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A COMBINED KAUP-NEWELL TYPE INTEGRABLE HAMILTONIAN HIERARCHY WITH FOUR POTENTIALS AND A HEREDITARY RECURSION OPERATOR

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Abstract. We aim to study a Kaup-Newell type matrix eigenvalue problem with four potentials, generated from a specific matrix Lie algebra, and compute an associated soliton hierarchy and its hereditary recursion operator and bi-Hamiltonian structure. The Liouville integrability of the resulting soliton hierarchy is a consequence of the bi-Hamiltonian structure. An illustrative example is explicitly worked out, providing a novel integrable model consisting of combined derivative nonlinear Schrödinger equations involving two arbitrary constants.

1. Introduction.

Integrable models comes in hierarchies which possess hereditary recursion operators [1, 2] and they are associated with Lax pairs of matrix eigenvalue problems [3]. Matrix eigenvalue problems are the key objects, which are primarily used to solve Cauchy problems by establishing inverse scattering transforms. From an intergrabiliy pespective, Hamiltonian structures, which connect symmetries with conserved quantities, are important and can also be generated from Lax pairs. Integrable models have diverse applications in physical sciences and engineering, such as water waves, nonlinear optics and quantum mechanics.

Among typical examples of integrable hierarchies are the Ablowitz-Kaup-Newell-Segur hierarchy [4] and its diverse hierarchies of integrable couplings [6]. Matrix Lie algebras provide a strong basis for studying integrable models within the zero curvature formulation [5, 6, 7]. The first and most important is to find spectral matrices while constructing integrable models. In this paper, we would like to propose a novel Kaup-Newell type 4×4 matrix eigenvalue problem and compute an associated integrable hierarchy.

The zero curvature formulation can be stated as follows (see [7, 8] for details). We denote a column potential vector by $u = (u_1, \cdots, u_q)^T$ and the spectral parameter

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by λ. Let $\tilde{g}$ be a given loop matrix algebra with the loop parameter λ. A matrix F_0 in $\tilde{g}$ is called to be pseudo-regular, if it satisfies

$$\mathrm{Im}\,\mathrm{ad}_{F_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{F_0} = \tilde{g},\ [\mathrm{Ker}\,\mathrm{ad}_{F_0}, \mathrm{Ker}\,\mathrm{ad}_{F_0}] = 0, \tag{1.1}$$

where ad_{F_0} denotes the adjoint action of F_0 on $\tilde{g}$. We always take one pseudo-regular matrix F_0 and q linear independent matrices $F_1, \cdots, F_q$ in $\tilde{g}$ to formulate a spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = F_0(\lambda) + u_1 F_1(\lambda) + \cdots + u_q F_q(\lambda). \tag{1.2}$$

Then try to solve the stationary zero curvature equation

$$Y_x = [\mathcal{M}, Y], \tag{1.3}$$

by assuming a solution Y of a Laurent series form $Y = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}$.

To determine the other parts of Lax pairs, we take an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (\lambda^m Y)_+ + \Delta_m = \sum_{n=0}^{m} \lambda^{m-n} Y^{[n]} + \Delta_m,\ m \geq 0, \tag{1.4}$$

where $\Delta_m \in \tilde{g},\ m \geq 0$, such that the zero curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}^{[m]}_x + [\mathcal{M}, \mathcal{N}^{[m]}] = 0,\ m \geq 0, \tag{1.5}$$

produce a hierarchy of integrable models:

$$u_{t_m} = X^{[m]} = X^{[m]}(u),\ m \geq 0. \tag{1.6}$$

The equations in (1.5) are the compatibility conditions of the spatial and temporal matrix eigenvalue problems:

$$\varphi_x = \mathcal{M}\varphi,\ \varphi_{t_m} = \mathcal{N}^{[m]}\varphi,\ m \geq 0. \tag{1.7}$$

During the process of finding a solution, one goes with a trial and error strategy.

The last step is to find a bi-Hamiltonian formulation for the resulting hierarchy (1.6), via computing a recursion operator and applying the so-called trace identity:

$$\frac{\delta}{\delta u} \int \mathrm{tr}\big(Y \frac{\partial \mathcal{M}}{\partial \lambda}\big)\, dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \mathrm{tr}\big(Y \frac{\partial \mathcal{M}}{\partial u}\big), \tag{1.8}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u, and κ is a constant, which can be computed from the solution Y. It finally follows that every member in the hierarchy has a bi-Hamiltonian formulation with a hereditary recursion operator and thus Liouville integrability (see, e.g., [7, 8, 9]).

Abundant hierarchies of Liouville integrable models are available in the literature [4]-[20]. One-component integrable hierarchies contain the Korteg-de Vries hierarchy, the nonlinear Schrödinger hierarchy and the modified Korteweg-de Vries hierarchy [1, 2]. The case of two components is most popular and the well-known examples are the Ablowitz-Kaup-Newell-Segur integrable hierarchy [4], the Heisenberg integrable hierarchy [21], the Kaup-Newell integrable hierarchy [22] and the Wadati-Konno-Ichikawa integrable hierarchy [23]. All those hierarchies are generated from 2×2 matrix eigenvalue problems. The case of higher-order spectral matrices create a high degree of difficulty.

In this paper, we aim to propose a specific 4×4 spectral matrix and generate a hierarchy of four-component Liouville integrable models within the zero curvature formulation. A hereditary recursion operator and a bi-Hamiltonian formulation are determined to show the Liouville integrability for the resulting soliton hierarchy.

An illustrative example, consisting of generalized combined integrable derivative nonlinear Schrödinger equations, is presented. A conclusion and concluding remarks are given in the last section.

2. **A four-component integrable hierarchy.**

A special matrix Lie algebra is our basis. Let δ be an arbitrary real number, and T be a square matrix of order $r \in \mathbb{N}$ such that

$$T^{-1} = T. \tag{2.1}$$

We define a set $\tilde{g}$ of block matrices as

$$\tilde{g} = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r\times 2r} \middle| A_4 = TA_1T^{-1},\ A_3 = \delta TA_2T^{-1} \right\}. \tag{2.2}$$

It is easy to see that this forms a matrix Lie algebra under the matrix commutator $[A, B] = AB - BA$. We will use this Lie algebra with $r = 2$, $\delta = 1$ and

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.3}$$

to formulate a specific 4×4 spectral matrix in our discussion.

We consider the case of four components. Let α_1 and α_2 be two arbitrary real numbers, and Let $u = u(x, t) = (u_1, u_2, u_3, u_4)^T$ be a column vector with four potentials. α_1 and α_2, two arbitrary real numbers, which satisfies

$$\alpha = \alpha_1 - \alpha_2 \neq 0. \tag{2.4}$$

Based on recent studies on matrix eigenvalue problems involving four potentials (see, e.g., [24, 25, 26] and [27, 28] for examples of matrix eigenvalue problems of arbitrary-order and fourth-order, respectively), we would like to introduce a matrix eigenvalue problem of the form:

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u, \lambda)\varphi,\ \mathcal{M} = \begin{bmatrix} \alpha_1\lambda^2 & \lambda u_1 & \lambda u_2 & 0 \\ \lambda u_3 & \alpha_2\lambda^2 & 0 & \lambda u_4 \\ \lambda u_4 & 0 & \alpha_2\lambda^2 & -\lambda u_3 \\ 0 & \lambda u_2 & -\lambda u_1 & \alpha_1\lambda^2 \end{bmatrix}, \tag{2.5}$$

where λ, as always, denotes the spectral parameter. This spectral matrix $\mathcal{M}$ is built from the above matrix Lie algebra $\tilde{g}$, and it is a kind of generalization of the 2×2 matrix Kaup-Newell eigenvalue problem [22]. Importantly, associated with this eigenvalue problem, an integrable hierarchy of bi-Hamiltonian equations can be generated. All equations in the hierarchy involve two arbitrary constants and possess particular combined structures.

To construct an associated integrable hierarchy, the first step is to solve the corresponding stationary zero curvature equation (1.3). We begin with

$$Y = \begin{bmatrix} a & b & e & f \\ c & -a & f & g \\ g & -f & -a & -c \\ -f & e & -b & a \end{bmatrix} = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}. \tag{2.6}$$

The reason to take this form is that with the spectral matrix $\mathcal{M}$, an arbitrary matrix in $\tilde{g}$ will generate a commutator matrix of the above form. Now based on

(2.5), we see that the corresponding stationary zero curvature equation (1.3) leads equivalently to

$$\begin{cases} a_x = \lambda c u_1 + \lambda g u_2 - \lambda b u_3 - \lambda e u_4, \\ b_x = \alpha\lambda^2 b - 2\lambda a u_1 - 2\lambda f u_2, \\ c_x = -\alpha\lambda^2 c + 2\lambda a u_3 - 2\lambda f u_4, \end{cases} \tag{2.7}$$

$$\begin{cases} e_x = \alpha\lambda^2 e + 2\lambda f u_1 - 2\lambda a u_2, \\ g_x = -\alpha\lambda^2 g + 2\lambda f u_3 + 2\lambda a u_4, \\ f_x = \lambda g u_1 - \lambda c u_2 + \lambda e u_3 - \lambda b u_4. \end{cases} \tag{2.8}$$

In order to compute a solution Y recursively, we assume that the basic objects of Y are taken as follows:

$$\begin{cases} a = \sum_{n\ge 0} \lambda^{-2n} a^{[n]},\ b = \sum_{n\ge 0} \lambda^{-2n-1} b^{[n]},\ c = \sum_{n\ge 0} \lambda^{-2n-1} c^{[n]}, \\ e = \sum_{n\ge 0} \lambda^{-2n-1} e^{[n]},\ f = \sum_{n\ge 0} \lambda^{-2n} f^{[n]},\ g = \sum_{n\ge 0} \lambda^{-2n-1} g^{[n]}. \end{cases} \tag{2.9}$$

Obviously, we can have two crucial relations:

$$\begin{cases} -\alpha\lambda a_x = u_3 b_x + u_1 c_x + u_4 e_x + u_2 g_x, \\ -\alpha\lambda f_x = u_4 b_x - u_2 c_x - u_3 e_x + u_1 g_x, \end{cases} \tag{2.10}$$

which enable us to get the recursion relations successfully. In this way, we can see that the above equations in (2.7) and (2.8) yield the two initial conditions:

$$\begin{cases} a_x^{[0]} = u_1 c^{[0]} + u_2 g^{[0]} - u_3 b^{[0]} - u_4 e^{[0]}, \\ f_x^{[0]} = u_1 g^{[0]} - u_2 c^{[0]} + u_3 e^{[0]} - u_4 b^{[0]}, \end{cases} \tag{2.11}$$

and the recursion relations which determine the Laurent series solution:

$$\begin{cases} a_x^{[n+1]} = -\frac{1}{\alpha}(u_3 b_x^{[n]} + u_1 c_x^{[n]} + u_4 e_x^{[n]} + u_2 g_x^{[n]}), \\ f_x^{[n+1]} = -\frac{1}{\alpha}(u_4 b_x^{[n]} - u_2 c_x^{[n]} - u_3 e_x^{[n]} + u_1 g_x^{[n]}), \end{cases} \tag{2.12}$$

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha}(b_x^{[n]} + 2u_1 a^{[n+1]} + 2u_2 f^{[n+1]}), \\ c^{[n+1]} = \frac{1}{\alpha}(-c_x^{[n]} + 2u_3 a^{[n+1]} - 2u_4 f^{[n+1]}), \end{cases} \tag{2.13}$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha}(e_x^{[n]} - 2u_1 f^{[n+1]} + 2u_2 a^{[n+1]}), \\ g^{[n+1]} = \frac{1}{\alpha}(-g_x^{[n]} + 2u_3 f^{[n+1]} + 2u_4 a^{[n+1]}), \end{cases} \tag{2.14}$$

where $n \ge 0$. Further solving (2.11), we obtain the initial data,

$$\begin{cases} b^{[0]} = \beta u_1 + \gamma u_2,\ c^{[0]} = \beta u_3 - \gamma u_4, \\ e^{[0]} = \beta u_2 - \gamma u_1,\ g^{[0]} = \beta u_4 + \gamma u_3, \\ a^{[0]} = \text{const.},\ f^{[0]} = \text{const.}, \end{cases} \tag{2.15}$$

where β and γ are two arbitrary constants. In our computation, we choose the zero constants of integration:

$$a^{[n]}|_{u=0} = 0,\ f^{[n]}|_{u=0} = 0,\ n \ge 1, \tag{2.16}$$

for the sake of brevity. The initial values for $a^{[0]}$ and $f^{[0]}$ don't affect all other coefficients in a Laurent series solution, but the two constants β and γ create the diversity of associated integrable models. Now, a direct computation tells that

$$\begin{cases} a^{[1]} = -\frac{1}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2], \\ f^{[1]} = -\frac{1}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2], \end{cases}$$

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}\{\beta u_{1,x} + \gamma u_{2,x} - \frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1 \\ \qquad -\frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_2\}, \\ c^{[1]} = \frac{1}{\alpha}\{-\beta u_{3,x} + \gamma u_{4,x} - \frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3 \\ \qquad +\frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_4\}, \end{cases}$$

$$\begin{cases} e^{[1]} = \frac{1}{\alpha}\{-\gamma u_{1,x} + \beta u_{2,x} + \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_1 \\ \qquad -\frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\}, \\ g^{[1]} = \frac{1}{\alpha}\{-\gamma u_{3,x} - \beta u_{4,x} - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_3 \\ \qquad -\frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\}. \end{cases}$$

Observing the above recursion relations carefully, one can make the following choice for the temporal matrix eigenvalue problems:

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(u,\lambda)\varphi,\ \mathcal{N}^{[m]} = \lambda(\lambda^{2m+1}Y)_+,\ m \geq 0, \tag{2.17}$$

where, as usual, the subscript + denotes the polynomial part of λ. This will successfully transform the zero curvature equations in (1.5) into a hierarchy of integrable models with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) = (b_x^{[m]}, e_x^{[m]}, c_x^{[m]}, g_x^{[m]})^T,\ m \geq 0, \tag{2.18}$$

or more concretely,

$$u_{1,t_m} = b_x^{[m]},\ u_{2,t_m} = e_x^{[m]},\ u_{3,t_m} = c_x^{[m]},\ u_{4,t_m} = g_x^{[m]},\ m \geq 0. \tag{2.19}$$

The first nonlinear example in this hierarchy is the model of combined integrable derivative nonlinear Schrödinger equations:

$$\begin{cases} u_{1,t_1} = \frac{1}{\alpha}(\beta u_{1,xx} + \gamma u_{2,xx}) - \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1\}_x \\ \qquad -\frac{2}{\alpha^2}\{(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_2\}_x, \\ u_{2,t_1} = \frac{1}{\alpha}(\gamma u_{1,xx} + \beta u_{2,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_1\}_x \\ \qquad -\frac{2}{\alpha^2}\{(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\}_x, \\ u_{3,t_1} = -\frac{1}{\alpha}(\beta u_{3,xx} + \gamma u_{4,xx}) - \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3\}_x \\ \qquad -\frac{2}{\alpha^2}\{(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_4\}_x, \\ u_{4,t_2} = -\frac{1}{\alpha}(\gamma u_{3,xx} + \beta u_{4,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_3\}_x \\ \qquad -\frac{2}{\alpha^2}\{(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\}_x. \end{cases} \tag{2.20}$$

This system provides a coupled integrable model with four components, which enlarges the category of coupled integrable models of nonlinear Schrödinger type equations (see, e.g., [25, 29, 30]). One interesting phenomenon is that each equation contains a linear combination of two derivative terms of the second order, and thus, we call such a model a combined model.

Two special cases, $\beta = 0$ and $\gamma = 0$, in the resulting hierarchy are interesting. They produce reduced hierarchies of uncombined integrable models.

Let us first take $\alpha = \beta = 1$ and $\gamma = 0$ in the model (2.20), we get a coupled integrable derivative nonlinear Schrödinger model:

$$\begin{cases} u_{1,t_1} = u_{1,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 + u_2u_3)u_2]_x, \\ u_{2,t_1} = u_{2,xx} - 2[(u_1u_4 + u_2u_3)u_1 + (u_1u_3 + u_2u_4)u_2]_x, \\ u_{3,t_1} = -u_{3,xx} - 2[(u_1u_3 + u_2u_4)u_3 + (u_1u_4 + u_2u_3)u_4]_x, \\ u_{4,t_1} = -u_{4,xx} - 2[(u_1u_4 + u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x. \end{cases} \tag{2.21}$$

Let us second take $\alpha = \gamma = 1$ and $\beta = 0$ in the model (2.20), we obtain another coupled integrable derivative nonlinear Schrödinger model:

$$\begin{cases} u_{1,t_1} = u_{2,xx} - 2[(u_1u_4 + u_2u_3)u_1 + (u_1u_3 + u_2u_4)u_2]_x, \\ u_{2,t_1} = u_{1,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 + u_2u_3)u_2]_x, \\ u_{3,t_1} = -u_{4,xx} - 2[(u_1u_4 + u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x, \\ u_{4,t_1} = -u_{3,xx} - 2[(u_1u_3 + u_2u_4)u_3 + (u_1u_4 + u_2u_3)u_4]_x. \end{cases} \tag{2.22}$$

It is worth pointing out that the resulting two models just exchange the first component with the second component and the third component with the fourth component in the vector fields on the right hand sides. Interestingly, those two models still commute with each other, i.e., one is a symmetry of the other.

3. **Recursion operator and bi-Hamiltonian formulation.** Let us show the Liouville integrability of the soliton hierarchy (2.19). To this end, we try to determine a hereditary recursion operator and furnish a Hamiltonian formulation by using the trace identity (1.8) in the case of the spatial matrix eigenvalue problem (2.5).

Noting the expression of the Laurent series solution Y by (2.6), we can easily compute that

$$\begin{cases} \operatorname{tr}\big(Y\dfrac{\partial \mathcal{M}}{\partial \lambda}\big) = 2(2\alpha\lambda a + bu_3 + cu_1 + eu_4 + gu_2), \\ \operatorname{tr}\big(Y\dfrac{\partial \mathcal{M}}{\partial u}\big) = 2(\lambda c, \lambda g, \lambda b, \lambda e)^T, \end{cases} \tag{3.1}$$

and then an application of the trace identity tells

$$\begin{aligned} &\frac{\delta}{\delta u}\int \lambda^{-2n-1}(2\alpha a^{[n+1]} + u_3b^{[n]} + u_4e^{[n]} + u_1c^{[n]} + u_2g^{[n]})\,dx \\ &= \lambda^{-\kappa}\frac{\partial}{\partial\lambda}\lambda^{\kappa-2n}(c^{[n]}, g^{[n]}, b^{[n]}, e^{[n]})^T,\ n \ge 0. \end{aligned} \tag{3.2}$$

Checking with $n = 1$ leads to $\kappa = 0$, and accordingly, one gets

$$\frac{\delta}{\delta u}\mathcal{H}^{[n]} = (c^{[n]}, g^{[n]}, b^{[n]}, e^{[n]})^T,\ n \ge 0, \tag{3.3}$$

where the Hamiltonian functionals are determined by

$$\begin{cases} \mathcal{H}^{[0]} \\ = \int \frac{1}{2}[u_1(\beta u_3 - \gamma u_4) + u_2(\beta u_4 + \gamma u_3) + u_3(\beta u_1 + \gamma u_2) + u_4(\beta u_2 - \gamma u_1)]\,dx, \\ \mathcal{H}^{[n]} = -\displaystyle\int \frac{1}{2n}(2\alpha a^{[n+1]} + u_3b^{[n]} + u_1c^{[n]} + u_4e^{[n]} + u_2g^{[n]})\,dx,\ n \ge 1. \end{cases} \tag{3.4}$$

This enables us to produce a Hamiltonian formulation for the soliton hierarchy (2.19):

$$u_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta u},\ m \geq 0, \tag{3.5}$$

where the Hamiltonian operator J_1 is given by

$$J_1 = \left[\begin{array}{c|c} 0 & \begin{matrix} \partial & 0 \\ 0 & \partial \end{matrix} \\ \hline \begin{matrix} \partial & 0 \\ 0 & \partial \end{matrix} & 0 \end{array}\right], \tag{3.6}$$

and the functionals $\mathcal{H}^{[m]}$ are defined by (3.4). A direct consequence of such a Hamiltonian is an interrelation $Z = J_1 \frac{\delta \mathcal{H}}{\delta u}$ between a symmetry Z and a conserved functional $\mathcal{H}$ of each model in the hierarchy.

As always, the characteristic commutative property for the vector fields $X^{[n]}$

$$[\![X^{[n_1]}, X^{[n_2]}]\!] = X^{[n_1]\prime}(u)[X^{[n_2]}] - X^{[n_2]\prime}(u)[X^{[n_1]}] = 0,\ n_1, n_2 \geq 0, \tag{3.7}$$

follows from an algebra of Lax operators:

$$[\![\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}]\!] = \mathcal{N}^{[n_1]\prime}(u)[X^{[n_2]}] - \mathcal{N}^{[n_2]\prime}(u)[X^{[n_1]}] + [\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}] = 0,\ n_1, n_2 \geq 0. \tag{3.8}$$

This can directly be derived from the relation between the isospectral zero curvature equations (see [35] for details).

Furthermore, from the recursion relation $X^{[m+1]} = \Phi X^{[m]}$, we can work out a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ [31] for the soliton hierarchy (2.19), which reads as follows:

$$\begin{cases} \Phi_{11} = \frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_3 + \partial u_2 \partial^{-1} u_4),\ \Phi_{12} = -\frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_4 - \partial u_2 \partial^{-1} u_3), \\ \Phi_{13} = -\frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_1 - \partial u_2 \partial^{-1} u_2),\ \Phi_{14} = -\frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_2 + \partial u_2 \partial^{-1} u_1); \end{cases} \tag{3.9}$$

$$\begin{cases} \Phi_{21} = \frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_4 - \partial u_2 \partial^{-1} u_3),\ \Phi_{22} = \frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_3 + \partial u_2 \partial^{-1} u_4), \\ \Phi_{23} = -\frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_2 + \partial u_2 \partial^{-1} u_1),\ \Phi_{24} = \frac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_1 - \partial u_2 \partial^{-1} u_2); \end{cases} \tag{3.10}$$

$$\begin{cases} \Phi_{31} = -\frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_3 - \partial u_4 \partial^{-1} u_4),\ \Phi_{32} = -\frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_4 + \partial u_4 \partial^{-1} u_3), \\ \Phi_{33} = -\frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_1 + \partial u_4 \partial^{-1} u_2),\ \Phi_{34} = -\frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_2 - \partial u_4 \partial^{-1} u_1); \end{cases} \tag{3.11}$$

$$\begin{cases} \Phi_{41} = -\frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_4 + \partial u_4 \partial^{-1} u_3),\ \Phi_{42} = \frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_3 - \partial u_4 \partial^{-1} u_4), \\ \Phi_{43} = \frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_2 - \partial u_4 \partial^{-1} u_1),\ \Phi_{44} = -\frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_1 + \partial u_4 \partial^{-1} u_2). \end{cases} \tag{3.12}$$

The hereditary property of Φ means [32] that it needs to satisfy

$$L_{\Phi X}\Phi = \Phi L_X \Phi, \tag{3.13}$$

where X is an arbitrary vector field and the Lie derivative $L_X \Phi$ is defined by

$$(L_X \Phi)S = \Phi[\![X, S]\!] - [\![X, \Phi S]\!]. \tag{3.14}$$

Note that an operator $\Psi = \Psi(x, t, u, u_x, \cdots)$ is a recursion operator of a given evolution equation $u_t = X(u)$ if and only if Ψ satisfies

$$\frac{\partial \Psi}{\partial t} + L_X \Psi = 0. \tag{3.15}$$

It is easy to check that $L_{X^{[0]}}\Phi = 0$, and based on this, we can compute that

$$L_{X^{[m]}}\Phi = L_{\Phi X^{[m-1]}}\Phi = \Phi L_{X^{[m-1]}}\Phi = \cdots = \Phi^m L_{X^{[0]}}\Phi = 0,\ m \ge 1. \tag{3.16}$$

Consequently, Φ is a common recursion operator for all models in the hierarchy (2.19). There are also symbolic algorithms for computing recursion operators of nonlinear partial differential equations directly (see, e.g., [33]).

With some direct analysis, we can further observe that J_1 and $J_2 = \Phi J_1$ constitute a Hamiltonian pair. Namely, an arbitrary linear combination J of J_1 and J_2 is again Hamiltonian, since it satisfies

$$\int (Z^{[1]})^T J'(u)[JZ^{[2]}]Z^{[3]}\,dx + \mathrm{cycle}(Z^{[1]}, Z^{[2]}, Z^{[3]}) = 0, \tag{3.17}$$

where $Z^{[i]}$'s are arbitrary vector fields. Accordingly, the hierarchy (2.19) possesses a bi-Hamiltonian structure [34]:

$$u_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta u} = J_2 \frac{\delta \mathcal{H}^{[m-1]}}{\delta u},\ \ m \ge 1. \tag{3.18}$$

It then follows that the associated Hamiltonian functionals commute with each other under the corresponding two Poisson brackets [7]:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = \int (\frac{\delta \mathcal{H}^{[n_1]}}{\delta u})^T J_1 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u}\,dx = 0,\ n_1, n_2 \ge 0, \tag{3.19}$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = \int (\frac{\delta \mathcal{H}^{[n_1]}}{\delta u})^T J_2 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u}\,dx = 0,\ n_1, n_2 \ge 0. \tag{3.20}$$

The bi-Hamiltonian formulation also implies the hereditariness of the recursion operator Φ.

To conclude, each model in the hierarchy (2.19) is Liouville integrable and possesses infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^\infty$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^\infty$. One particular illustrative integrable model is the system in (2.20), which adds to the existing category of nonlinear combined Liouville integrable Hamiltonian models with four components.

4. **Concluding remarks.** From a specific 4×4 matrix eigenvalue problem, we have generated a hierarchy of four-component Liouville integrable models within the zero curvature formulation. The success lies at defining a particular Laurent series solution of the corresponding stationary zero curvature equation recursively. The resulting integrable hierarchy possesses a hereditary recursion operator and a bi-Hamiltonian structure, and thus it is Liouville integrable.

It would be very interesting to find what kind of mathematical structures of soliton solutions there could exist in the obtained integrable models. Various powerful and effective approaches are available for use, which include the Riemann-Hilbert technique [36], the Darboux transformation [37, 38, 39], the Zakharov-Shabat dressing method [40], and the determinant approach [41]. In addition to solitons, lump, kink, breather and rogue wave solutions, particularly their interaction solutions (see, e.g., [42]-[49]), are of much interest, and it is possible to compute them from soliton solutions by conducting wave number reductions. Nonlocal reduced integrable models have become another hot topic in the study of integrable models. The key is to conduct nonlocal group reductions or similarity transformations for matrix eigenvalue problems (see, e.g., [50, 51, 52]). Solitons in the nonlocal case are significantly important in mathematics as well as physics.

Integrable models are of great interest, and they appear in various areas of mathematical physics, including classical mechanics, quantum mechanics and statistical mechanics. The mathematical solvability of those models provides a unique window into the underlying principles that govern the dynamics of complex nonlinear physical systems.

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REFERENCES

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Studies in Applied Mathematics, 4, SIAM, Philadelphia, PA, 1981.

[2] A. Das, *Integrable Models*, World Scientific, Teaneck, NJ, 1989.

[3] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.*, **21** (1968), 467-490.

[4] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier Analysis for nonlinear problems, *Stud. Appl. Math.*, **53** (1974), 249-315.

[5] V. Drinfel'd and V. V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, *J. Sov. Math.*, **30** (1985), 1975-2036.

[6] W. X. Ma, Integrable couplings and matrix loop algebras, in: *Nonlinear and Modern Mathematical Physics* (ed. W. X. Ma and D. Kaup), AIP Conference Proceedings 1562, American Institute of Physics, Melville, NY, 2013.

[7] G. Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy, *J. Phys. A: Math. Gen.*, **22** (1989), 2375-2392.

[8] W. X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, *Chin. Ann. Math. Ser. A*, **13** (1992), 115-123.

[9] C. S. Liu, How many first integrals imply integrability in infinite-dimensional Hamilton system, *Rep. Math. Phys.*, **67** (2011), 109-123.

[10] M. Antonowicz and A. P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, *Phys. D*, **28** (1987), 345-357.

[11] T. Xia, F. Yu and Y. Zhang, The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions, *Phys. A*, **343** (2004), 238-246.

[12] W. X. Ma, A soliton hierarchy associated with so(3,$\mathbb{R}$), *Appl. Math. Comput.*, **220** (2013), 117-122.

[13] S. Manukure, Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints, *Commun. Nonlinear Sci. Numer. Simul.*, **57** (2018), 125-135.

[14] S. Shen, C. Li, Y. Jin and W. X. Ma, Completion of the Ablowitz-Kaup-Newell-Segur integrable coupling, *J. Math. Phys.*, **59** (2018), 103503.

[15] T. Liu and T. Xia, Multi-component generalized Gerdjikov-Ivanov integrable hierarchy and its Riemann-Hilbert problem, *Nonlinear Anal. Real World Appl.*, **68** (2022), 103667.

[16] V. S. Gerdjikov, Nonlinear evolution equations related to Kac-Moody algebras $A_r^{(1)}$: Spectral aspects, *Turkish J. Math.*, **46** (2022), 1828-1844.

[17] H. Wang and Y. Zhang, Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations, *J. Comput. Appl. Math.*, **420** (2023), 114812.

[18] W. X. Ma, AKNS type reduced integrable hierarchies with Hamiltonian formulations, *Rom. J. Phys.*, **68** (2023), 116.

[19] J. Y. Yang and W. X. Ma, Four-component Liouville integrable models and their bi-Hamiltonian formulations, *Rom. J. Phys.*, **69** (2024), 101.

[20] W. X. Ma, A four-component hierarchy of combined integrable equations with bi-Hamiltonian formulations, *Appl. Math. Lett.*, **153** (2024), 109025.

[21] L. A. Takhtajan, Integration of the continuous Heisenberg spin chain through the inverse scattering method, *Phys. Lett. A*, **64** (1977), 235-237.

[22] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.*, **19** (1978), 798-801.

[23] M. Wadati, K. Konno, and Y. H. Ichikawa, New integrable nonlinear evolution equations, *J. Phys. Soc. Jpn.*, **47** (1979), 1698-1700.

[24] W. X. Ma, A Liouville integrable hierarchy with four potentials and its bi-Hamiltonian structure, *Rom. Rep. Phys.*, **75** (2023), 115.

[25] W. X. Ma, Four-component integrable hierarchies of Hamiltonian equations with $(m+n+2)$th-order Lax pairs, *Theoret. Math. Phys.*, **216** (2023), 1180-1188.

[26] W. X. Ma, Four-component integrable hierarchies and their Hamiltonian structures, *Commun. Nonlinear Sci. Numer. Simul.*, **126** (2023), 107460.

[27] Y.-F. Zhang, A few expanding integrable models, Hamiltonian structures and constrained flows, *Commun. Theor. Phys.*, **55** (2011), 273-290.

[28] Zhaqilao, A generalized AKNS hierarchy, bi-Hamiltonian structure, and Darboux transformation, *Commun. Nonlinear Sci. Numer. Simul.*, **17** (2012), 2319-2332.

[29] W. X. Ma, Novel Liouville integrable Hamiltonian models with six components and three signs, *Chin. J. Phys.*, **86** (2023), 292-299.

[30] W. X. Ma, A combined Liouville integrable hierarchy associated with a fourth-order matrix spectral problem, *Commun. Theor. Phys.*, **76** (2024), 075001..

[31] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D*, **4** (1981), 47-66.

[32] B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Anal. Theory, Methods Appl.*, **3** (1979), 849-862.

[33] D. E. Baldwin and W. Hereman, A symbolic algorithm for computing recursion operators of nonlinear partial differential equations, *Int. J. Comput. Math.*, **87** (2010), 1094-1119.

[34] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.*, **19** (1978), 1156-1162.

[35] W. X. Ma, The algebraic structure of zero curvature representationn and application to coupled KdV systems, *J. Phys. A: Math. Gen.*, **26** (1993), 2573-2582.

[36] S. Novikov, S. V. Manakov, L. P. Pitaevskiĭ and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Consultantn Bureau, New York, 1984.

[37] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer-Verlag, Berlin, 1991.

[38] X. Geng, R. Li and B. Xue, A vector general nonlinear Schrödinger equation with $(m + n)$ components, *J. Nonlinear Sci.*, **30** (2020), 991-1013.

[39] W. X. Ma, Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations, *Chaos Solitons Fractals*, **180** (2024), 114539.

[40] E. V. Doktorov and S. B. Leble, *A Dressing Method in Mathematical Physics*, Springer, Dordrecht, 2007.

[41] T. Aktosun, T. Busse, F. Demontis and C. van der Mee, Symmetries for exact solutions to the nonlinear Schrödinger equation, *J. Phys. A: Math. Theoret.*, **43** (2010), 025202.

[42] L. Cheng, Y. Zhang and M.-J. Lin, Lax pair and lump solutions for the (2+1)-dimensional DJKM equation associated with bilinear Bäcklund transformations, *Anal. Math. Phys.*, **9** (2019), 1741-1752.

[43] T. A. Sulaiman, A. Yusuf, A. Abdeljabbar and M. Alquran, Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation, *J. Geom. Phys.*, **169** (2021), 104347.

[44] A. Yusuf, T. A. Sulaiman, A. Abdeljabbam and M. Alquran, Breather waves, analytical solutions and conservation lawn using Lie-Bäcklund symmetries to the (2+1)-dimensional Chaffee-Infante equation, *J. Ocean Eng. Sci.*, **8** (2023), 145-151.

[45] S. Manukure, A. Chowdhury and Y. Zhou, Complexiton solutions to the asymmetric Nizhnik-Novikov-Veselov equation, *Internat. J. Modern Phys. B*, **33** (2019), 1950098.

[46] Y. Zhou, S. Manukure and M. McAnally, Lump and rogue wave solutions to a (2+1)-dimensional Boussinesq type equation, *J. Geom. Phys.*, **167** (2021), 104275.

[47] W. X. Ma, Lump waves in a spatial symmetric nonlinear dispersive wave model in (2+1)-dimensions, *Mathematics*, **11** (2023), 4664.

[48] S. Manukure and Y. Zhou, A study of lump and line rogue wave solutions to a (2+1)-dimensional nonlinear equation, *J. Geom. Phys.*, **167** (2021), 104274.

[49] S. X. Yang, Y. F. Wang and X. Zhang, Conservation laws, Darboux transformation and localized waves for the N-coupled nonautonomous Gross-Pitaevskii equations in the Bose-Einstein condensates, *Chaos Solitons Fractals*, **169** (2023), 113272.
[50] W. X. Ma, Reduced nonlocal integrable mKdV equations of type $(-\lambda, \lambda)$ and their exact soliton solutions, *Commun. Theoret. Phys.*, **74** (2022), 065002.
[51] W. X. Ma, Soliton solutions to constrained nonlocal integrable nonlinear Schrödinger hierarchies of type $(-\lambda, \lambda)$, *Int. J. Geom. Methods Mod. Phys.*, **20** (2023), 2350098.
[52] W. X. Ma, Integrable non-local nonlinear Schrödinger hierarchies of type $(-\lambda^*, \lambda)$ and soliton solutions, *Rep. Math. Phys.*, **92** (2023), 19-36.

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