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COMMUTING INTEGRABLE MODELS GENERATED FROM A GENERALIZED AKNS EIGENVALUE PROBLEM

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Abstract. This study aims to propose a specific eigenvalue problem involving a 4×4 matrix and construct a hierarchy of related commuting integrable models endowed with a bi-Hamiltonian structure. The Lax pair compatibility condition ensures the Liouville integrability, while the trace identity provides the Hamiltonian structure. Specific integrable equations of second- and third-order are explicitly derived, demonstrating the integrable hierarchy.

1. **Introduction.** In soliton theory, Lax pairs play an essential role in exploring integrable models. The concept of a Lax pair [13] involves formulating a pair of eigenvalue problems, that correspond to a given nonlinear model. By constructing a suitable Lax pair, we can derive a compatible set of model equations endowed with significant integrability features, including infinitely many conservation laws and symmetries. These equations exhibit soliton solutions, facilitating their analysis through analytical techniques and offering profound insights into their dynamics [2].

To construct integrable models using Lax pairs, we typically begin with a column potential vector u and introduce an eigenvalue parameter k. The process involves defining a pair of linear differential equations, known as the Lax pair, which are interconnected through a compatibility condition crucial for ensuring the integrability of the corresponding nonlinear equations. Specifically, the Lax pair comprises two eigenvalue equations:

$$\phi_x = \mathcal{U}(u,k)\phi, \ \ \phi_t = \mathcal{V}(u,k)\phi, \tag{1}$$

where ϕ represents the eigenfunction, $\mathcal{U}(u,k)$ denotes the spatial spectral matrix, and $\mathcal{V}(u,k)$ signifies the temporal spectral matrix. These matrices are dependent on both the potential vector u and the eigenvalue parameter k. The zero curvature condition, or compatibility condition, is expressed as:

$$\mathcal{U}_t - \mathcal{V}_x + [\mathcal{U},\mathcal{V}] = 0, \tag{2}$$

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where $[\mathcal{U},\mathcal{V}]=\mathcal{U}\mathcal{V}-\mathcal{V}\mathcal{U}$ denotes the commutator of $\mathcal{U}$ and $\mathcal{V}$. This condition ensures that the eigenfunction ϕ evolves consistently in both the spatial and temporal directions, thereby leading to an integrable system.

To illustrate, let's consider the AKNS (Ablowitz-Kaup-Newell-Segur) system, which serves as a prominent framework for generating integrable equations. The AKNS system, introduced by Ablowitz, Kaup, Newell, and Segur [1], defines the matrices $\mathcal{U}$ and $\mathcal{V}$ as follows:

$$\mathcal{U}(u,k)=\begin{bmatrix} -k & p \\ q & k \end{bmatrix},\ \mathcal{V}(u,k)=\begin{bmatrix} -A & B \\ C & A \end{bmatrix}, \tag{3}$$

where p and q are components of the potential vector u, and A,B, and C are functions of u and k. The specific forms of A,B, and C depend on the particular integrable model under consideration. The zero curvature condition (2) derived from these matrices leads to a set of nonlinear partial differential equations for p and q. For example, in the case of the nonlinear Schrödinger (NLS) equation, the condition results in:

$$\begin{cases} p_t=-p_{xx}+2p^2q, \\ q_t=q_{xx}-2pq^2. \end{cases} \tag{4}$$

Solving these equations reveals the integrable structure of the system, characterized by soliton solutions, infinite symmetries, and conserved quantities. This approach highlights how the AKNS system, through its Lax pair formulation and the zero curvature condition, facilitates the exploration and understanding of integrable models in nonlinear physics and mathematics.

By choosing suitable forms for $\mathcal{U}$ and $\mathcal{V}$, various integrable models can be derived, each possessing distinctive properties and being amenable to powerful analytical techniques such as the inverse scattering transform. Examples include: the sine-Gordon equation, the Korteweg-de Vries (KdV) equation, and other well-known integrable models. These models exhibit remarkable integrable properties, allowing for the study of soliton solutions, infinite symmetries, and conserved quantities.

Hamiltonian structures play a fundamental role in the study of integrable systems, offering a framework to investigate their integrability. One effective method to generate Hamiltonian structures involves utilizing the trace identity [39, 14] or the variational identity [31]. Among these, the trace identity stands out as a robust technique in this context.

The trace identity, as detailed in references [39, 14], is formulated as:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\big(\mathcal{W}\frac{\partial \mathcal{U}}{\partial k}\big)\,dx=k^{-\tau}\frac{\partial}{\partial k}k^{\tau}\operatorname{tr}\big(\mathcal{W}\frac{\partial \mathcal{U}}{\partial u}\big), \tag{5}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u and tr represents the trace of a matrix, and τ is a constant independent of the eigenvalue parameter k. Here, $\mathcal{W}$ satisfies the stationary zero curvature equation

$$\mathcal{W}_x=[\mathcal{U},\mathcal{W}], \tag{6}$$

where $\mathcal{U}$ is the spectral matrix. The trace identity establishes a connection between the variational derivative of an integral involving the eigenvalue parameter k and the trace of a matrix expression. This connection is pivotal in linking the eigenvalue problem with the Hamiltonian structure of the system, providing insights into its integrable properties.

A wide array of Liouville integrable hierarchies of soliton Hamiltonian equations can be constructed using the Lax pair formulation mentioned earlier, utilizing loop algebras derived from both special linear algebras (as discussed in works such as [1]) and special orthogonal algebras (as illustrated in references like [15, 16, 17]). These hierarchies play a crucial role in the field of integrable models, offering a structured framework to investigate the solutions and properties of soliton equations.

This paper introduces a novel spectral matrix and employs the Lax pair formulation to construct a Liouville integrable hierarchy consisting of four-component bi-Hamiltonian equations. The resulting soliton equations feature bi-Hamiltonian structures, which are substantiated using the trace identity. Several illustrative examples are presented, including four-component coupled integrable NLS equations and modified Korteweg-de Vries (mKdV) equations. These examples showcase the versatility and applicability of the proposed framework in generating integrable models. In the concluding section, the paper summarizes its findings and provides summary remarks that highlight the potential for future research directions.

2. **Commuting integrable models with Hamiltonian structures.** Building upon research into non-perturbation integrable couplings within the AKNS hierarchy using the Lax pair formulation [45], we explore a newly proposed matrix eigenvalue problem formulated as follows:

$$\phi_x = \mathcal{U}\phi = \mathcal{U}(u,k)\phi,\ \mathcal{U} = \begin{bmatrix} \gamma_1 k & u_1 & -\delta_1 k & -u_3 \\ u_2 & \gamma_2 k & -u_4 & -\delta_2 k \\ \delta_1 k & u_3 & \gamma_1 k & u_1 \\ u_4 & \delta_2 k & u_2 & \gamma_2 k \end{bmatrix}, \tag{7}$$

where k designates once more the eigenvalue parameter and u stands for the dependent variable consisting of four components:

$$u = u(x,t) = (u_1, u_2, u_3, u_4)^T. \tag{8}$$

If the bottom-left 2×2 block is set to zero, this spectral problem, with parameters $\gamma_1 = -\gamma_2 = -1$ and $\delta_1 = -\delta_2 = 1$, aligns with the one discussed in the aforementioned reference. To ensure the generation of an integrable hierarchy via the Lax pair formulation from this new spectral problem, a necessary and sufficient condition must be imposed:

$$\gamma^2 + \delta^2 \neq 0,\ \gamma = \gamma_1 - \gamma_2,\ \delta = \delta_1 - \delta_2. \tag{9}$$

When $\delta_1 = \delta_2 = 0$ and $u_3 = u_4 = 0$, the spectral problem transforms into two identical forms of the standard AKNS eigenvalue problem [1], thereby introducing a generalized variant of the AKNS eigenvalue problem.

To construct a corresponding four-component Liouville integrable hierarchy, we begin by solving the associated stationary zero curvature equation (6) through the determination of a targeted Laurent series solution:

$$\mathcal{W} = \begin{bmatrix} a & b & -e & -f \\ c & -a & -g & e \\ e & f & a & b \\ g & -e & c & -a \end{bmatrix} = \sum_{n\geq 0} k^{-n}\mathcal{W}^{\{n\}}, \tag{10}$$

with six fundamental components assumed to be expanded in Laurent series of the eigenvalue parameter k:

$$\begin{array}{l} a = \sum_{n\geq 0} k^{-n} a^{\{n\}},\ b = \sum_{n\geq 0} k^{-n} b^{\{n\}},\ c = \sum_{n\geq 0} k^{-n} c^{\{n\}}, \\ e = \sum_{n\geq 0} k^{-n} e^{\{n\}},\ f = \sum_{n\geq 0} k^{-n} f^{\{n\}},\ g = \sum_{n\geq 0} k^{-n} g^{\{n\}}. \end{array} \tag{11}$$

It is apparent that the corresponding associated stationary zero curvature equation (6) results in the following relationships:

$$\left\{\begin{array}{l} a_x = u_1 c - u_2 b - u_3 g + u_4 f, \\ b_x = \gamma k b - \delta k f - 2u_1 a + 2u_3 e, \\ c_x = -\gamma k c + \delta k g + 2u_2 a - 2u_4 e, \\ e_x = u_1 g - u_2 f + u_3 c - u_4 b, \\ f_x = \delta k b + \gamma k f - 2u_1 e - 2u_3 a, \\ g_x = -\delta k c - \gamma k g + 2u_2 e + 2u_4 a. \end{array}\right. \tag{12}$$

This provides

$$k \begin{bmatrix} \gamma & -\delta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} b \\ f \end{bmatrix} = \begin{bmatrix} b_x + 2u_1 a - 2u_3 e \\ f_x + 2u_1 e + 2u_3 a \end{bmatrix} \tag{13}$$

and

$$k \begin{bmatrix} \gamma & -\delta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} -c_x + 2u_2 a - 2u_4 e \\ -g_x + 2u_2 e + 2u_4 a \end{bmatrix}. \tag{14}$$

Therefore, the condition (9), ensuring the invertibility of the coefficient matrix in the aforementioned systems, is both necessary and sufficient to ensure the recursive determination of a Laurent series solution $\mathcal{W}$. Based on (13) and (14), it is evident that the system (12) yields the initial requirements:

$$a_x^{\{0\}} = e_x^{\{0\}} = 0,\ b^{\{0\}} = c^{\{0\}} = f^{\{0\}} = g^{\{0\}} = 0, \tag{15}$$

and the iterative relationships employed to define the Laurent series solution:

$$\left\{\begin{array}{l} b^{\{n+1\}} = \frac{\gamma}{\gamma^2+\delta^2}(b_x^{\{n\}} + 2u_1 a^{\{n\}} - 2u_3 e^{\{n\}}) + \frac{\delta}{\gamma^2+\delta^2}(f_x^{\{n\}} + 2u_1 e^{\{n\}} + 2u_3 a^{\{n\}}), \\ f^{\{n+1\}} = -\frac{\delta}{\gamma^2+\delta^2}(b_x^{\{n\}} + 2u_1 a^{\{n\}} - 2u_3 e^{\{n\}}) + \frac{\gamma}{\gamma^2+\delta^2}(f_x^{\{n\}} \\ \qquad +2u_1 e^{\{n\}} + 2u_3 a^{\{n\}}), \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{l} c^{\{n+1\}} = \frac{\gamma}{\gamma^2+\delta^2}(-c_x^{\{n\}} + 2u_2 a^{\{n\}} - 2u_4 e^{\{n\}}) \\ \qquad + \frac{\delta}{\gamma^2+\delta^2}(-g_x^{\{n\}} + 2u_2 e^{\{n\}} + 2u_4 a^{\{n\}}), \\ g^{\{n+1\}} = -\frac{\delta}{\gamma^2+\delta^2}(-c_x^{\{n\}} + 2u_2 a^{\{n\}} - 2u_4 e^{\{n\}}) \\ \qquad + \frac{\gamma}{\gamma^2+\delta^2}(-g_x^{\{n\}} + 2u_2 e^{\{n\}} + 2u_4 a^{\{n\}}), \end{array}\right. \tag{17}$$

$$\left\{\begin{array}{l} a_x^{\{n+1\}} = u_1 c^{\{n+1\}} - u_2 b^{\{n+1\}} - u_3 g^{\{n+1\}} + u_4 f^{\{n+1\}}, \\ e_x^{\{n+1\}} = u_1 g^{\{n+1\}} - u_2 f^{\{n+1\}} + u_3 c^{\{n+1\}} - u_4 b^{\{n+1\}}, \end{array}\right. \tag{18}$$

where $n \geq 0$. As customary, to determine a particular Laurent series solution, we introduce arbitrary constant initial data:

$$a^{\{0\}} = \frac{1}{2}\mu,\ e^{\{0\}} = \frac{1}{2}\nu, \tag{19}$$

and assume the integration constants are set to zero:

$$a^{\{n\}}|_{u=0} = 0,\ e^{\{n\}}|_{u=0} = 0,\ n \geq 1. \tag{20}$$

Under these conditions, one can deduce all sequences of $\{a^{\{n\}}, b^{\{n\}}, c^{\{n\}}, e^{\{n\}}, f^{\{n\}}, g^{\{n\}}\}$ for $n \geq 1$. The initial sequence reads

$$\left\{\begin{array}{l} b^{\{1\}} = \frac{1}{\gamma^2+\delta^2}[\gamma(\mu u_1 - \nu u_3) + \delta(\nu u_1 + \mu u_3)], \\ f^{\{1\}} = \frac{1}{\gamma^2+\delta^2}[\gamma(\nu u_1 + \mu u_3) - \delta(\mu u_1 - \nu u_3)], \\ c^{\{1\}} = \frac{1}{\gamma^2+\delta^2}[\gamma(\mu u_2 - \nu u_4) + \delta(\nu u_2 + \mu u_4)], \\ g^{\{1\}} = \frac{1}{\gamma^2+\delta^2}[\gamma(\nu u_2 + \mu u_4) - \delta(\mu u_2 - \nu u_4)], \\ a^{\{1\}} = e^{\{1\}} = 0. \end{array}\right.$$

The subsequent sequence reads

$$\left\{\begin{array}{l} b^{\{2\}}=\frac{1}{(\gamma^2+\delta^2)^2}(p_{2,1}u_{1,x}+p_{2,2}u_{3,x}),\\ f^{\{2\}}=\frac{1}{(\gamma^2+\delta^2)^2}(-p_{2,2}u_{1,x}+p_{2,1}u_{3,x}),\\ c^{\{2\}}=\frac{1}{(\gamma^2+\delta^2)^2}(-p_{2,1}u_{2,x}-p_{2,2}u_{4,x}),\\ g^{\{2\}}=\frac{1}{(\gamma^2+\delta^2)^2}(p_{2,2}u_{2,x}-p_{2,1}u_{4,x}), \end{array}\right.$$

$$\left\{\begin{array}{l} a^{\{2\}}=-\frac{1}{(\gamma^2+\delta^2)^2}[(p_{2,1}u_2+p_{2,2}u_4)u_1+(p_{2,2}u_2-p_{2,1}u_4)u_3],\\ e^{\{2\}}=-\frac{1}{(\gamma^2+\delta^2)^2}[(-p_{2,2}u_2+p_{2,1}u_4)u_1+(p_{2,1}u_2+p_{2,2}u_4)u_3], \end{array}\right.$$

where $p_{2,1}$ and $p_{2,2}$ are two special polynomials of second-order in terms of γ and δ:

$$p_{2,1}=\gamma^2\mu+2\gamma\delta\nu-\delta^2\mu,\ \ p_{2,2}=-\gamma^2\nu+2\gamma\delta\mu+\delta^2\nu. \tag{21}$$

The following sequence reads

$$\left\{\begin{array}{l} b^{\{3\}}=\frac{1}{(\gamma^2+\delta^2)^3}[p_{3,1}u_{1,xx}+p_{3,2}u_{3,xx}-2(p_{3,1}u_2+p_{3,2}u_4)u_1^2\\ \qquad -4(p_{3,2}u_2-p_{3,1}u_4)u_1u_3+2(p_{3,1}u_2+p_{3,2}u_4)u_3^2],\\ f^{\{3\}}=\frac{1}{(\gamma^2+\delta^2)^3}[-p_{3,2}u_{1,xx}+p_{3,1}u_{3,xx}+2(p_{3,2}u_2-p_{3,1}u_4)u_1^2\\ \qquad -4(p_{3,1}u_2+p_{3,2}u_4)u_1u_3-2(p_{3,2}u_2-p_{3,1}u_4)u_3^2], \end{array}\right.$$

$$\left\{\begin{array}{l} c^{\{3\}}=\frac{1}{(\gamma^2+\delta^2)^3}[p_{3,1}u_{2,xx}+p_{3,2}u_{4,xx}-2(p_{3,1}u_1+p_{3,2}u_3)u_2^2\\ \qquad -4(p_{3,2}u_1-p_{3,1}u_3)u_2u_4+2(p_{3,1}u_1+p_{3,2}u_3)u_4^2],\\ g^{\{3\}}=\frac{1}{(\gamma^2+\delta^2)^3}[-p_{3,2}u_{2,xx}+p_{3,1}u_{4,xx}+2(p_{3,2}u_1-p_{3,1}u_3)u_2^2\\ \qquad -4(p_{3,1}u_1-p_{3,2}u_3)u_2u_4-2(p_{3,2}u_1-p_{3,1}u_3)u_4^2], \end{array}\right.$$

$$\left\{\begin{array}{l} a^{\{3\}}=\frac{1}{(\gamma^2+\delta^2)^3}[(-p_{3,1}u_2-p_{3,2}u_4)u_{1,x}+(p_{3,1}u_1+p_{3,2}u_3)u_{2,x}\\ \qquad -(p_{3,2}u_2-p_{3,1}u_4)u_{3,x}+(p_{3,2}u_1-p_{3,1}u_3)u_{4,x}],\\ e^{\{3\}}=\frac{1}{(\gamma^2+\delta^2)^3}[(p_{3,2}u_2-p_{3,1}u_4)u_{1,x}-(p_{3,2}u_1-p_{3,1}u_3)u_{2,x}\\ \qquad -(p_{3,1}u_2+p_{3,2}u_4)u_{3,x}+(p_{3,1}u_1+p_{3,2}u_3)u_{4,x}], \end{array}\right.$$

where $p_{3,1}$ and $p_{3,2}$ are two special polynomials of third-order in terms of γ and δ:

$$p_{3,1}=\gamma^3\mu+3\gamma^2\delta\nu-3\gamma\delta^2\mu-\delta^3\nu,\ \ p_{3,2}=-\gamma^3\nu+3\gamma^2\delta\mu+3\gamma\delta^2\nu-\delta^3\mu. \tag{22}$$

The ensuing sequence reads

$$\left\{\begin{array}{l} b^{\{4\}}=\frac{1}{(\gamma^2+\delta^2)^4}[p_{4,1}u_{1,xxx}+p_{4,2}u_{3,xxx}\\ \qquad -6(p_{4,1}u_1u_2+p_{4,2}u_1u_4+p_{4,2}u_2u_3-p_{4,1}u_3u_4)u_{1,x}\\ \qquad -6(p_{4,2}u_1u_2-p_{4,1}u_1u_4-p_{4,1}u_2u_3-p_{4,2}u_3u_4)u_{3,x}],\\ f^{\{4\}}=\frac{1}{(\gamma^2+\delta^2)^4}[-p_{4,2}u_{1,xxx}+p_{4,1}u_{3,xxx}\\ \qquad +6(p_{4,2}u_1u_2-p_{4,1}u_1u_4-p_{4,1}u_2u_3-p_{4,2}u_3u_4)u_{1,x}\\ \qquad -6(p_{4,1}u_1u_2+p_{4,2}u_1u_4+p_{4,2}u_2u_3-p_{4,1}u_3u_4)u_{3,x}], \end{array}\right.$$

$$\left\{\begin{array}{l} c^{\{4\}}=\frac{1}{(\gamma^2+\delta^2)^4}[-p_{4,1}u_{2,xxx}-p_{4,2}u_{4,xxx}\\ \qquad +6(p_{4,1}u_1u_2+p_{4,2}u_1u_4+p_{4,2}u_2u_3-p_{4,1}u_3u_4)u_{2,x}\\ \qquad +6(p_{4,2}u_1u_2-p_{4,1}u_1u_4-p_{4,1}u_2u_3-p_{4,2}u_3u_4)u_{4,x}],\\ g^{\{4\}}=\frac{1}{(\gamma^2+\delta^2)^4}[p_{4,2}u_{2,xxx}-p_{4,1}u_{4,xxx}\\ \qquad -6(p_{4,2}u_1u_2-p_{4,1}u_1u_4-p_{4,1}u_2u_3-p_{4,2}u_3u_4)u_{2,x}\\ \qquad +6(p_{4,1}u_1u_2+p_{4,2}u_1u_4+p_{4,2}u_2u_3-p_{4,1}u_3u_4)u_{4,x}], \end{array}\right.$$

$$
\begin{aligned}
a^{\{4\}} = \tfrac{1}{(\gamma^2+\delta^2)^4}[& -(p_{4,1}u_2 + p_{4,2}u_4)u_{1,xx} - (p_{4,1}u_1 + p_{4,2}u_3)u_{2,xx} \\
& -(p_{4,2}u_2 - p_{4,1}u_4)u_{3,xx} - (p_{4,2}u_1 - p_{4,1}u_3)u_{4,xx} \\
& +p_{4,1}u_{1,x}u_{2,x} + p_{4,2}u_{1,x}u_{4,x} + p_{4,2}u_{2,x}u_{3,x} - p_{4,1}u_{3,x}u_{4,x} \\
& +3(p_{4,1}u_2^2 + 2p_{4,2}u_2u_4 - p_{4,1}u_4^2)u_1^2 \\
& +6(p_{4,2}u_2^2 - 2p_{4,1}u_2u_4 - p_{4,2}u_4^2)u_1u_3 \\
& -3(p_{4,1}u_2^2 + 2p_{4,2}u_2u_4 - p_{4,1}u_4^2)u_3^2], \\
e^{\{4\}} = \tfrac{1}{(\gamma^2+\delta^2)^4}[& (p_{4,2}u_2 - p_{4,1}u_4)u_{1,xx} + (p_{4,2}u_1 - p_{4,1}u_3)u_{2,xx} \\
& -(p_{4,1}u_2 + p_{4,2}u_4)u_{3,xx} - (p_{4,1}u_1 + p_{4,2}u_3)u_{4,xx} \\
& -p_{4,2}u_{1,x}u_{2,x} + p_{4,1}u_{1,x}u_{4,x} + p_{4,1}u_{2,x}u_{3,x} + p_{4,2}u_{3,x}u_{4,x} \\
& -3(p_{4,2}u_2^2 - 2p_{4,1}u_2u_4 - p_{4,2}u_4^2)u_1^2 \\
& +6(p_{4,1}u_2^2 + 2p_{4,2}u_2u_4 - p_{4,1}u_4^2)u_1u_3 \\
& +3(p_{4,2}u_2^2 - 2p_{4,1}u_2u_4 - p_{4,2}u_4^2)u_3^2],
\end{aligned}
$$

where $p_{4,1}$ and $p_{4,2}$ are two special polynomials of fourth-order in terms of γ and δ:

$$
\begin{cases} p_{4,1} = \gamma^4\mu + 4\gamma^3\delta\nu - 6\gamma^2\delta^2\mu - 4\gamma\delta^3\nu + \delta^4\mu, \\ p_{4,2} = -\gamma^4\nu + 4\gamma^3\delta\mu + 6\gamma^2\delta^2\nu - 4\gamma\delta^3\mu - \delta^4\nu. \end{cases} \tag{23}
$$

Based on these computations, we can set $\Delta_m = 0$ for all $m \geq 0$, thereby defining the temporal matrix eigenvalue problems:

$$
\phi_{t_m} = \mathcal{V}^{\{m\}}\phi = \mathcal{V}^{\{m\}}(u,k)\phi,\ \mathcal{V}^{\{m\}} = (k^m\mathcal{W})_+ = \sum_{n=0}^{m} k^n \mathcal{W}^{\{m-n\}},\ m \geq 0. \tag{24}
$$

These equations form the temporal matrix eigenvalue problems within the Lax pair formulation. The conditions ensuring solvability of the spatial and temporal matrix eigenvalue problems in (7) and (24) are represented by the following zero curvature equations:

$$
\mathcal{U}_{t_m} - \mathcal{V}_x^{\{m\}} + [\mathcal{U}, \mathcal{V}^{\{m\}}] = 0,\ m \geq 0. \tag{25}
$$

These compatibility equations establish a hierarchy of integrable models with four dependent variables:

$$
\begin{aligned}
u_{t_m} = \mathcal{K}^{\{m\}} = (&\gamma b^{\{m+1\}} - \delta f^{\{m+1\}}, -\gamma c^{\{m+1\}} + \delta g^{\{m+1\}}, \\
&\delta b^{\{m+1\}} + \gamma f^{\{m+1\}}, -\delta c^{\{m+1\}} - \gamma g^{\{m+1\}})^T,
\end{aligned} \tag{26}
$$

or more specifically,

$$
\begin{cases} u_{1,t_m} = \gamma b^{\{m+1\}} - \delta f^{\{m+1\}}, \\ u_{2,t_m} = -\gamma c^{\{m+1\}} + \delta g^{\{m+1\}}, \\ u_{3,t_m} = \delta b^{\{m+1\}} + \gamma f^{\{m+1\}}, \\ u_{4,t_m} = -\delta c^{\{m+1\}} - \gamma g^{\{m+1\}}, \end{cases} \tag{27}
$$

in which $m \geq 0$.

For instance, this soliton hierarchy includes various coupled systems of integrable NLS equations and coupled systems of integrable mKdV equations. If we set

$$
\gamma = 1,\ \delta = 0,\ \mu = 1,\ \nu = 0, \tag{28}
$$

we obtain a coupled system of integrable NLS equations:

$$
\begin{cases} u_{1,t_2} = u_{1,xx} - 2u_1^2u_2 + 4u_1u_3u_4 + 2u_2u_3^2, \\ u_{2,t_2} = -u_{2,xx} + 2u_1u_2^2 - 2u_1u_4^2 - 4u_2u_3u_4, \\ u_{3,t_2} = u_{3,xx} - 2u_1^2u_4 - 4u_1u_2u_3 + 2u_3^2u_4, \\ u_{4,t_2} = -u_{4,xx} + 4u_1u_2u_4 + 2u_2^2u_3 - 2u_3u_4^2, \end{cases} \tag{29}
$$

and a coupled system of integrable mKdV equations:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} - 6(u_1u_2 - u_3u_4)u_{1,x} + 6(u_1u_4 + u_2u_3)u_{3,x}, \\ u_{2,t_3} = u_{2,xxx} - 6(u_1u_2 - u_3u_4)u_{2,x} + 6(u_1u_4 + u_2u_3)u_{4,x}, \\ u_{3,t_3} = u_{3,xxx} - 6(u_1u_4 + u_2u_3)u_{1,x} - 6(u_1u_2 - u_3u_4)u_{3,x}, \\ u_{4,t_3} = u_{4,xxx} - 6(u_1u_4 + u_2u_3)u_{2,x} - 6(u_1u_2 - u_3u_4)u_{4,x}. \end{cases} \tag{30}$$

If we set

$$\gamma = 1,\ \delta = 0,\ \mu = 1,\ \nu = 1, \tag{31}$$

we obtain a coupled system of combined integrable NLS equations:

$$\begin{cases} u_{1,t_2} = u_{1,xx} - u_{3,xx} - 2(u_2 - u_4)u_1^2 + 4(u_2 + u_4)u_1u_3 + 2(u_2 - u_4)u_3^2, \\ u_{2,t_2} = -u_{2,xx} + u_{4,xx} + 2(u_1 - u_3)u_2^2 - 4(u_1 + u_3)u_2u_4 - 2(u_1 - u_3)u_4^2, \\ u_{3,t_2} = u_{1,xx} + u_{3,xx} - 2(u_2 + u_4)u_1^2 - 4(u_2 - u_4)u_1u_3 + 2(u_2 + u_4)u_3^2, \\ u_{4,t_2} = -u_{2,xx} - u_{4,xx} + 2(u_1 + u_3)u_2^2 + 4(u_1 - u_3)u_2u_4 - 2(u_1 + u_3)u_4^2, \end{cases} \tag{32}$$

and a coupled system of combined integrable mKdV equations:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} - u_{3,xxx} - 6[u_1(u_2 - u_4) - u_3(u_2 + u_4)]u_{1,x} \\ \qquad\quad +6[u_1(u_2 + u_4) + u_3(u_2 - u_4)]u_{3,x}, \\ u_{2,t_3} = u_{2,xxx} - u_{4,xxx} - 6[u_1(u_2 - u_4) - u_3(u_2 + u_4)]u_{2,x} \\ \qquad\quad +6[u_1(u_2 + u_4) + u_3(u_2 - u_4)]u_{4,x}, \\ u_{3,t_3} = u_{1,xxx} + u_{3,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 - u_4)]u_{1,x} \\ \qquad\quad -6[u_1(u_2 - u_4) - u_3(u_2 + u_4)]u_{3,x}, \\ u_{4,t_3} = u_{2,xxx} + u_{4,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 - u_4)]u_{2,x} \\ \qquad\quad -6[u_1(u_2 - u_4) - u_3(u_2 + u_4)]u_{4,x}. \end{cases} \tag{33}$$

These four systems exemplify typical coupled integrable models, thereby broadening the scope of coupled integrable NLS equations and mKdV equations (see, e,g., [18, 19, 20]).

3. **Recursion operator and bi-Hamiltonian structures.**

Introducing bi-Hamiltonian structures into the soliton hierarchy (27) can be accomplished by applying the classical trace identity (5) to the spatial matrix eigenvalue problem (7).

The trace identity utilizes the solution $\mathcal{W}$ defined by (10). This approach allows for the straightforward determination of Hamiltonian structures within the hierarchy of soliton models. By applying the classical trace identity to the spatial matrix eigenvalue problem, one can systematically derive the Hamiltonian densities and associated flows. Concretely, we have

$$\operatorname{tr}\big(\mathcal{W}\frac{\partial \mathcal{U}}{\partial k}\big) = 2\gamma a - 2\delta e,\ \operatorname{tr}\big(\mathcal{W}\frac{\partial \mathcal{U}}{\partial u}\big) = (2c, 2b, -2g, -2f)^T, \tag{34}$$

and consequently, the classical trace identity gives

$$\begin{aligned} &\frac{\delta}{\delta u}\int k^{-(n+1)}(\gamma a^{\{n+1\}} - \delta e^{\{n+1\}})\,dx \\ &= k^{-\tau}\frac{\partial}{\partial k}k^{\tau-n}(c^{\{n\}}, b^{\{n\}}, -g^{\{n\}}, -f^{\{n\}})^T,\ n \ge 0. \end{aligned} \tag{35}$$

A check with $n = 2$ results in $\tau = 0$, and as a consequence, one obtains

$$\frac{\delta}{\delta u}\mathcal{H}^{\{n\}} = (c^{\{n+1\}}, b^{\{n+1\}}, -g^{\{n+1\}}, -f^{\{n+1\}})^T,\ n \ge 0, \tag{36}$$

in which the required Hamiltonian quantities are evaluated as follows:

$$\mathcal{H}^{\{n\}} = -\int \frac{\gamma a^{\{n+2\}} - \delta e^{\{n+2\}}}{n+1}\, dx,\ n \geq 0. \tag{37}$$

This enables us to establish the Hamiltonian structures for the soliton hierarchy (27):

$$u_{t_m} = \mathcal{K}^{\{m\}} = J_1 \frac{\delta \mathcal{H}^{\{m\}}}{\delta u},\ J_1 = \begin{bmatrix} 0 & \gamma & 0 & \delta \\ -\gamma & 0 & -\delta & 0 \\ 0 & \delta & 0 & -\gamma \\ -\delta & 0 & \gamma & 0 \end{bmatrix},\ m \geq 0, \tag{38}$$

where J_1 is evidently Hamiltonian, and $\mathcal{H}^{[m]}$ are the functionals defined by (37). It is important to emphasize that Hamiltonian structures manifest a crucial property: the interrelation $S = J_1 \frac{\delta \mathcal{H}}{\delta u}$ between a conserved quantity $\mathcal{H}$ and a symmetry S within the same nonlinear model.

The standard soliton theory asserts that the vector fields $\mathcal{K}^n$ commute:

$$[\![\mathcal{K}^{\{n_1\}}, \mathcal{K}^{\{n_2\}}]\!] = \mathcal{K}^{\{n_1\}\prime}(u)[\mathcal{K}^{\{n_2\}}] - \mathcal{K}^{\{n_2\}\prime}(u)[\mathcal{K}^{\{n_1\}}] = 0,\ n_1, n_2 \geq 0. \tag{39}$$

This commutativity arises from the algebra of temporal spectral matrices:

$$\begin{aligned} &[\![\mathcal{V}^{\{n_1\}}, \mathcal{V}^{\{n_2\}}]\!] \\ &= \mathcal{V}^{\{n_1\}\prime}(u)[\mathcal{K}^{\{n_2\}}] - \mathcal{V}^{\{n_2\}\prime}(u)[\mathcal{K}^{\{n_1\}}] + [\mathcal{V}^{\{n_1\}}, \mathcal{V}^{\{n_2\}}] = 0,\ n_1, n_2 \geq 0. \end{aligned} \tag{40}$$

This property can be verified directly by examining the relationship between the isospectral zero curvature equations (see [21] for details).

Furthermore, employing the recursion relation $\mathcal{K}^{m+1} = \Phi \mathcal{K}^m$ involves a straightforward albeit lengthy computation, yielding a recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$, which can be proven to be hereditary [8], in the context of the soliton hierarchy (27). This hereditary recursion operator Φ reads:

$$\left\{ \begin{aligned} \Phi_{11} &= \tfrac{1}{\gamma^2+\delta^2}(\gamma\partial - 2\gamma u_1 \partial^{-1} u_2 + 2\gamma u_3 \partial^{-1} u_4 - 2\delta u_1 \partial^{-1} u_4 - 2\delta u_3 \partial^{-1} u_2), \\ \Phi_{12} &= \tfrac{1}{\gamma^2+\delta^2}(-2\gamma u_1 \partial^{-1} u_1 + 2\gamma u_3 \partial^{-1} u_3 - 2\delta u_1 \partial^{-1} u_3 - 2\delta u_3 \partial^{-1} u_1), \\ \Phi_{13} &= \tfrac{1}{\gamma^2+\delta^2}(\delta\partial + 2\gamma u_1 \partial^{-1} u_4 + 2\gamma u_3 \partial^{-1} u_2 - 2\delta u_1 \partial^{-1} u_2 + 2\delta u_3 \partial^{-1} u_4), \\ \Phi_{14} &= \tfrac{1}{\gamma^2+\delta^2}(2\gamma u_1 \partial^{-1} u_3 + 2\gamma u_3 \partial^{-1} u_1 - 2\delta u_1 \partial^{-1} u_1 + 2\delta u_3 \partial^{-1} u_3); \end{aligned} \right. \tag{41}$$

$$\left\{ \begin{aligned} \Phi_{21} &= \tfrac{1}{\gamma^2+\delta^2}(2\gamma u_2 \partial^{-1} u_2 - 2\gamma u_4 \partial^{-1} u_4 + 2\delta u_2 \partial^{-1} u_4 + 2\delta u_4 \partial^{-1} u_2), \\ \Phi_{22} &= \tfrac{1}{\gamma^2+\delta^2}(-\gamma\partial + 2\gamma u_2 \partial^{-1} u_1 - 2\gamma u_4 \partial^{-1} u_3 + 2\delta u_2 \partial^{-1} u_3 + 2\delta u_4 \partial^{-1} u_1), \\ \Phi_{23} &= \tfrac{1}{\gamma^2+\delta^2}(-2\gamma u_2 \partial^{-1} u_4 - 2\gamma u_4 \partial^{-1} u_2 + 2\delta u_2 \partial^{-1} u_2 - 2\delta u_4 \partial^{-1} u_4), \\ \Phi_{24} &= \tfrac{1}{\gamma^2+\delta^2}(-\delta\partial - 2\gamma u_2 \partial^{-1} u_3 - 2\gamma u_4 \partial^{-1} u_1 + 2\delta u_2 \partial^{-1} u_1 - 2\delta u_4 \partial^{-1} u_3); \end{aligned} \right. \tag{42}$$

$$\left\{ \begin{aligned} \Phi_{31} &= \tfrac{1}{\gamma^2+\delta^2}(-\delta\partial - 2\gamma u_1 \partial^{-1} u_4 - 2\gamma u_3 \partial^{-1} u_2 + 2\delta u_1 \partial^{-1} u_2 - 2\delta u_3 \partial^{-1} u_4), \\ \Phi_{32} &= \tfrac{1}{\gamma^2+\delta^2}(-2\gamma u_1 \partial^{-1} u_3 - 2\gamma u_3 \partial^{-1} u_1 + 2\delta u_1 \partial^{-1} u_1 - 2\delta u_3 \partial^{-1} u_3), \\ \Phi_{33} &= \tfrac{1}{\gamma^2+\delta^2}(\gamma\partial - 2\gamma u_1 \partial^{-1} u_2 + 2\gamma u_3 \partial^{-1} u_4 - 2\delta u_1 \partial^{-1} u_4 - 2\delta u_3 \partial^{-1} u_2), \\ \Phi_{34} &= \tfrac{1}{\gamma^2+\delta^2}(-2\gamma u_1 \partial^{-1} u_1 + 2\gamma u_3 \partial^{-1} u_3 - 2\delta u_1 \partial^{-1} u_3 - 2\delta u_3 \partial^{-1} u_1); \end{aligned} \right. \tag{43}$$

and

$$\begin{cases} \Phi_{41} = \frac{1}{\gamma^2+\delta^2}(2\gamma u_2\partial^{-1}u_4 + 2\gamma u_4\partial^{-1}u_2 - 2\delta u_2\partial^{-1}u_2 + 2\delta u_4\partial^{-1}u_4), \\ \Phi_{42} = \frac{1}{\gamma^2+\delta^2}(\delta\partial + 2\gamma u_2\partial^{-1}u_3 + 2\gamma u_4\partial^{-1}u_1 - 2\delta u_2\partial^{-1}u_1 + 2\delta u_4\partial^{-1}u_3), \\ \Phi_{43} = \frac{1}{\gamma^2+\delta^2}(2\gamma u_2\partial^{-1}u_2 - 2\gamma u_4\partial^{-1}u_4 + 2\delta u_2\partial^{-1}u_4 + 2\delta u_4\partial^{-1}u_2), \\ \Phi_{44} = \frac{1}{\gamma^2+\delta^2}(-\gamma\partial + 2\gamma u_2\partial^{-1}u_1 - 2\gamma u_4\partial^{-1}u_3 + 2\delta u_2\partial^{-1}u_3 + 2\delta u_4\partial^{-1}u_1). \end{cases} \tag{44}$$

Let's outline the process of computing this recursion operator using the recursion relations given in (16), (17) and (18). Assuming $\mathcal{K}^{\{m\}} = (\mathcal{K}_1^{\{m\}}, \mathcal{K}_2^{\{m\}}, \mathcal{K}_3^{\{m\}}, \mathcal{K}_4^{\{m\}})^T$, $m \geq 0$, we will specifically focus on the computation of the third component of $\mathcal{K}^{m+1}$:

$$\begin{aligned} &\mathcal{K}_3^{\{m+1\}} = \delta b^{\{m+2\}} + \gamma f^{\{m+2\}} \\ &= \tfrac{1}{\gamma^2+\delta^2}[\gamma\delta(b_x^{\{m+1\}} + 2u_1a^{\{m+1\}} - 2u_3e^{\{m+1\}}) + \delta^2(f_x^{\{m+1\}} + 2u_1e^{\{m+1\}} \\ &\quad +2u_3a^{\{m+1\}}) - \gamma\delta(b_x^{\{m+1\}} + 2u_1a^{\{m+1\}} - 2u_3e^{\{m+1\}}) + \gamma^2(f_x^{\{m+1\}} \\ &\quad +2u_1e^{\{m+1\}} + 2u_3a^{\{m+1\}})] \\ &= \tfrac{1}{\gamma^2+\delta^2}[\gamma\mathcal{K}_{3,x}^{\{m\}} - \delta\mathcal{K}_{1,x}^{\{m\}} + 2(\gamma u_1 + \delta u_3)(\delta a^{\{m+1\}} + \gamma e^{\{m+1\}}) \\ &\quad +2(\gamma u_3 - \delta u_1)(-\delta e^{\{m+1\}} + \gamma a^{\{m+1\}})]. \end{aligned}$$

On the other hand, we can have

$$\begin{aligned} &\delta a^{\{m+1\}} + \gamma e^{\{m+1\}} = -\partial^{-1}(u_1\mathcal{K}_4^{\{m\}} + u_2\mathcal{K}_3^{\{m\}} + u_3\mathcal{K}_2^{\{m\}} + u_4\mathcal{K}_1^{\{m\}}), \\ &-\delta e^{\{m+1\}} + \gamma a^{\{m+1\}} = -\partial^{-1}(u_1\mathcal{K}_2^{\{m\}} + u_2\mathcal{K}_1^{\{m\}} - u_3\mathcal{K}_4^{\{m\}} - u_4\mathcal{K}_3^{\{m\}}). \end{aligned}$$

This results in the third row of the recursion operator Φ, as defined by (43). The remaining rows can be obtained using a similar approach.

The recursion operator, defined by (41), (42), (43) and (44), incorporates two constant parameters, γ and δ, which are not simultaneously zero, highlighting the diverse nature of the recursion structure within the integrable hierarchy. Despite the nonlocal nature of the recursion operator, the locality of the isospectral flows ($k_{t_m} = 0$) is preserved. This preservation ensures that each flow within the hierarchy maintains the integrable structure, thereby guaranteeing that the resulting soliton equations remain solvable using inverse scattering techniques and other methods applicable to local equations.

Moreover, through detailed analysis, it becomes evident that J_1 and $J_2 = \Phi J_1$ constitute a Hamiltonian pair. Consequently, the soliton hierarchy (27) manifests the following bi-Hamiltonian structures [33]:

$$u_{t_m} = \mathcal{K}^{\{m\}} = J_1\frac{\delta\mathcal{H}^{\{m\}}}{\delta u} = J_2\frac{\delta\mathcal{H}^{\{m-1\}}}{\delta u}, \quad m \geq 1. \tag{45}$$

It can then be observed that the resulting Hamiltonian quantities commute under their respective Poisson brackets:

$$\{\mathcal{H}^{\{n_1\}}, \mathcal{H}^{\{n_2\}}\}_{J_i} = 0, \ n_1, n_2 \geq 0, \ i = 1, 2, \tag{46}$$

where

$$\{\mathcal{H}, \mathcal{I}\}_{J_i} = \int (\frac{\delta\mathcal{H}}{\delta u})^T J_i \frac{\delta\mathcal{I}}{\delta u}\, dx, \ i = 1, 2. \tag{47}$$

The relationships expressed in (39) and either (46) or (47) imply that all isospectral flows inherently possess infinitely many conserved quantities and symmetries. Moreover, leveraging the recursion and bi-Hamiltonian structures allows for effective computation and utilization of these conserved quantities and symmetries. This property is pivotal for the practical application and analysis of integrable models, ensuring that their solutions exhibit well-defined physical behavior and can be systematically studied.

In summary, the soliton hierarchy (27) demonstrates distinct bi-Hamiltonian structures, affirming Liouville integrability. Each model within this hierarchy possesses infinitely many commuting conserved quantities $\mathcal{H}^{\{n\}}{}_{n=0}^{\infty}$ and symmetries $\mathcal{K}^{\{n\}}{}_{n=0}^{\infty}$. The specific examples presented in (29), (30), (32), and (33) illustrate unique nonlinear coupled Liouville integrable models with bi-Hamiltonian structures, contributing significantly to current literature discussions (see, for instance, [42, 22, 23]).

4. Conclusion and discussions.

This research investigates integrable hierarchies and their connection to matrix eigenvalue problems formulated using zero curvature equations. The primary focus is on generating integrable models that exhibit bi-Hamiltonian structures, which are crucial for understanding the inherent dynamics of these systems.

Employing Laurent series solutions to solve the stationary zero curvature equation has proven to be a robust method, facilitating the exploration of integrability characteristics in the models under study. Additionally, applying the trace identity to the matrix isospectral eigenvalue problem offers deeper insights into the bi-Hamiltonian structures inherent in these systems.

The concrete examples presented in this research illustrate specific coupled systems of nonlinear integrable models, both uncoupled and combined. These examples not only showcase the practical application of the theoretical framework discussed earlier but also underscore the integrability and complex structure of the resulting equations.

Exploring the structures of explicit soliton solutions within resulting integrable models is a focal point, employing advanced methodologies in soliton theory such as the Zakharov-Shabat dressing method [6], Riemann-Hilbert technique [37], determinant approach [3], and Darboux transformation [36, 11, 24, 32]. Additionally, significant solutions such as breather, kink, anti-kink, lump, rogue wave solutions, and mixed solutions can be derived through specific reductions of solitons (see, e.g., [5, 38, 35, 46, 43]). Novel reduced integrable equations involving reflection points can also arise from nonlocal reduced matrix eigenvalue problems under similarity transformations (see, for example, [25, 26, 27]).

Increasing the number of dependent variables in the spatial spectral matrix can indeed lead to the generation of larger integrable models (see, for instance, [44, 12, 10]). These expansions result in systems with richer dynamics and more complex interactions between variables (see, for example, [28, 29]). Incorporating additional variables can lead to new phenomena such as the emergence of hereditary recursion operators and higher-order matrix eigenvalue problems [30]. However, it is important to note that as the number of dependent variables increases, the complexity of the resulting equations also intensifies, posing challenges in their analysis and comprehension. Nonetheless, the exploration of larger integrable models remains a productive area of research, offering profound insights into the fundamental principles governing nonlinear dynamics and integrability in mathematical physics.

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