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Real Reduced Matrix mKdV Integrable Hierarchies Under Two Local Group Reductions

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Abstract. We propose a kind of reduced Ablowitz-Kaup-Newell-Segur matrix spectral problems under two local group reductions by similarity transformations. Associated integrable hierarchies of matrix mKdV type integrable models are presented, which amend the complex matrix mKdV integrable hierarchies. Zero curvature equations are key objects in generating integrable models.

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1. Introduction

The zero curvature formulation provides a systematical scheme to generate integrable models [6]. The key is to choose a matrix spectral problem and then an associated hierarchy of integrable models can be computed via zero curvature equations. The inverse scattering transform exactly uses the matrix spectral problem to solve Cauchy problems of integrable models, the evolution of the scattering data being determined by the associated temporal matrix spectral problems [4].

Matrix spectral problems with free potentials are standard and natural. But reduced matrix spectral problems are more restrictive and harder to apply. A idea of using similarity transformations is adopted for formulating reduced matrix spectral problems, which lead to integrable hierarchies (see, e.g., [14]). The aim of using similarity transformations is to make it easier to achieve to keep the corresponding zero curvature equations invariant

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and so generate integrable models. Two such typical kinds of integrable models are the nonlinear Schrödinger equations and the modified Korteweg-de Vries equation. Both of them are generated from the Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems by taking one similarity transformation. Moreover, taking a pair of similarity transformations can engender more diverse integrable models. Some difficulty might be involved, since two reductions on potentials, corresponding to the pair of similarity transformations, bring new requirements on balancing associated zero curvature equations.

Very recently, the idea of taking similarity transformations has also been applied to construction of nonlocal integrable models [3]. Three kinds of reduced integrable nonlinear Schrödinger type equations, and two kinds of reduced integrable modified Kortweweg-de Vries type equations have been proposed and classified [16]. The inverse scattering transform has also been developed to solve nonlocal integrable models (see, e.g., [2,10,15,22]). There are other efficient approaches which attempt nonlocal integrable models, and particularly, construct soliton solutions. The Hirota bilinear method, Darboux transformation, Bäclund transforms and the Riemann-Hilbert technique have been proved to be powerful and many theories have been proposed for different reduced integrable models, both local and non-local (see, for example, [7–9,14,26,33]).

In this paper, we would like to propose a pair of local group reductions by similarity transformations for the AKNS matrix spectral problems to generate reduced integrable models. The rest of the paper is organized as follows. In Section 2, we recall the AKNS matrix spectral problems and their associated hierarchies of matrix integrable models to prepare the subsequent analyses. In Section 3, we consider two local group reductions by similarity transformations for the AKNS matrix spectral problems simultaneously and generate reduced local hierarchies of real matrix mKdV integrable models. In Section 4, we illustrate the presented formulation with concrete examples, which present abundant reduced AKNS matrix spectral problems and reduced corresponding matrix integrable models, including novel mKdV type integrable models. In the last section, we summarize the results and give some concluding remarks.

2. The Standard AKNS Matrix Integrable Hierarchies

Let $m,n \geq 1$ be two arbitrarily given natural numbers. For each pair of $m,n \geq 1$, we state the AKNS matrix spectral problems and the associated AKNS hierarchies of matrix integrable models, to facilitate the subsequent analyses.

First, we denote the spectral parameter by λ, and assume that p and q are two submatrix potentials

$$p = p(x,t) = (p_{jk})_{m\times n}, \quad q = q(x,t) = (q_{kj})_{n\times m}. \tag{2.1}$$

The standard matrix AKNS spectral problems reads

$$-i\phi_x = U\phi, \quad U = U(u,\lambda) = (\lambda\Lambda + P), \tag{2.2}$$

and

$$-i\phi_t = V^{[r]}\phi, \quad V^{[r]} = V^{[r]}(u,\lambda) = \left(\lambda^r\Omega + Q^{[r]}\right), \quad r \geq 0, \tag{2.3}$$

where $u=u(p,q)$ is the potential consisting of the two submatrix potentials p and q. In the above Lax pair of matrix spectral problems, the $(m+n)$-th order square matrices, Λ and Ω, are defined by

$$\Lambda=\mathrm{diag}(\alpha_1 I_m,\alpha_2 I_n),\quad \Omega=\mathrm{diag}(\beta_1 I_m,\beta_2 I_n),$$

where I_k is the identity matrix of size k, and α_1,α_2 and β_1,β_2 are two pairs of arbitrarily given distinct real constants, which will show the diversity of matrix spectral problems but do not have a serious effect on associated integrable models. The other two $(m+n)$-th order square matrices, P and $Q^{[r]}$, are given by

$$P=P(u)=\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix},$$

which is called the potential matrix, and

$$Q^{[r]}=\sum_{s=0}^{r-1}\lambda^s\begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.4}$$

with $a^{[s]},b^{[s]},c^{[s]}$ and $d^{[s]}$ being determined recursively via

$$b^{[0]}=0,\quad c^{[0]}=0,\quad a^{[0]}=\beta_1 I_m,\quad d^{[0]}=\beta_2 I_n, \tag{2.5}$$

and

$$\begin{cases} b^{[s+1]}=\dfrac{1}{\alpha}\left(-ib_x^{[s]}-pd^{[s]}+a^{[s]}p\right), \\ c^{[s+1]}=\dfrac{1}{\alpha}\left(ic_x^{[s]}+qa^{[s]}-d^{[s]}q\right), \\ a_x^{[s+1]}=i\left(pc^{[s+1]}-b^{[s+1]}q\right), \\ d_x^{[s+1]}=i\left(qb^{[s+1]}-c^{[s+1]}p\right), \end{cases} \quad s\geq 0, \tag{2.6}$$

where $\alpha=\alpha_1-\alpha_2$ and zero constants of integration are taken in computing $a^{[s]}$ and $d^{[s]}$. Obviously, we can work out

$$Q^{[1]}=\frac{\beta}{\alpha}P,\quad Q^{[2]}=\frac{\beta}{\alpha}\lambda P-\frac{\beta}{\alpha^2}I_{m,n}\left(P^2+iP_x\right),$$
$$Q^{[3]}=\frac{\beta}{\alpha}\lambda^2 P-\frac{\beta}{\alpha^2}\lambda I_{m,n}\left(P^2+iP_x\right)-\frac{\beta}{\alpha^3}\left(i[P,P_x]+P_{xx}+2P^3\right),$$

where $\beta=\beta_1-\beta_2$ and $I_{m,n}=\mathrm{diag}(I_m,-I_n)$. We can readily see from the recursive relations in (2.6) with (2.5) that

$$W=\sum_{s\geq 0}\lambda^{-s}W^{[s]}=\sum_{s\geq 0}\lambda^{-s}\begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{2.7}$$

provides a Laurent series solution to the stationary zero curvature equation

$$W_x=i[U,W],$$

where U is the spectral matrix in (2.2). Such a formal series solution is a crucial object to generate integrable hierarchies (see, e.g., [24,31,37] for examples).

Now, it directly follows that for each pair of $m,n\geq 1$, the compatibility conditions of the two matrix spectral problems in (2.2) and (2.3), which are the zero curvature equations

$$U_t - V_x^{[r]} + i\big[U, V^{[r]}\big] = 0, \quad r\geq 0,$$

determine one matrix AKNS integrable hierarchy

$$p_t = i\alpha b^{[r+1]}, \quad q_t = -i\alpha c^{[r+1]}, \quad r\geq 0. \tag{2.8}$$

The case of $m = n = 1$ gives rise to the typical AKNS integrable hierarchy with two scalar potentials [1]. By applying the trace identity [30] as in [17], each system in this AKNS matrix integrable hierarchy can be showed to possess a bi-Hamiltonian structure and infinitely many symmetries and conserved quantities (see, e.g., [12,32,38] for more examples).

It is easy to see that the first and second nonlinear (corresponding to $r = 2,3$) integrable models in (2.8) are the AKNS matrix nonlinear Schrödinger equations

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \quad q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq),$$

and the AKNS matrix modified Korteweg-de Vries equations

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp),\\ q_t &= -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x), \end{aligned}$$

where p and q are the two submatrix potentials given by (2.1). More examples of matrix AKNS integrable models could be found in [21].

3. Real Reduced Matrix mKdV Integrable Hierarchies

3.1. Reduced AKNS matrix spectral problems

Let Σ_1 and Σ_2 constant invertible symmetric matrices of orders m and n, respectively, and Δ_1 and Δ_2, constant invertible matrices of orders m and n, respectively. We make the two bigger invertible constant matrices of order $m+n$ of the form

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

For a given AKNS spectral matrix U in (2.2), we consider a pair of group reductions by similarity transformations

$$\Sigma U(\lambda)\Sigma^{-1} = -U^\top(-\lambda) = -\big(U(-\lambda)\big)^\top, \tag{3.1}$$

$$\Delta U(\lambda)\Delta^{-1} = U(\lambda). \tag{3.2}$$

In (3.1), $\top$ denotes the matrix transpose. These two reductions show the two simultaneous invariance properties (see also [18]).

Noting the specific form of the spectral matrix U, we can show that these two group reductions equivalently generate

$$\Sigma P \Sigma^{-1} = -P^{\top}, \quad \Delta P \Delta^{-1} = P,$$

respectively. Obviously, these require the following corresponding constraints for the two submatrix potentials p and q:

$$p = -\Sigma_1^{-1} q^{\top} \Sigma_2 \quad \text{or} \quad q = -\Sigma_2^{-1} p^{\top} \Sigma_1, \tag{3.3}$$

and

$$p = \Delta_1 p \Delta_2^{-1}, \quad q = \Delta_2 q \Delta_1^{-1}. \tag{3.4}$$

Consequently, from (3.3) and (3.4), the first submatrix potential p is required to satisfy

$$\Delta_1 p = p \Delta_2, \quad \Sigma_1^{-1} \Delta_1^{\top} \Sigma_1 p = p \Sigma_2^{-1} \Delta_2^{\top} \Sigma_2, \tag{3.5}$$

or the second submatrix potential q is required to satisfy

$$q \Delta_1 = \Delta_2 q, \quad q \Sigma_1^{-1} \Delta_1^{\top} \Sigma_1 = \Sigma_2^{-1} \Delta_2^{\top} \Sigma_2 q. \tag{3.6}$$

Therefore, under both group reductions in (3.1) and (3.2), we have a class of reduced AKNS matrix spectral problems

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ -\Sigma_2^{-1} p^{\top} \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \tag{3.7}$$

where the submatrix potential p is required to satisfy the constraints in (3.5), or equivalently,

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \alpha_1 \lambda I_m & -\Sigma_1^{-1} q^{\top} \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \tag{3.8}$$

where the submatrix potential q is required to satisfy the constraints in (3.6).

3.2. Real reduced matrix mKdV integrable hierarchies

Under the two group reductions in (3.1) and (3.2), we can prove that

$$\begin{aligned} &\Sigma W(\lambda) \Sigma^{-1} = W^{\top}(-\lambda) = \big(W(-\lambda)\big)^{\top}, \\ &\Delta W(\lambda) \Delta^{-1} = W(\lambda), \end{aligned}$$

where W is determined by (2.7). It follows from these invariance properties that for each $r \geq 0$, we have the following one pair of invariance properties:

$$\begin{aligned} &\Sigma V^{[2s+1]}(\lambda) \Sigma^{-1} = -V^{[2s+1]\top}(-\lambda) = -\big(V^{[2s+1]}(-\lambda)\big)^{\top}, \\ &\Delta V^{[2s+1]}(\lambda) \Delta^{-1} = V^{[2s+1]}(\lambda), \end{aligned}$$

which are equivalent to

$$\Sigma Q^{[2s+1]}(\lambda)\Sigma^{-1} - Q^{[2s+1]\top}(-\lambda) = -\big(Q^{[2s+1]}(-\lambda)\big)^\top,$$
$$\Delta Q^{2s+1]}(\lambda)\Delta^{-1} = Q^{[2s+1]}(\lambda),$$

where $s \geq 0$ and $V^{[2s+1]}$ and $Q^{[2s+1]}$ are defined in (2.3) and (2.4), respectively. Then, consequently, we see that under the potential constraints (3.3) and (3.4),

$$\begin{aligned}
&\Sigma\big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(\lambda)\Sigma^{-1} \\
&= -\big(U_t^\top + V_x^{[2s+1]\top} + i[U^T, V^{[2s+1]\top}]\big)(-\lambda), \\
&\Delta\big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(\lambda)\Delta^{-1} \\
&= \big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(\lambda),
\end{aligned}$$

where $s \geq 0$, and thus the matrix AKNS integrable models in (2.8) with $r = 2s+1$, $s \geq 0$, become a hierarchy of real reduced AKNS matrix mKdV integrable models

$$p_t = i\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^\top\Sigma_1}, \quad s \geq 0, \tag{3.9}$$

where the submatrix potential p is a reduced $m \times n$ matrix potential being subject to (3.5), or equivalently,

$$q_t = -i\alpha c^{[2s+2]}|_{p=-\Sigma_1^{-1}q^\top\Sigma_2}, \quad s \geq 0, \tag{3.10}$$

where the submatrix potential q is a reduced $n \times m$ matrix potential being subject to (3.6). Moreover, every member in the reduced hierarchy (3.9) or (3.10) possesses a Lax pair consisting of the reduced matrix spectral problems in (2.2) and (2.3) with $r = 2s+1$, $s \geq 0$, and has a hierarchy of commuting symmetries and conserved densities, which are reduced from those for the matrix integrable AKNS equations in (2.8) with $r = 2s+1$, $s \geq 0$. The matrix spectral problems (3.7) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^\top\Sigma_1}\phi, \quad s \geq 0, \tag{3.11}$$

constitute a Lax pair for the reduced hierarchy (3.9), or equivalently, the matrix spectral problems (3.8) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^\top\Sigma_2}\phi, \quad s \geq 0, \tag{3.12}$$

constitute a Lax pair for the reduced hierarchy (3.10).

Noting that Σ_1 and Σ_2 are arbitrary invertible constant symmetric matrices of orders m and n, respectively, and Δ_1 and Δ_2 are arbitrary invertible constant matrices of orders m and n, respectively, we can present abundant reduced hierarchies of matrix mKdV integrable models.

4. Illustrative Examples

In this section, we would like to compute a few examples to illustrate the preceding analyses. We will focus on two simple cases.

4.1. Case of $m = 1$ and $n = 2$

In the case of $m = 1$ and $n = 2$, we would like to present two examples. First, let us choose

$$\Sigma_1 = 1, \quad \Sigma_2 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix},$$

where δ and σ take on values of either 1 or −1. Then we obtain

$$p = (p_1, \delta p_1), \quad q = -\sigma p^\top = -\sigma(p_1, \delta p_1)^\top, \tag{4.1}$$

and the reduced matrix spectral problem becomes

$$-i\phi_x = U|_{q=-\Sigma_2^{-1}p^\top\Sigma_1}\phi = \begin{bmatrix} \alpha_1\lambda & p_1 & \delta p_1 \\ -\sigma p_1 & \alpha_2\lambda & 0 \\ -\sigma\delta p_1 & 0 & \alpha_2\lambda \end{bmatrix}\phi.$$

Upon taking those choices for p and q in (4.1), we see that the 3rd-order reduced integrable model presents exactly the mKdV equation

$$p_{1,t} = -\frac{\beta}{\alpha^3}\left(p_{1,xxx} - 12\sigma p_1^2 p_{1,x}\right).$$

Second, let us choose

$$\Sigma_1 = 1, \quad \Sigma_2 = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix},$$

where δ and σ take on values of either 1 or −1. Then, we arrive at

$$p = (p_1, \delta p_1), \quad q = -\sigma\delta p^\top = -\sigma(\delta p_1, p_1)^\top, \tag{4.2}$$

and the reduced matrix spectral problem becomes

$$-i\phi_x = U|_{q=-\Sigma_2^{-1}p^\top\Sigma_1}\phi = \begin{bmatrix} \alpha_1\lambda & p_1 & \delta p_1 \\ -\sigma\delta p_1 & \alpha_2\lambda & 0 \\ -\sigma p_1 & 0 & \alpha_2\lambda \end{bmatrix}\phi.$$

Now taking those choices for p and q in (4.2), we see that the 3rd-order reduced integrable model is precisely the mKdV equation

$$p_{1,t} = -\frac{\beta}{\alpha^3}\left(p_{1,xxx} - 12\sigma\delta p_1^2 p_{1,x}\right).$$

To sum up, we have shown that the mKdV equation possesses different 3×3 matrix Lax pairs, which provides supplements to the 2×2 matrix Lax pairs in the existing literature [4].

4.2. Case of $m=2$ and $n=2$

In the case of $m=n=2$, we would like to present several examples below.

Let us choose a general set of

$$\Sigma_1=\begin{bmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{bmatrix},\quad \Sigma_2=\begin{bmatrix} 0 & \sigma_3 \\ \sigma_4 & 0 \end{bmatrix},\quad \Delta_1=\begin{bmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{bmatrix},\quad \Delta_2=\begin{bmatrix} 0 & \delta_3 \\ \delta_4 & 0 \end{bmatrix},$$

where each of σ_i and δ_i takes on values of either 1 or -1 and their products are assumed to be 1

$$\delta_1\delta_2\delta_3\delta_4=1,\quad \sigma_1\sigma_2\sigma_3\sigma_4=1,$$

which comes from the two group reductions. Then, we get

$$p=\begin{bmatrix} p_1 & p_2 \\ \delta_2\delta_3 p_2 & \delta_2\delta_4 p_1 \end{bmatrix},\quad q=-\begin{bmatrix} \delta_2\delta_4\sigma_2\sigma_4 p_1 & \sigma_1\sigma_4 p_2 \\ \delta_2\delta_3\sigma_2\sigma_3 p_2 & \sigma_1\sigma_3 p_1 \end{bmatrix},$$

and so the reduced matrix spectral problem takes the form

$$-i\phi_x=\begin{bmatrix} \alpha_1\lambda & 0 & p_1 & p_2 \\ 0 & \alpha_1\lambda & \delta_2\delta_3 p_2 & \delta_2\delta_4 p_1 \\ -\delta_2\delta_4\sigma_2\sigma_4 p_1 & -\sigma_1\sigma_4 p_2 & \alpha_2\lambda & 0 \\ -\delta_2\delta_3\sigma_2\sigma_3 p_2 & -\sigma_1\sigma_3 p_1 & 0 & \alpha_2\lambda \end{bmatrix}\phi.$$

Particularly, if we firstly choose

$$\begin{aligned} &\delta_1=-\delta_2=\delta_3=-\delta_4=1, \\ &\sigma_1=\sigma_2,\ \ \sigma_3=\sigma_4,\ \ \sigma_1\sigma_3=\pm 1, \end{aligned}$$

then we have

$$p=\begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix},\quad q=\mp p=\mp\begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix}.$$

The two reduced coupled mKdV integrable models read

$$\begin{cases} p_{1,t}=-\dfrac{\beta}{\alpha^3}\left[p_{1,xxx}\mp 6\left(p_1^2-p_2^2\right)p_{1,x}\pm 12p_1p_2p_{2,x}\right], \\ p_{2,t}=-\dfrac{\beta}{\alpha^3}\left[p_{2,xxx}\mp 12p_1p_2p_{1,x}\pm 6\left(p_2^2-p_1^2\right)p_{2,x}\right]. \end{cases}$$

If we secondly choose

$$\begin{aligned} &\delta_1=-\delta_2=\delta_3=-\delta_4=1, \\ &\sigma_1=-\sigma_2,\ \ \sigma_3=-\sigma_4,\ \ \sigma_1\sigma_3=\pm 1, \end{aligned}$$

then we have

$$p=\begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix},\quad q=\mp p^T=\mp\begin{bmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{bmatrix}.$$

Such a reduction on p has also been discussed in [28,29]. The two reduced coupled mKdV integrable models read

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}\left[p_{1,xxx} \mp 6\left(p_1^2+p_2^2\right)p_{1,x}\right], \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}\left[p_{2,xxx} \mp 6\left(p_1^2+p_2^2\right)p_{2,x}\right]. \end{cases}$$

Finally, let us choose

$$\delta_1 = \delta_2 = -\delta_3 = -\delta_4 = -1,$$

and then we obtain

$$p = \begin{bmatrix} p_1 & p_2 \\ -p_2 & -p_1 \end{bmatrix}.$$

Further, let us choose

$$\begin{aligned} &\sigma_1 = \sigma_2, \ \ \sigma_3 = \sigma_4, \ \ \sigma_1\sigma_3 = \pm 1, \\ &\sigma_1 = -\sigma_2, \ \ \sigma_3 = -\sigma_4, \ \ \sigma_1\sigma_3 = \pm 1, \end{aligned}$$

and so we get

$$q = \pm \begin{bmatrix} p_1 & -p_2 \\ p_2 & -p_1 \end{bmatrix}, \quad q = \pm p = \pm \begin{bmatrix} p_1 & p_2 \\ -p_2 & -p_1 \end{bmatrix},$$

respectively. The two pairs of corresponding real reduced coupled mKdV integrable models read

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}\left[p_{1,xxx} \pm 6\left(p_1^2+p_2^2\right)p_{1,x} \pm 12 p_1 p_2 p_{2,x}\right], \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}\left[p_{2,xxx} \pm 12 p_1 p_2 p_{1,x} \pm 6\left(p_1^2+p_2^2\right)p_{2,x}\right], \end{cases}$$

and

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}\left[p_{1,xxx} \pm 6\left(p_1^2-p_2^2\right)p_{1,x}\right], \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}\left[p_{2,xxx} \mp 6\left(p_2^2-p_1^2\right)p_{2,x}\right]. \end{cases}$$

5. Conclusion and Remarks

Two local group reductions by similarity transformations have been introduced and analyzed, which reduce the AKNS matrix spectral problems, and associated real reduced matrix mKdV integrable hierarchies have been constructed, which consist of commuting flows. Illustrative examples of reduced AKNS matrix spectral problems and real reduced mKdV integrable models have been presented. One of the two group reductions leads to a constraint on the two submatrix potentials in the original AKNS matrix spectral problems, and the other engenders a constraint on one of the two submatrix potentials. The paper provides a novel kind of pairs of group reductions from the ones discussed in the literature [19,20], which consist of one local and one nonlocal group reductions.

Soliton type solutions could be generated through various approaches, such as the Darboux transformation, the Hirota bilinear tool, Bäcklund transforms and the Wronskian determinant technique. Rational solutions (see, e.g., [5]), lump wave solutions (see, e.g., [13, 23, 27]), breather wave and rogue wave solutions (see, e.g., [25, 35, 36]) and interaction solutions (see, e.g., [11]) are among interesting solutions. On the other hand, Riemann-Hilbert problems are applied to construction of soliton type solutions to integrable models with multiple poles of the scattering coefficients [34]. We point out that reduced integrable models correspond to balancing different potentials in the original equations, and thus, they need to satisfy constraint conditions and are more difficult to get and solve. It is expected that our analysis could help classify integrable models as well as enrich the field of integrable models from a reduction point of view.

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