t) \end{bmatrix}, \; s = \begin{bmatrix} -\delta r_{21}^*(-x,t) & -\delta r_{11}^*(-x,t) \\ \sigma r_{21}^* & \sigma r_{11}^* \end{bmatrix}, \tag{55}$$

respectively, where $\sigma = \sigma_1\sigma_2$ and $\delta = \delta_1\delta_2$. These enable us to obtain three classes of two-component non-local reverse-space integrable NLS equations:

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} - 2\delta(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))r_{11} \\ \qquad -2\delta(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{11}(-x,t)], \\ r_{12,t} = -\dfrac{\beta}{\alpha^2} i[r_{12,xx} - 2\delta(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))r_{12} \\ \qquad -2\delta(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{12}(-x,t)], \end{cases} \tag{56}$$

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} + 2\sigma(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{11} \\ \qquad +2\sigma(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))r_{11}(-x,t)], \\ r_{12,t} = -\dfrac{\beta}{\alpha^2} i[r_{12,xx} + 2\sigma(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{12} \\ \qquad +2\sigma(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))r_{12}(-x,t)], \end{cases} \tag{57}$$

and

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} - 2\delta(r_{11}r_{21}^*(-x,t) + r_{11}(-x,t)r_{21}^*)r_{11} \\ \qquad -2\delta(r_{11}r_{11}^*(-x,t) + r_{11}(-x,t)r_{11}^*)r_{21}], \\ r_{21,t} = -\dfrac{\beta}{\alpha^2} i[r_{21,xx} - 2\delta(r_{21}r_{21}^*(-x,t) + r_{21}(-x,t)r_{21}^*)r_{11} \\ \qquad -2\delta(r_{21}r_{11}^*(-x,t) + r_{21}(-x,t)r_{11}^*)r_{21}], \end{cases} \tag{58}$$

respectively, where $\delta = \delta_1\delta_2 = \pm 1$ and $\sigma = \sigma_1\sigma_2 = \pm 1$.

Obviously, the non-linearity patterns in these four examples, (49), (56), (57) and (58), are different from the ones in (42) and (45).

Type $(\lambda^*, -\lambda)$ Reduced NLS Hierarchies

Assume that Σ_1 and Σ_2 are one pair of constant invertible Hermitian matrices of sizes p and q, respectively, and Δ_1 and Δ_2 are another pair of constant invertible symmetric matrices of sizes p and q, respectively. Let us introduce the second pair of local and non-local group constraints for the spectral matrix U:

$$U^\dagger(x,t,\lambda^*) = (U(x,t,\lambda^*))^\dagger = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{59}$$

and

$$U^T(x,-t,-\lambda) = (U(x,-t,-\lambda))^T = -\Delta U(x,t,\lambda)\Delta^{-1}, \tag{60}$$

where the two constant invertible matrices, Σ and Δ, are defined by (19).

We will see soon that these two constraints are the key in constructing reduced integrable NLS equations.

It is easy to note that the above two group constraints equivalently require

$$R^\dagger(x,t) = \Sigma R(x,t)\Sigma^{-1}, \tag{61}$$

and

$$R^T(x,-t) = -\Delta R(x,t)\Delta^{-1}, \tag{62}$$

respectively. More precisely, they equivalently engender the following constraints on the matrix potentials r and s:

$$s(x,t) = \Sigma_2^{-1} r^\dagger(x,t)\Sigma_1, \tag{63}$$

and

$$s(x,t) = -\Delta_2^{-1} r^T(x,-t)\Delta_1, \tag{64}$$

respectively. It therefore follows that the matrix potential r must satisfy an additional constraint:

$$\Sigma_2^{-1} r^\dagger(x,t)\Sigma_1 = -\Delta_2^{-1} r^T(x,-t)\Delta_1, \tag{65}$$

or the matrix potential s must satisfy a similar additional constraint:

$$\Sigma_1^{-1} s^\dagger(x,t)\Sigma_2 = -\Delta_1^{-1} s^T(x,-t)\Delta_2, \tag{66}$$

to ensure that both group constraints in (59) and (60) are satisfied.

Furthermore, under the pair of group constraints in (59) and (60), we can prove that

$$\begin{cases} W^\dagger(x,t,\lambda^*) = (W(x,t,\lambda^*))^\dagger = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(x,-t,-\lambda) = (W(x,-t,-\lambda))^T = \Delta W(x,t,\lambda)\Delta^{-1}. \end{cases} \tag{67}$$

This ensures that

$$\begin{cases} V^{[2n]\dagger}(x,t,\lambda^*) = (V^{[2n]}(x,t,\lambda^*))^\dagger = \Sigma V^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2n]T}(x,-t,-\lambda) = (V^{[2n]}(x,-t,-\lambda))^T = \Delta V^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{68}$$

and

$$\begin{cases} S^{[2n]\dagger}(x,t,\lambda^*) = (S^{[2n]}(x,t,\lambda^*))^\dagger = \Sigma S^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ S^{[2n]T}(x,-t,-\lambda) = (R^{[2n]}(x,-t,-\lambda))^T = \Delta S^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{69}$$

where $n \ge 0$.

Following those explored properties of the Lax pairs, we know that under the pair of potential constraints (63) and (64), the integrable matrix AKNS equations in (15) with $m = 2n,\ n \ge 0$, are reduced to a hierarchy of non-local reverse-time integrable NLS type equations:

$$r_t = i\alpha b^{[2n+1]}|_{s=\Sigma_2^{-1} r^\dagger \Sigma_1 = -\Delta_2^{-1} r^T(x,-t)\Delta_1},\ n \ge 0, \tag{70}$$

where r is a $p \times q$ reduced matrix potential satisfying (24), Σ_1 and Σ_2 are a pair of arbitrary invertible Hermitian matrices of sizes p and q, respectively, and Δ_1 and Δ_2 are a pair of arbitrary invertible symmetric matrices of sizes p and q, respectively. Moreover, each reduced equation in the hierarchy (70) possesses a Lax pair consisting of the reduced spatial and temporal matrix eigenvalue problems in (5) with $m = 2n,\ n \geq 0$, and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (15) with $m = 2n,\ n \geq 0$.

If we set $n = 1$, i.e., $m = 2$, then the reduced non-local reverse-time integrable NLS type equations in (70) with $n = 1$ produce a kind of reduced non-local reverse-time integrable NLS equations:

$$r_t = -\frac{\beta}{\alpha^2} i(r_{xx} + 2r\Sigma_2^{-1} r^{\dagger} \Sigma_1 r) = -\frac{\beta}{\alpha^2} i(r_{xx} - 2r\Delta_2^{-1} r^T(x, -t)\Delta_1 r), \tag{71}$$

where r is a $p \times q$ reduced matrix potential satisfying (65).

We can similarly work out a few examples by taking different values for p, q and appropriate choices for Σ, Δ, to illustrate these novel reduced non-local reverse-time integrable NLS equations.

First, we consider the case of $p = 1$ and $q = 2$. The choices of (32) and (36) can produce the reduced potential matrix R:

$$R = \begin{bmatrix} 0 & r_1 & -\sigma\delta r_1^*(x, -t) \\ \sigma r_1^* & 0 & 0 \\ -\delta r_1(x, -t) & 0 & 0 \end{bmatrix}, \tag{72}$$

and

$$R = \begin{bmatrix} 0 & r_1 & -\sigma\delta r_1^*(x, -t) \\ -\delta r_1(x, -t) & 0 & 0 \\ \sigma r_1^* & 0 & 0 \end{bmatrix}, \tag{73}$$

respectively. The corresponding resulting non-local reverse-time integrable NLS equations read

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\sigma(r_1 r_1^* + r_1(x, -t) r_1^*(x, -t)) r_1], \tag{74}$$

and

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} - 2\delta(r_1 r_1(x, -t) + r_1^* r_1^*(x, -t)) r_1], \tag{75}$$

respectively, where $\sigma = \pm 1$, $\delta = \pm 1$, and r_1^* denotes the complex conjugate of r_1. The first pair of equations has been studied in [24], but the second pair of equations is new.

Second, we consider the case of $p = 1$ and $q = 4$. The choices of (39) and (43) can produce the reduced potential matrix R:

$$R = \begin{bmatrix} 0 & r_1 & -\sigma_1\delta_1 r_1^*(x, -t) & r_3 & -\sigma_2\delta_2 r_3^*(x, -t) \\ \sigma_1 r_1^* & 0 & 0 & 0 & 0 \\ -\delta_1 r_1(x, -t) & 0 & 0 & 0 & 0 \\ \sigma_2 r_3^* & 0 & 0 & 0 & 0 \\ -\delta_2 r_3(x, -t) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{76}$$

and

$$R=\begin{bmatrix} 0 & r_1 & -\sigma_1\delta_1 r_1^*(x,-t) & r_3 & -\sigma_2\delta_2 r_3^*(x,-t) \\ -\delta_1 r_1(x,-t) & 0 & 0 & 0 & 0 \\ \sigma_1 r_1^* & 0 & 0 & 0 & 0 \\ -\delta_2 r_3(x,-t) & 0 & 0 & 0 & 0 \\ \sigma_2 r_3^* & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{77}$$

respectively. The corresponding two classes of two-component non-local reverse-time integrable NLS equations read

$$\begin{cases} r_{1,t}=-\dfrac{\beta}{\alpha^2}i[r_{1,xx}+2\sigma_1(r_1r_1^*+r_1(x,-t)r_1^*(x,-t))r_1 \\ \qquad +2\sigma_2(r_3r_3^*+r_3(x,-t)r_3^*(x,-t))r_1], \\ r_{3,t}=-\dfrac{\beta}{\alpha^2}i[r_{3,xx}+2\sigma_1(r_1r_1^*+r_1(x,-t)r_1^*(x,-t))r_3 \\ \qquad +2\sigma_2(r_3r_3^*+r_3(x,-t)r_3^*(x,-t))r_3], \end{cases} \tag{78}$$

and

$$\begin{cases} r_{1,t}=-\dfrac{\beta}{\alpha^2}i[r_{1,xx}-2\delta_1(r_1r_1(x,-t)+r_1^*r_1^*(x,-t))r_1 \\ \qquad -2\delta_2(r_3r_3(x,-t)+r_3^*r_3^*(x,-t))r_1], \\ r_{3,t}=-\dfrac{\beta}{\alpha^2}i[r_{3,xx}-2\delta_1(r_1r_1(x,-t)+r_1^*r_1^*(x,-t))r_3 \\ \qquad -2\delta_2(r_3r_3(x,-t)+r_3^*r_3^*(x,-t))r_3], \end{cases} \tag{79}$$

respectively, where σ_j and δ_j are real constants satisfying $\sigma_j^2=\delta_j^2=1,\ j=1,2$.

Third, we consider a more general case of $p=2$ and $q=2$. Similar arguments with (46), (50), (51) and (52) can generate

$$r=\begin{bmatrix} r_{11} & -\sigma\delta r_{11}^*(x,-t) \\ r_{21} & -\sigma\delta r_{21}^*(x,-t) \end{bmatrix},\ s=\begin{bmatrix} \sigma r_{21}^* & \sigma r_{11}^* \\ -\delta r_{21}(x,-t) & -\delta r_{11}(x,-t) \end{bmatrix}, \tag{80}$$

$$r=\begin{bmatrix} r_{11} & r_{12} \\ -\sigma\delta r_{11}^*(x,-t) & -\sigma\delta r_{12}^*(x,-t) \end{bmatrix},\ s=\begin{bmatrix} -\delta r_{12}(x,-t) & \sigma r_{12}^* \\ -\delta r_{11}(x,-t) & \sigma r_{11}^* \end{bmatrix}, \tag{81}$$

$$r=\begin{bmatrix} r_{11} & r_{12} \\ -\sigma\delta r_{11}^*(x,-t) & -\sigma\delta r_{12}^*(x,-t) \end{bmatrix},\ s=\begin{bmatrix} \sigma r_{12}^* & -\delta r_{12}(x,-t) \\ \sigma r_{11}^* & -\delta r_{11}(x,-t) \end{bmatrix}, \tag{82}$$

and

$$r=\begin{bmatrix} r_{11} & -\sigma\delta r_{11}^*(x,-t) \\ r_{21} & -\sigma\delta r_{21}^*(x,-t) \end{bmatrix},\ s=\begin{bmatrix} -\delta r_{21}(x,-t) & -\delta r_{11}(x,-t) \\ \sigma r_{21}^* & \sigma r_{11}^* \end{bmatrix}, \tag{83}$$

respectively, where $\sigma=\sigma_1\sigma_2$ and $\delta=\delta_1\delta_2$. These enables us to obtain four classes of two-component non-local reverse-time integrable NLS equations:

$$\begin{cases} r_{11,t}=-\dfrac{\beta}{\alpha^2}i[r_{11,xx}+2\sigma(r_{11}r_{21}^*+r_{11}^*(x,-t)r_{21}(x,-t))r_{11} \\ \qquad +2\sigma(r_{11}r_{11}^*+r_{11}(x,-t)r_{11}^*(x,-t))r_{21}], \\ r_{21,t}=-\dfrac{\beta}{\alpha^2}i[r_{21,xx}+2\sigma(r_{21}r_{21}^*+r_{21}(x,-t)r_{21}^*(x,-t))r_{11} \\ \qquad +2\sigma(r_{11}^*r_{21}+r_{11}(x,-t)r_{21}^*(x,-t))r_{21}], \end{cases} \tag{84}$$

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} - 2\delta(r_{11}r_{12}(x,-t) + r_{12}r_{11}(x,-t))r_{11} \\ \qquad -2\delta(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{11}^*(x,-t)], \\ r_{12,t} = -\frac{\beta}{\alpha^2} i[r_{12,xx} - 2\delta(r_{11}r_{12}(x,-t) + r_{12}r_{11}(x,-t))r_{12} \\ \qquad -2\delta(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{12}^*(x,-t)], \end{cases} \tag{85}$$

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} + 2\sigma(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{11} \\ \qquad +2\sigma(r_{11}r_{12}(x,-t) + r_{12}r_{11}(x,-t))r_{11}^*(x,-t)], \\ r_{12,t} = -\frac{\beta}{\alpha^2} i[r_{12,xx} + 2\sigma(r_{11}r_{12}^* + r_{12}r_{11}^*)r_{12} \\ \qquad +2\sigma(r_{11}r_{12}(x,-t) + r_{12}r_{11}(x,-t))r_{12}^*(x,-t)], \end{cases} \tag{86}$$

and

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} - 2\delta(r_{11}r_{21}(x,-t) + r_{11}^*(x,-t)r_{21}^*)r_{11} \\ \qquad -2\delta(r_{11}r_{11}(x,-t) + r_{11}^*(x,-t)r_{11}^*)r_{21}], \\ r_{21,t} = -\frac{\beta}{\alpha^2} i[r_{21,xx} - 2\delta(r_{21}r_{21}(x,-t) + r_{21}^*(x,-t)r_{21}^*)r_{11} \\ \qquad -2\delta(r_{21}r_{11}(x,-t) + r_{21}^*(x,-t)r_{11}^*)r_{21}], \end{cases} \tag{87}$$

respectively, where $\sigma = \sigma_1\sigma_2 = \pm 1$ and $\delta = \delta_1\delta_2 = \pm 1$. Again, the non-linearity patterns in these four examples, (84), (85), (86) and (87), are different from the ones in (78) and (79).

Type (λ^*, λ) Reduced NLS Hierarchies

Similarly, assume that Σ_1 and Σ_2 are a pair of constant invertible Hermitian matrices of sizes p and q, respectively, and Δ_1 and Δ_2 are another pair of constant invertible symmetric matrices of sizes p and q, respectively. Let us propose the third pair of local and non-local group constraints for the spectral matrix U:

$$U^\dagger(x,t,\lambda^*) = (U(x,t,\lambda^*))^\dagger = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{88}$$

and

$$U^T(-x,-t,\lambda) = (U(-x,-t,\lambda))^T = \Delta U(x,t,\lambda)\Delta^{-1}, \tag{89}$$

where the two constant invertible matrices, Σ and Δ, are defined by (19).

We will see later that these two constraints play a crucial role in formulating reduced integrable NLS equations.

Equivalently, to satisfy the above two group constraints, we need to impose the two constraints on the potential matrix R:

$$R^\dagger(x,t) = \Sigma R(x,t)\Sigma^{-1}, \tag{90}$$

and

$$R^T(-x,-t) = \Delta R(x,t)\Delta^{-1}, \tag{91}$$

respectively. More precisely, they lead equivalently to the following constraints on the matrix potentials r and s:

$$s(x,t) = \Sigma_2^{-1} r^\dagger(x,t)\Sigma_1, \tag{92}$$

and

$$s(x,t)=\Delta_2^{-1}r^T(-x,-t)\Delta_1, \tag{93}$$

respectively. It therefore follows that the matrix potential r must satisfy an additional constraint:

$$\Sigma_2^{-1}r^\dagger(x,t)\Sigma_1=\Delta_2^{-1}r^T(-x,-t)\Delta_1, \tag{94}$$

or the matrix potential s must satisfy a similar additional constraint:

$$\Sigma_1^{-1}s^\dagger(x,t)\Sigma_1=\Delta_1^{-1}s^T(-x,-t)\Delta_1, \tag{95}$$

to guarantee that both group constraint requirements in (88) and (89) are met.

Moreover, it is straightforward to show that under the group constraints in (88) and (89), we have

$$\begin{cases} W^\dagger(x,t,\lambda^*)=(W(x,t,\lambda^*))^\dagger=\Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(-x,-t,\lambda)=(W(-x,-t,\lambda))^T=\Delta W(x,t,\lambda)\Delta^{-1}. \end{cases} \tag{96}$$

This assures us that

$$\begin{cases} V^{[2n]\dagger}(x,t,\lambda^*)=(V^{[2n]}(x,t,\lambda^*))^\dagger=\Sigma V^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2n]T}(-x,-t,\lambda)=(V^{[2n]}(-x,-t,\lambda))^T=\Delta V^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{97}$$

and

$$\begin{cases} S^{[2n]\dagger}(x,t,\lambda^*)=(S^{[2n]}(x,t,\lambda^*))^\dagger=\Sigma S^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ S^{[2n]T}(-x,-t,\lambda)=(S^{[2n]}(-x,-t,\lambda))^T=\Delta S^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{98}$$

where $n\geq 0$.

Accordingly, based on the explored properties of the Lax pairs, we see that the pair of potential constraints (92) and (93) reduces the integrable matrix AKNS equations in (15) with $m=2n,\ n\geq 0$, to a hierarchy of non-local reverse-spacetime integrable NLS type equations:

$$r_t=i\alpha b^{[2n+1]}|_{s=\Sigma_2^{-1}r^\dagger\Sigma_1=\Delta_2^{-1}r^T(-x,-t)\Delta_1},\ n\geq 0, \tag{99}$$

where r is a $p\times q$ reduced matrix potential satisfying (24), Σ_1 and Σ_2 are a pair of arbitrary invertible Hermitian matrices of sizes p and q, respectively, and Δ_1 and Δ_2 are a pair of arbitrary invertible symmetric matrices of sizes p and q, respectively. As consequences of the two group constraints, each reduced equation in the hierarchy (99) possesses a Lax pair consisting of the reduced spatial and temporal matrix eigenvalue problems in (5) with $m=2n,\ n\geq 0$, and infinitely many symmetries and conservation laws which are reduced from those for the integrable matrix AKNS equations in (15) with $m=2n,\ n\geq 0$.

If we set $n=1$, i.e., $m=2$, then the reduced non-local reverse-spacetime integrable NLS type equations in (99) with $n=1$ produce a kind of reduced non-local reverse-spacetime integrable NLS equations:

$$r_t=-\frac{\beta}{\alpha^2}i(r_{xx}+2r\Sigma_2^{-1}r^\dagger\Sigma_1 r)=-\frac{\beta}{\alpha^2}i(r_{xx}+2r\Delta_2^{-1}r^T(-x,-t)\Delta_1 r), \tag{100}$$

where r is a $p\times q$ reduced matrix potential satisfying (94).

Let us now present a few examples with different values for p,q and appropriate choices for Σ,Δ, to illustrate these reduced non-local reverse-spacetime integrable NLS equations.

First, we consider the case of $p = 1$ and $q = 2$. The choices of (32) and (36) can yield the reduced potential matrix R:

$$R = \begin{bmatrix} 0 & r_1 & \sigma\delta r_1^*(-x,-t) \\ \sigma r_1^* & 0 & 0 \\ \delta r_1(-x,-t) & 0 & 0 \end{bmatrix}, \tag{101}$$

and

$$R = \begin{bmatrix} 0 & r_1 & \sigma\delta r_1^*(-x,-t) \\ \delta r_1(-x,-t) & 0 & 0 \\ \sigma r_1^* & 0 & 0 \end{bmatrix}, \tag{102}$$

respectively. Furthermore, the corresponding resulting non-local reverse-spacetime integrable NLS equations read

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\sigma(r_1 r_1^* + r_1(-x,-t) r_1^*(-x,-t)) r_1], \tag{103}$$

and

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\delta(r_1 r_1(-x,-t) + r_1^* r_1^*(-x,-t)) r_1], \tag{104}$$

respectively, where $\sigma = \pm 1$, $\delta = \pm 1$ and r_1^* denotes the complex conjugate of r_1. The first pair of equations has been studied in [24], but the second pair is new.

Second, we consider the case of $p = 1$ and $q = 4$. The choices of (39) and (43) can generate the reduced potential matrix R:

$$R = \begin{bmatrix} 0 & r_1 & \sigma_1\delta_1 r_1^*(-x,-t) & r_3 & \sigma_2\delta_2 r_3^*(-x,-t) \\ \sigma_1 r_1^* & 0 & 0 & 0 & 0 \\ \delta_1 r_1(-x,-t) & 0 & 0 & 0 & 0 \\ \sigma_2 r_3^* & 0 & 0 & 0 & 0 \\ \delta_2 r_3(-x,-t) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{105}$$

and

$$R = \begin{bmatrix} 0 & r_1 & \sigma_1\delta_1 r_1^*(-x,-t) & r_3 & \sigma_2\delta_2 r_3^*(-x,-t) \\ \delta_1 r_1(-x,-t) & 0 & 0 & 0 & 0 \\ \sigma_1 r_1^* & 0 & 0 & 0 & 0 \\ \delta_2 r_3(-x,-t) & 0 & 0 & 0 & 0 \\ \sigma_2 r_3^* & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{106}$$

respectively. The corresponding two classes of two-component non-local reverse-spacetime integrable NLS equations are

$$\begin{cases} r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\sigma_1(r_1 r_1^* + r_1(-x,-t) r_1^*(-x,-t)) r_1 \\ \qquad + 2\sigma_2(r_3 r_3^* + r_3(-x,-t) r_3^*(-x,-t)) r_1], \\ r_{3,t} = -\frac{\beta}{\alpha^2} i[r_{3,xx} + 2\sigma_1(r_1 r_1^* + r_1(-x,-t) r_1^*(-x,-t)) r_3 \\ \qquad + 2\sigma_2(r_3 r_3^* + r_3(-x,-t) r_3^*(-x,-t)) r_3], \end{cases} \tag{107}$$

and

$$\begin{cases} r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\delta_1(r_1 r_1(-x,-t) + r_1^* r_1^*(-x,-t))r_1 \\ \qquad +2\delta_2(r_3 r_3(-x,-t) + r_3^* r_3^*(-x,-t))r_1], \\ r_{3,t} = -\frac{\beta}{\alpha^2} i[r_{3,xx} + 2\delta_1(r_1 r_1(-x,-t) + r_1^* r_1^*(-x,-t))r_3 \\ \qquad +2\sigma_2(r_3 r_3(-x,-t) + r_3^* r_3^*(-x,-t))r_3], \end{cases} \tag{108}$$

respectively, where and σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1,\ j = 1, 2$.

Third, we consider a more general case of $p = 2$ and $q = 2$. Similar deductions associated with (46), (50), (51) and (52) can produce the pairs of reduced matrix potentials:

$$r = \begin{bmatrix} r_{11} & \sigma\delta r_{11}^*(-x,-t) \\ r_{21} & \sigma\delta r_{21}^*(-x,-t) \end{bmatrix},\ s = \begin{bmatrix} \sigma r_{21}^* & \sigma r_{11}^* \\ \delta r_{21}(-x,-t) & \delta r_{11}(-x,-t) \end{bmatrix}, \tag{109}$$

$$r = \begin{bmatrix} r_{11} & r_{12} \\ \sigma\delta r_{11}^*(-x,-t) & \sigma\delta r_{12}^*(-x,-t) \end{bmatrix},\ s = \begin{bmatrix} \delta r_{12}(-x,-t) & \sigma r_{12}^* \\ \delta r_{11}(-x,-t) & \sigma r_{11}^* \end{bmatrix}, \tag{110}$$

$$r = \begin{bmatrix} r_{11} & r_{12} \\ \sigma\delta r_{11}^*(-x,-t) & \sigma\delta r_{12}^*(-x,-t) \end{bmatrix},\ s = \begin{bmatrix} \sigma r_{12}^* & \delta r_{12}(-x,-t) \\ \sigma r_{11}^* & \delta r_{11}(-x,-t) \end{bmatrix}, \tag{111}$$

and

$$r = \begin{bmatrix} r_{11} & \sigma\delta r_{11}^*(-x,-t) \\ r_{21} & \sigma\delta r_{21}^*(-x,-t) \end{bmatrix},\ s = \begin{bmatrix} \delta r_{21}(-x,-t) & \delta r_{11}(-x,-t) \\ \sigma r_{21}^* & \sigma r_{11}^* \end{bmatrix}, \tag{112}$$

respectively, where $\sigma = \sigma_1\sigma_2$ and $\delta = \delta_1\delta_2$. These formulations enable us to obtain the following four classes of two-component non-local reverse-spacetime integrable NLS equations:

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} + 2\sigma(r_{11} r_{21}^* + r_{11}^*(-x,-t) r_{21}(-x,-t))r_{11} \\ \qquad +2\sigma(r_{11} r_{11}^* + r_{11}(-x,-t) r_{11}^*(-x,-t))r_{21}], \\ r_{21,t} = -\frac{\beta}{\alpha^2} i[r_{21,xx} + 2\sigma(r_{21} r_{21}^* + r_{21}(-x,-t) r_{21}^*(-x,-t))r_{11} \\ \qquad +2\sigma(r_{11}^* r_{21} + r_{11}(-x,-t) r_{21}^*(-x,-t))r_{21}], \end{cases} \tag{113}$$

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} + 2\delta(r_{11} r_{12}(-x,-t) + r_{12} r_{11}(-x,-t))r_{11} \\ \qquad +2\delta(r_{11} r_{12}^* + r_{12} r_{11}^*) r_{11}^*(-x,-t)], \\ r_{12,t} = -\frac{\beta}{\alpha^2} i[r_{12,xx} + 2\delta(r_{11} r_{12}(-x,-t) + r_{12} r_{11}(-x,-t))r_{12} \\ \qquad +2\delta(r_{11} r_{12}^* + r_{12} r_{11}^*) r_{12}^*(-x,-t)], \end{cases} \tag{114}$$

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} + 2\sigma(r_{11} r_{12}^* + r_{12} r_{11}^*) r_{11} \\ \qquad +2\sigma(r_{11} r_{12}(-x,-t) + r_{12} r_{11}(-x,-t)) r_{11}^*(-x,-t)], \\ r_{12,t} = -\frac{\beta}{\alpha^2} i[r_{12,xx} + 2\sigma(r_{11} r_{12}^* + r_{12} r_{11}^*) r_{12} \\ \qquad +2\sigma(r_{11} r_{12}(-x,-t) + r_{12} r_{11}(-x,-t)) r_{12}^*(-x,-t)], \end{cases} \tag{115}$$

and

$$\begin{cases} r_{11,t} = -\frac{\beta}{\alpha^2} i[r_{11,xx} + 2\delta(r_{11}r_{21}(-x,-t) + r_{11}^*(-x,-t)r_{21}^*)r_{11} \\ \qquad +2\delta(r_{11}r_{11}(-x,-t) + r_{11}^*(-x,-t)r_{11}^*)r_{21}], \\ r_{21,t} = -\frac{\beta}{\alpha^2} i[r_{21,xx} + 2\delta(r_{21}r_{21}(-x,-t) + r_{21}^*(-x,-t)r_{21}^*)r_{11} \\ \qquad +2\delta(r_{21}r_{11}(-x,-t) + r_{21}^*(-x,-t)r_{11}^*)r_{21}], \end{cases} \tag{116}$$

respectively, where $\sigma = \sigma_1\sigma_2 = \pm 1$ and $\delta = \delta_1\delta_2 = \pm 1$. Obviously, the non-linearity patterns in these four examples, (113), (114), (115) and (116), are different from the ones in (107) and (108).

Concluding Remarks

Two simultaneous group constraints of the AKNS matrix eigenvalue problems, of which one is local while the other is non-local, were introduced and preformed. All reduced non-local integrable equations are classified into type $(\lambda^*, -\lambda^*)$, type $(\lambda^*, -\lambda)$ and type (λ^*, λ) reduced non-local integrable NLS hierarchies. All obtained non-local integrable equations are PT-symmetric and the non-locality types involved include reverse-space, reverse-time and reverse-spacetime.

We remark that we can generate many other examples of two-component non-local integrable NLS equations, if other choices are taken for Σ and Δ in the two group constraints. It is particularly important to construct soliton solutions to such novel non-local integrable equations by the Darboux transformation and the Riemann-Hilbert technique. In addition, it is of great interest to explore other kinds of reduced non-local integrable equations associated with other classes of matrix eigenvalue problems (see, e.g., [25]), and or generated from two non-local group constraints (see, e.g., [26]).

Further related studies in this realm will provide good supplements to the theory of non-local integrable equations.

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Declarations

Conflicts of interest The author declares that there is no conflict of interest.

References

1. Ma, W.X.: Application of the Riemann-Hilbert approach to the multicomponent AKNS integrable hierarchies. Nonlinear Anal. Real World Appl. **47**, 1–17 (2019)
2. Ma, W.X.: A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations. Mod. Phys. Lett. B **36**, to appear (2022). https://doi.org/10.1142/S0217984922500944
3. Ablowitz, M.J., Musslimani, Z.H.: Integrable nonlocal nonlinear equations. Stud. Appl. Math. **139**, 7–59 (2017)

4. Ma, W.X.: Nonlocal PT-symmetric integrable equations and related Riemann-Hilbert problems. Partial Differ. Equ. Appl. Math. **4**, 100190 (2021)
5. Ma, W.X.: Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations. Appl. Math. Lett. **102**, 106161 (2020)
6. Ma, W.X., Huang, Y.H., Wang, F.D.: Inverse scattering transforms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies. Stud. Appl. Math. **145**, 563–585 (2020)
7. Ma, W.X.: Riemann-Hilbert problems and soliton solutions of type $(\lambda^*, -\lambda^*)$ reduced nonlocal integrable mKdV hierarchies. Math. **10**, 870 (2022)
8. Ma, W.X.: Type $(-\lambda, -\lambda^*)$ reduced nonlocal integrable mKdV equations and their soliton solutions. Appl. Math. Lett. **131**, 108074 (2022)
9. Ablowitz, M.J., Musslimani, Z.H.: Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. Nonlinearity **29**, 915–946 (2016)
10. Ling, L.M., Ma, W.X.: Inverse scattering and soliton solutions of nonlocal complex reverse-spacetime modified Korteweg-de Vries hierarchies. Symmetry **13**, 512 (2021)
11. Ma, W.X.: The algebraic structure of zero curvature representations and application to coupled KdV systems. J. Phys. A Math. Gen. **26**, 2573–2582 (1993)
12. Ma, W.X.: Integrable nonlocal PT-symmetric modified Korteweg-de Vries equations associated with so(3,$\mathbb{R}$). Symmetry **13**, 2205 (2021)
13. Ji, J.L., Zhu, Z.N.: On a nonlocal modified Korteweg-de Vries equation: Integrability, Darboux transformation and soliton solutions. Commun. Nonlinear Sci. Numer. Simul. **42**, 699–708 (2017)
14. Song, C.Q., Xiao, D.M., Zhu, Z.N.: Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation. Commun. Nonlinear Sci. Numer. Simul. **45**, 13–28 (2017)
15. Gürses, M., Pekcan, A.: Nonlocal nonlinear Schrödinger equations and their soliton solutions. J. Math. Phys. **59**, 051501 (2018)
16. Feng, B.F., Luo, X.D., Ablowitz, M.J., Musslimani, Z.H.: General soliton solution to a nonlocal nonlinear Schrödinger equation with zero and nonzero boundary conditions. Nonlinearity **31**, 5385–5409 (2018)
17. Gürses, M., Pekcan, A.: Nonlocal modified KdV equations and their soliton solutions by Hirota method. Commun. Nonlinear Sci. Numer. Simul. **67**, 427–448 (2019)
18. Yang, J.: General N-solitons and their dynamics in several nonlocal nonlinear Schrödinger equations. Phys. Lett. A **383**, 328–337 (2019)
19. Ma, W.X.: Inverse scattering and soliton solutions of nonlocal reverse-spacetime nonlinear Schrödinger equations. Proc. Amer. Math. Soc. **149**, 251–263 (2021)
20. Adjiri, A., Ahmed, A., Ma, W.X.: Riemann-Hilbert problems of a nonlocal reverse-time six-component AKNS system of fourth order and its exact soliton solutions. Int. J. Mod. Phys. B **35**, 2150035 (2022)
21. Ma, W.X.: Riemann-Hilbert problems and soliton solutions of nonlocal reverse-time NLS hierarchies. Acta Math. Sci. **42B**, 127–140 (2022)
22. Ma, W.X.: Riemann-Hilbert problems and inverse scattering of nonlocal real reverse-spacetime matrix AKNS hierarchies. Physica D **430**, 133078 (2022)
23. Ma, W.X.: Riemann-Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction. Math. Meth. Appl. Sci. **42**, 1099–1113 (2019)
24. Yang, J.: Phyiscally significant nonlocal nonlinear Schrödinger equation and its soliton solutions. Phys. Rev. E **98**, 042202 (2018)
25. Ma, W.X.: Integrable nonlocal nonlinear Schrödinger equations associated with so(3,$\mathbb{R}$). Proc. Amer. Math. Soc. Ser. B **9**, 1–11 (2022)
26. Ma, W.X.: Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions. J. Geom. Phys. **177**, 104522 (2022)