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# Global Behavior of a Higher-Order Nonlinear Difference Equation with Many Arbitrary Multivariate Functions

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**Abstract.** Let $k \geq 0$ and $l \geq 2$ be integers, $c$ a nonnegative number and $f$ an arbitrary multivariate function such that $f(x_1, x_2, x_3, \cdots, x_l) \geq x_1 + x_2$ for $x_1, x_2 \geq 0$. This work deals with the higher-order nonlinear difference equation

$$z_{n+1} = \frac{(c+1)z_n z_{n-k} + c[f(z_n, z_{n-k}, w_3, \cdots, w_l)) - z_n - z_{n-k}] + 2c^2}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l)) + c}, \quad n \geq 0,$$

where $z_{-k}, z_{-k+1}, \cdots, z_0$ are positive initial values and $w_i$, $3 \leq i \leq l$, arbitrary functions of variables $z_{n-k}, z_{n-k+1}, \cdots, z_n$. All solutions of this equation are classified into three groups, according to their asymptotic behavior, and a decreasing and increasing characteristic of oscillatory solutions is also explored. Finally, the global asymptotic stability of the positive equilibrium solution $\bar{z} = c$ is exhibited by establishing a strong negative feedback property.

**AMS subject classifications**: 39A11, 39A30, 39A10

**Key words**: Difference equation, positive equilibrium, oscillatory solution, strong negative feedback, global asymptotic stability.

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## 1. Introduction

Many problems in probability, biology, computer science, digital signal processing and economics involve difference equations. Difference equations are connected with differential equations as discrete mathematics is connected with continuous mathematics. Differential equations, even supposedly elementary ones, can often be hard and overwhelming. By contrast, elementary difference equations are relatively easy to deal with.

Nevertheless, the solution structure is well studied only for linear difference equations [3], whereas for nonlinear ones, various properties of solutions are usually observed and guessed only via numerical simulations but not by rigorous mathematical analysis [4,13,15,28]. Therefore, it is fundamentally important to provide a qualitative analysis on nonlinear difference equations, especially their global behavior. This is the main topic of the current study. For related analytical studies on rational difference equations, the reader may consult Refs. [1,5,6,8,9,11,16,18,26]. Asymptotic behavior of eigenfunctions also plays a crucial role in determining scattering data in Riemann-Hilbert problems and matrix spectral problems [22,23] and representing algebro-geometric solutions of integrable equations [19,20].

Let $k \geq 0$ and $l \geq 2$ be integers, $c$ a nonnegative number and $f$ an arbitrary multivariate function such that

$$f(x_1, x_2, x_3, \cdots, x_l) \geq x_1 + x_2, \quad \text{when} \;\; x_1, x_2 \geq 0. \tag{1.1}$$

We would like to study a higher-order nonlinear difference equation involving many arbitrary multivariate functions, — viz.

$$z_{n+1} = \frac{(c+1)z_n z_{n-k} + c[f(z_n, z_{n-k}, w_3, \cdots, w_l) - z_n - z_{n-k}] + 2c^2}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) + c}, \quad n \geq 0, \tag{1.2}$$

with positive initial values $z_{-k}, z_{-k+1}, \cdots, z_0$ and arbitrary multivariate functions $w_i$, $3 \leq i \leq l$, of variables $z_{n-k}, z_{n-k+1}, \cdots, z_n$. The positivity of the initial values and the property (1.1) guarantee the existence of positive solutions for the Eq. (1.2). Moreover, a direct computation can show that the Eq. (1.2) possesses only one positive equilibrium (steady state) solution $\bar{z} = c$.

Observe that the transformation

$$z_n = \frac{c}{y_n}, \quad n \geq -k, \tag{1.3}$$

puts (1.2) into the equivalent difference equation

$$y_{n+1} = \frac{c(y_n y_{n-k} + c) + f(c/y_n, c/y_{n-k}) y_n y_{n-k}}{c(2y_n y_{n-k} - y_n - y_{n-k} + 1) + f(c/y_n, c/y_{n-k}) y_n y_{n-k} + c^2}, \quad n \geq 0, \tag{1.4}$$

where $f = f(x_1, x_2)$ is assumed. Obviously, under (1.3), the positive equilibrium solution $\bar{z} = c$ of (1.2) becomes the positive equilibrium solution $\bar{y} = 1$ of the Eq. (1.4). Upon taking a reduction with $c = 1$ and $f(x_1, x_2) = x_1 + x_2$, we obtain the rational difference equation

$$y_{n+1} = \frac{(y_n + 1)(y_{n-k} + 1)}{2(y_n y_{n-k} + 1)}, \quad n \geq 0. \tag{1.5}$$

The difference equation (1.2) can generate many other rational difference equations, which amend the classes of rational difference equations with globally asymptotically stable equilibria in [2].

In this paper, we analyse the higher-order nonlinear difference equation (1.2) and classify all its positive solutions into three groups according to their asymptotic behavior. For the group of oscillatory solutions, a decreasing and increasing characteristic is explored. The global asymptotic stability, implying the global attractivity of the positive equilibrium solution $\bar{z} = c$ is then verified, and four illustrative examples are presented to further demonstrate the global behavior of solutions. A few concluding remarks are given in the last section.

## 2. Global Behavior

### 2.1. Classification of solutions

From the Eq. (1.2) itself, we can derive

$$z_{n+1} - c = \frac{(z_n - c)(z_{n-k} - c)}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) + c}, \quad n \geq 0, \tag{2.1}$$

$$z_{n+1} - z_n = \frac{(c - z_n)[z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) - z_{n-k} + 2c]}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) + c}, \quad n \geq 0, \tag{2.2}$$

and

$$z_{n+1} - z_{n-k} = \frac{(c - z_{n-k})[z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) - z_n + 2c]}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) + c}, \quad n \geq 0. \tag{2.3}$$

The following results are an immediate consequence of the Eqs. (2.2) and (2.3).

**Theorem 2.1** (Properties of solutions)**.** *Let* $\{z_n\}_{n=-k}^{\infty}$ *be a solution of the nonlinear difference equation* (1.2)*. Then*

$$\begin{cases} z_{n+1} > z_n, & \text{if } z_n < c, \\ z_{n+1} < z_n, & \text{if } z_n > c, \end{cases} \tag{2.4}$$

*and*

$$\begin{cases} z_{n+1} > z_{n-k}, & \text{if } z_{n-k} < c, \\ z_{n+1} < z_{n-k}, & \text{if } z_{n-k} > c, \end{cases} \tag{2.5}$$

*where* $n \geq 0$.

If $k = 0$, the nonlinear difference equation (1.2) becomes the first-order difference equation

$$z_{n+1} = \frac{(c+1)z_n^2 + c[f(z_n, z_n, w_3, \cdots, w_l) - z_n - z_n] + 2c^2}{z_n^2 + f(z_n, z_n, w_3, \cdots, w_l) + c}, \quad n \geq 0. \tag{2.6}$$

From this or due to (2.1), we obtain $z_{n+1} \geq c$ for $n \geq 0$. Further if $n \geq 1$, then $z_{n+1} \leq z_n$ by (2.2). Thus, $\{z_n\}_{n=1}^{\infty}$ decays to $c$ as $n \to \infty$.

Generally, it follows directly from the Eq. (2.1) and the inequalities (2.4) that there are three groups of solutions of (1.2) described by the following theorem.

**Theorem 2.2** (Classification of solutions). *Let* $k \geq 1$. *Suppose that* $\{z_n\}_{n=-k}^{\infty}$ *is a solution of the higher-order nonlinear difference equation* (1.2). *Then*

*(i) the solution eventually equals c, more precisely* $z_n = c,\ n \geq m$, *which occurs when* $z_m = c$ *for some* $m \geq 0$;

*(ii) the solution is eventually greater than c, more precisely* $c < z_{n+1} < z_n,\ n \geq m+k$, *which occurs when* $z_m, z_{m+1}, \cdots, z_{m+k} > c$ *for some* $m \geq -k$;

*(iii) the solution oscillates about c with at most* $k+1$ *consecutive and increasing terms less than c and at most k consecutive and decreasing terms greater than c.*

Note that it follows from (2.1) that another solution situation that a solution of (1.2) is eventually less than $c$ does not occur.

A solution $\{z_n\}_{n=-k}^{\infty}$ in the group *(iii)* in Theorem 2.2 is called an oscillatory solution. The decreasing and increasing characteristic of the oscillatory solutions in Theorem 2.2 can be demonstrated as follows.

Let $n_1, n_2 \geq 0$ be integers such that $n_1 < n_2$. If $n_2 = n_1 + 1$, the monotonicity simply follows from (2.4). Let us now assume that $n_2 \geq n_1 + 2$. Using the Eq. (2.2), we can compute that

$$\begin{aligned} z_{n_2} - z_{n_1} &= (z_{n_2} - z_{n_1+1}) + (z_{n_1+1} - z_{n_1}) \\ &= \sum_{j=n_1+1}^{n_2-1} (z_{j+1} - z_j) + (z_{n_1+1} - z_{n_1}) = D + (z_{n_1+1} - z_{n_1}), \end{aligned} \tag{2.7}$$

where

$$D = \sum_{j=n_1+1}^{n_2-1} \frac{(c - z_j)[z_j z_{j-k} + f(z_j, z_{j-k}) - z_{j-k} + 2c]}{z_j z_{j-k} + f(z_j, z_{j-k}) + c}. \tag{2.8}$$

Considering the first case of $z_n > c, n_1 \leq n \leq n_2$, we note that the term $D$ in (2.8) is negative, and thus (2.7) yields $z_{n_2} < z_{n_1}$. Similarly in the second case of $z_n < c, n_1 \leq n \leq n_2$, $D$ in (2.8) is positive, and thus (2.7) implies $z_{n_2} > z_{n_1}$.

## 2.2. Global asymptotic stability

Since a globally attractive equilibrium of the first-order difference equations cannot be unstable [27], the equilibrium $\bar{z} = c$ of the Eq. (2.6) is globally asymptotically stable.

We can verify this global asymptotic stability result in general case $k \geq 1$ for the higher-order nonlinear difference equation (1.2) by establishing an associated strong negative feedback property [2] (for a generalization of this property, the reader can consult [14]).

**Theorem 2.3** (Global asymptotic stability). *The positive equilibrium solution $\bar{z} = c$ of the higher-order nonlinear difference equation* (1.2) *is globally asymptotically stable.*

*Proof.* It follows immediately from the Eq. (1.2) that

$$\frac{c^2}{z_{n-k}} - z_{n+1} = \frac{(c - z_{n-k})h_n}{z_{n-k}(z_n z_{n-k} + g_n + c)}, \quad n \geq 0,$$

where

$$g_n := f(z_n, z_{n-k}, w_3, \cdots, w_l), \quad h_n := (c+1)z_n z_{n-k} + c g_n - c z_{n-k} + c^2, \quad n \geq 0.$$

Together with (2.3), it yields

$$(z_{n-k} - z_{n+1})\left(\frac{c^2}{z_{n-k}} - z_{n+1}\right) = -\frac{(c - z_{n-k})^2(z_n z_{n-k} + g_n - z_n + 2c)h_n}{z_{n-k}(z_n z_{n-k} + g_n + c)^2}, \quad n \geq 0.$$

This clearly tells the strong negative feedback property

$$(z_{n-k} - z_{n+1})\left(\frac{c^2}{z_{n-k}} - z_{n+1}\right) \leq 0, \quad n \geq 0,$$

where the equality holds for all $n \geq 0$ if and only if $z_n = c, n \geq -k$, namely, $\{z_n\}_{n=-k}^{\infty}$ is the positive equilibrium solution $\bar{z} = c$. Therefore, by a stability theorem in [2], the positive equilibrium solution $\bar{z} = c$ of the Eq. (1.2) is globally asymptotically stable. This completes the proof. □

## 2.3. Illustrative examples

To illustrate the global asymptotic stability property in Theorem 2.3, we present two sets of specific examples associated with two special choices of $c$ and $f$, — viz.

$$c = 5, \quad f(x_1, x_2) = x_1^3 + x_1 x_2 + x_1 + x_2,$$

and

$$c = 9, \quad f(x_1, x_2) = x_2^5 + 3x_1^2 x_2 + x_1 + x_2.$$

In the first choice, we take

$$k = 5, \quad z_{-5} = 3, \quad z_{-4} = 7, \quad z_{-3} = \frac{6}{5},$$
$$z_{-2} = 2, \quad z_{-1} = 6, \quad z_0 = \frac{7}{2},$$

and

$$k = 7, \quad z_{-7} = 6, \quad z_{-6} = 3, \quad z_{-5} = 8, \quad z_{-4} = 4,$$
$$z_{-3} = \frac{11}{5}, \quad z_{-2} = \frac{17}{3}, \quad z_{-1} = \frac{9}{2}, \quad z_0 = \frac{28}{9}.$$

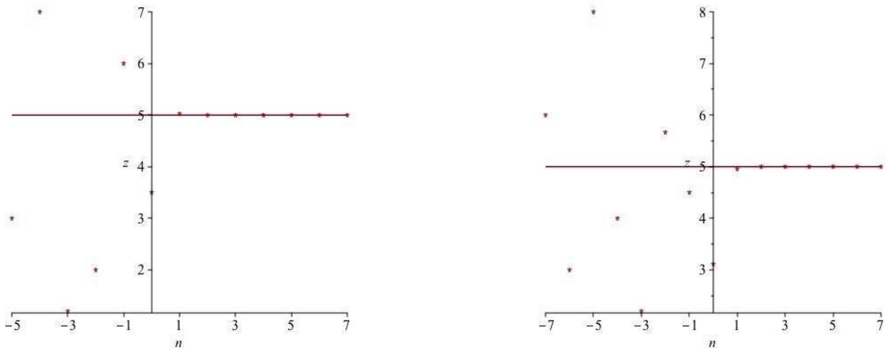


Figure 1: Profiles of $\{z_n\}_{n=-k}^{\infty}$ with $c=5$, $f=x_1^3+x_1x_2+x_1+x_2$. Left: $k=5$. Right: $k=7$.

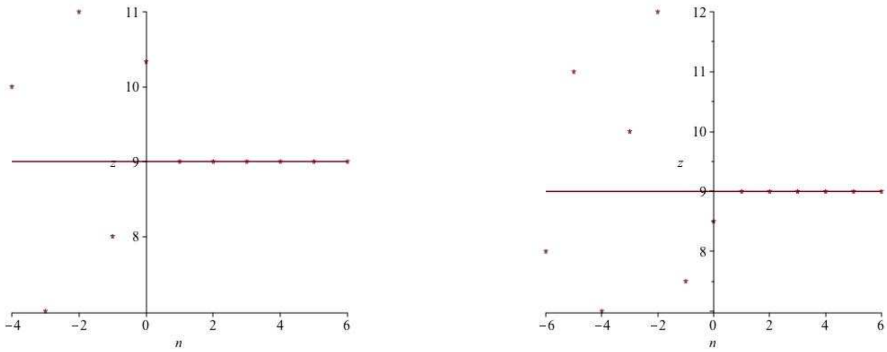


Figure 2: Profiles of $\{z_n\}_{n=-k}^{\infty}$ with $c=9$, $f=x_1^5+3x_1^2x_2+x_1+x_2$. Left: $k=4$. Right: $k=6$.

The corresponding plots are displayed in Fig. 1.

In the second choice, we take

$$k=4, \quad z_{-4}=10, \quad z_{-3}=7,$$
$$z_{-2}=11, \quad z_{-1}=8, \quad z_0=\frac{31}{3},$$

and

$$k=6, \quad z_{-6}=8, \quad z_{-5}=11, \quad z_{-4}=7,$$
$$z_{-3}=10, \quad z_{-2}=12, \quad z_{-1}=\frac{15}{2}, \quad z_0=\frac{17}{2}.$$

The corresponding plots are displayed in Fig. 2. The graphs show high rates of convergence in all four cases.

## 3. Concluding Remarks

We have proved that there exist three groups of solutions to the higher-order nonlinear difference equation (1.2) with many arbitrary multivariate functions. A decreasing and increasing characteristic of the oscillatory solutions was explored and the global asymptotic stability of the unique positive equilibrium solution was shown.

In the special case of $c = 1$ and $f = x_1 + x_2$, from the theorems in Section 2, we can obtain the corresponding results on global behavior of the rational difference equation

$$z_{n+1} = \frac{2(z_n z_{n-k} + 1)}{z_n z_{n-k} + z_n + z_{n-k} + 1}, \quad n \geq 0,$$

or equivalently, the rational difference equation (1.5). There have also been similar studies on global behavior of polynomial difference equations — cf. [17], on rational difference equations or systems — cf. Refs. [1, 5–9, 11, 16, 18, 26] and [10, 12], and other recent studies on positive rational function solutions, called lump solutions, to both linear and nonlinear partial differential equations [21, 24, 25].

Let $k \geq 1$. For an oscillatory solution $\{z_n\}_{n=-k}^{\infty}$ of the Eq. (1.2), we introduce the two sets

$$N_g = \{n \,|\, z_n > c \text{ and } n \geq 0\}, \quad N_l = \{n \,|\, z_n < c \text{ and } n \geq 0\}.$$

Since $\{z_n\}_{n=-k}^{\infty}$ is oscillatory, Theorem 2.2 tells that both sets, $N_g$ and $N_l$, contain infinitely many subsets of consecutive integers, on each of which the solution $z_n$ is decreasing or increasing. An interesting question is what condition on the function $f$ guarantees that $z_n$ is decreasing on the whole set $N_g$ and increasing on the whole set $N_l$.

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## References

[1] R. Abu-Saris, C. Çinar and I. Yalçinkaya, *On the asymptotic stability of $x_{n+1} = (a + x_n x_{n-k})/(x_n + x_{n-k})$*, Comput. Math. Appl. **56**, 1172–1175 (2008).

[2] A.M. Amleh, N. Kruse and G. Ladas, *On a class of difference equations with strong negative feedback*, J. Differ. Equ. Appl. **5**, 497–515 (1999).

[3] P.M. Batchelder, *An Introduction to linear difference equations*, Dover Publications (1967).

[4] E. Camouzis and G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*, Chapman and Hall/CRC (2007).

[5] E. Camouzis, G. Ladas, I.W. Rodrigues and S. Northshield, *The rational recursive sequence $x_{n+1} = (\beta x_n^2)/(1 + x_{n-1}^2)$*, Comput. Math. Appl. **28**, 37–43 (1994).

[6] D.M. Chen, X.Y. Li and Y.Q. Wang, *Dynamics for nonlinear difference equation $x_{n+1} = (\alpha x_{n-k})/(\beta + \gamma x_{n-1}^p)$*, Adv. Difference Equ. **2009**, 235691 (2009).

[7] C. Çinar, *On the solutions of the difference equation $x_{n+1} = x_{n-1}/(-1 + a x_n x_{n-1})$*, Appl. Math. Comput. **158**, 793–797 (2004).

[8] E.M. Elsayed, *New method to obtain periodic solutions of period two and three of a rational difference equation*, Nonlinear Dynam. **79**, 241–250 (2015).
[9] A. Gelişken, C. Çinar and A.S. Kurbanli, *On the asymptotic behavior and periodic nature of a difference equation with maximum*, Comput. Math. Appl. **59**, 898–902 (2010).
[10] M. Gümüş, *The global asymptotic stability of a system of difference equations*, J. Differ. Equ. Appl. **24**, 976–991 (2018).
[11] M. Gümüş and R. Abo-Zeid, *On the solutions of a* $(2K+2)$*th order difference equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **25**, 129–143 (2018).
[12] M. Gümüş and Ö. Öcalan, *The qualitative analysis of a rational system of difference equations*, J. Fract. Calc. Appl. **9**, 113–126 (2018).
[13] V.L. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer (1993).
[14] N. Kruse and T. Nesemann, *Global asymptotic stability in some discrete dynamical systems*, J. Math. Anal. Appl. **235**, 151–158 (1997).
[15] M.R.S. Kulenović and G. Ladas, *Dynamics of second order rational difference equations: with open problems and conjectures*, Chapman and Hall/CRC (2001).
[16] X. Li and D. Zhu, *Global asymptotic stability in a rational equation*, J. Differ. Equ. Appl. **9**, 833–839 (2003).
[17] X. Li and D. Zhu, *Global asymptotic stability for two recursive difference equations*, Appl. Math. Comput. **150**, 481–492 (2004).
[18] X. Li and D. Zhu, *Two rational recursive sequences*, Comput. Math. Appl. **47**, 1487–1494 (2004).
[19] W.X. Ma, *Trigonal curves and algebro-geometric solutions to soliton hierarchies I*, Proc. Roy. Soc. A **473**, 20170232 (2017).
[20] W.X. Ma, *Trigonal curves and algebro-geometric solutions to soliton hierarchies II*, Proc. Roy. Soc. A **473**, 20170233 (2017).
[21] W.X. Ma, *Abundant lumps and their interaction solutions of (3+1)-dimensional linear PDEs*, J. Geom. Phys. **133**, 10–16 (2018).
[22] W.X. Ma, *Application of the Riemann-Hilbert approach to the multicomponent AKNS integrable hierarchies*, Nonlinear Anal. Real. World Appl. **47**, 1–17 (2019).
[23] W.X. Ma, *The inverse scattering transform and soliton solutions of a combined modified Korteweg-de Vries equation*, J. Math. Anal. Appl. **471**, 796–811 (2019).
[24] W.X. Ma, J. Lie and C.M. Khalique, *A study on lump solutions to a generalized Hirota-Satsuma-Ito equation in (2+1)-dimensions*, Complexity **2018**, 9059858 (2018).
[25] W.X. Ma and Y. Zhou, *Lump solutions to nonlinear partial differential equations via Hirota bilinear forms*, J. Diff. Equ. **264**, 2633–2659 (2018).
[26] Ö. Öcalan, *Dynamics of the difference equation* $x_{n+1} = p_n + x_{n-k}/x_n$ *with a period-two coefficient*, Appl. Math. Comput. **228**, 31–37 (2014).
[27] H. Sedaghat, *The impossibility of unstable, globally attracting fixed points for continuous mappings of the line*, Amer. Math. Monthly **104**, 356–358 (1997).
[28] H. Sedaghat, *Nonlinear difference equations: theory with applications to social science models*, Springer (2003).