DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **11**, Number **4**, August **2018**

doi:10.3934/dcdss.2018044

pp. **707–721**

CONSERVATION LAWS BY SYMMETRIES AND ADJOINT SYMMETRIES

Wen-Xiu Ma

Department of Mathematics, Zhejiang Normal University
Jinhua 321004, Zhejiang, China
Department of Mathematics and Statistics, University of South Florida Tampa
FL 33620, USA
College of Mathematics and Systems Science
Shandong University of Science and Technology
Qingdao 266590, Shandong, China
International Institute for Symmetry Analysis and Mathematical Modelling
Department of Mathematical Sciences, North-West University, Mafikeng Campus
Private Bag X2046, Mmabatho 2735, South Africa

Abstract. Conservation laws are fomulated for systems of differential equations by using symmetries and adjoint symmetries, and an application to systems of evolution equations is made, together with illustrative examples. The formulation does not require the existence of a Lagrangian for a given system, and the presented examples include computations of conserved densities for the heat equation, Burgers' equation and the Korteweg-de Vries equation.

1. **Introduction.** For systems of Euler-Lagrange equations, Noether's theorem shows that symmetries lead to conservation laws [3, 24]. The Lagrangian formulation of the systems under consideration is essential in presenting conservation laws from symmetries, and many physically important examples can be found in [3, 7, 23, 24]. Is it possible to extend such a connection between conservation laws and symmetries for systems of non-Euler-Lagrange equations? A definite answer has been given in the case of systems of discrete evolution equations [20]. We would, in this paper, like to formulate a similar theory of conservation laws for systems of differential equations and apply it to systems of evolution equations. More precisely, we want to exhibit that pairs of symmetries and adjoint symmetries lead to conservation laws for whatever systems of differential equations.

For totally nondegenerate systems of differential equations, conservation laws are classified by characteristic forms [24]. The characteristics, also called the multipliers [1], of conservation laws are adjoint symmetries, and thus, the existence of an adjoint symmetry is necessary for a totally nondegenerate system of differential equations to admit a conservation law. Adjoint symmetries generate conservation laws, when the variational derivatives of product functionals of the adjoint symmetries and the systems under consideration vanish [1]. For systems of evolution equations, functionals are conserved if and only if their variational derivatives are adjoint symmetries [20, 21]. Nonlinear self-adjointness has also been introduced on the basis of adjoint systems to construct conservation laws for nonlinearly self-adjoint

2010 *Mathematics Subject Classification.* 70S10, 37K05, 35Q51.

Key words and phrases. Conservation law, symmetry, evolution equation.

* Corresponding author: Wen-Xiu Ma.

systems of differential equations, where the nonlinear self-adjointness means that the second set of dependent variables in an adjoint system stands for an adjoint symmetry [8, 9]. We will directly utilize both symmetries and adjoint symmetries to produce conservation laws for systems of differential equations, without using any Lagrangian or extended Lagrangian.

This paper is organized as follows. In section 2, a formulation of conservation laws is furnished for systems of differential equations, regardless of the existence of a Lagrangian. In section 3, an application is made for systems of evolution equations, and in section 4, the resulting theory is used to compute conserved densities for the heat equation, Burgers' equation and the Korteweg-de Vries equation, along with many new conserved densities. Finally in section 5, a few of concluding remarks are given with some discussion.

2. **A formulation of conservation laws.** We will use a plain language to formulate the correspondence between conservation laws and pairs of symmetries and adjoint symmetries, although there is a geometrical language.

Let $x = (x^1, \cdots, x^p) \in \mathbb{R}^p$, $u = (u^1, \cdots, u^q)^T,\ u^i = u^i(x),\ 1 \le i \le q$, and

$$u^i_\alpha = D^\alpha u^i,\ D^\alpha = \partial_1^{\alpha_1} \cdots \partial_p^{\alpha_p},\ \partial_j = \frac{\partial}{\partial x^j},\ 1 \le i \le q,\ 1 \le j \le p, \tag{2.1}$$

for $\alpha = (\alpha_1, \cdots, \alpha_p)$ with non-negative integers $\alpha_i,\ 1 \le i \le p$. Assume that $\mathcal{A}$ denotes the space of all functions $f(x, u, \cdots, u^{(n)})$ with $n \ge 0$, where f is a smooth function of the involved variables and $u^{(n)}$ is the set of n-th order partial derivatives of u with respect to x, and $\mathcal{B}$ denotes the space of all smooth functions of x, u and derivatives of u with respect to x to some finite order. The locality means that a function $f(u)$ depends locally on u with respect to x, i.e., any value $(f(u))(x)$ is completely determined by the value of u in a sufficiently small region of x. Any functions in $\mathcal{A}$, particularly differential polynomial functions, are local. The space $\mathcal{B}$ contains nonlocal functions, and simple examples are $x^l \int_0^{x^k} \partial_j u^i \, dx^k,\ j \ne k$. We use $\mathcal{A}^r$ and $\mathcal{B}^r$ to denote the r-th order tensor products of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}^r = \{(f_1, \cdots, f_r)^T \,|\, f_i \in \mathcal{A},\ 1 \le i \le r\},\ \mathcal{B}^r = \{(f_1, \cdots, f_r)^T \,|\, f_i \in \mathcal{B},\ 1 \le i \le r\}. \tag{2.2}$$

Two functions $f_1, f_2 \in \mathcal{B}$ are said to be equivalent and denoted by $f_1 \sim f_2$, if there exist $g_i \in \mathcal{B},\ 1 \le i \le p$, such that

$$f_1 - f_2 = \sum_{i=1}^p \partial_i g_i.$$

This is an equavalence relation. Each equivalence class is called a functional and the class that $P \in \mathcal{B}$ belongs to is denoted by $\int P\, dx$. The inner products are defined by

$$\langle X, Y \rangle = \int \sum_{i=1}^s X_i Y_i \, dx,\ X = (X_1, \cdots, X_s)^T,\ Y = (Y_1, \cdots, Y_s)^T \in \mathcal{B}^s,\ s \ge 1,$$

and the adjoint operator $\Phi^* : \mathcal{B}^s \to \mathcal{B}^r$ of a linear operator $\Phi : \mathcal{B}^r \to \mathcal{B}^s$ is determined by

$$\langle X, \Phi^* Y \rangle = \langle \Phi X, Y \rangle,\ X \in \mathcal{B}^r,\ Y \in \mathcal{B}^s. \tag{2.3}$$

For any vector function $X = X(u) = (X_1, \cdots, X_r)^T \in \mathcal{A}^r$, we introduce its Gateaux operator

$$X' = X'(u) = (V_j(X_i))_{r\times q} = \begin{pmatrix} V_1(X_1) & V_2(X_1) & \cdots & V_q(X_1) \\ V_1(X_2) & V_2(X_2) & \cdots & V_q(X_2) \\ \vdots & \vdots & \ddots & \vdots \\ V_1(X_r) & V_2(X_r) & \cdots & V_q(X_r) \end{pmatrix}, \tag{2.4}$$

where

$$V_i(X_j) = \sum_{\alpha\geq 0} \frac{\partial X_j}{\partial u^i_\alpha} D^\alpha,$$

with $\alpha \geq 0$ meaning that all components $\alpha_i \geq 0,\ 1 \leq i \leq p$. The adjoint operator $(X^{'})^*$ of X' is given by

$$(X^{'})^* = (X^{'})^*(u) = (V_i^*(X_j))_{q\times r},\ V_i^*(X_j) = \sum_{\alpha\geq 0}(-D)^\alpha \frac{\partial X_j}{\partial u^i_\alpha}, \tag{2.5}$$

where $(-D)^\alpha = (-\partial_1)^{\alpha_1}\cdots(-\partial_p)^{\alpha_p}$.

Now, for a given integer $l \geq 1$, let us consider a system of differential equations

$$\Delta(u) = 0,\ \Delta \in \mathcal{A}^l. \tag{2.6}$$

Its linearized system and adjoint linearized system are defined by

$$\Delta'(u)\sigma(u) = 0,\ \sigma \in \mathcal{B}^q, \tag{2.7}$$

$$(\Delta^{'})^*(u)\rho(u) = 0,\ \rho \in \mathcal{B}^l, \tag{2.8}$$

respectively. Here Δ' and $(\Delta^{'})^*$ denote the Gateaux operator of Δ and its adjoint operator, respectively. It is easy to observe that

$$\Delta'(u)\sigma(u) = \Delta'(u)[\sigma(u)] := \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}(\Delta(u+\varepsilon\sigma(u))).$$

Definition 2.1. A vector function $\sigma \in \mathcal{B}^q$ is called a symmetry of (2.6), if it satisfies (2.7) when u solves (2.6). A vector function $\rho \in \mathcal{B}^l$ is called an adjoint symmetry of (2.6), if it satisfies (2.8) when u solves (2.6).

Definition 2.2. If a total divergence relation

$$\mathrm{Div}\, P = \sum_{i=1}^p \partial_i P_i = 0,\ P = (P_1, \cdots, P_p)^T \in \mathcal{B}^p, \tag{2.9}$$

holds for all solutions of (2.6), then (2.9) is called a conservation law and P a conserved vector of (2.6).

Conservation laws include familiar concepts of conservation of mass, energy and momentum arising in physical applications. In the case of Laplace's equation, the equation itself is a conservation law, since

$$\Delta u = \mathrm{Div}\,(\nabla u) = 0$$

for all solutions. For a system of ordinary differential equations involving a single independent variable $x \in \mathbb{R}$, a conservation law $D_x P = 0$ requires that $P(x, u^{(u)})$ be constant for all solutions of the system. Thus, a conservation law for a system of ordinary differential equations is equivalent to the classical notion of a first integral or constant of motion of the system.

In a dynamical problem, one of the independent variables is distinguished as the time t, the remaining variables $x=(x^1,\cdots,x^p)$ being spatial variables. Then, a conservation law takes the form

$$D_tT+\mathrm{Div}\,X=0,$$

in which Div stands for the spatial divergence of $X\in\mathcal{B}^p$ with respect to x. Here T is called a conserved density and X, the associated flux vector corresponding to T.

Conservation laws of a totally nondegenerate system of differential equations (2.6) are classified through characteristic forms [24]:

$$\mathrm{Div}\,P=Q^T\Delta, \tag{2.10}$$

where $Q\in\mathcal{B}^l$. Such a vector function Q is called the characteristic of the associated conservation law. For a normal, totally nondegenerate system of differential equations, equivalent conservation laws correspond to equivalent characteristics [24].

We would like to formulate conservation laws for a general system of differential equations (2.6) by using pairs of symmetries and adjoint symmetries. In our expressions of construction, we will use the following assumption for brevity: an empty product of derivatives $\partial_i^{\alpha_i}$, $\alpha_i\ge 0$, $1\le i\le p$, is understood to be the identity operator. For example, $\Pi_{i=r}^s\partial_i^2$ implies the identity operator, when $r>s$.

Lemma 2.1. *Let f and g be two smooth functions in variables $x^1,\cdots,x^p$. Then for any $\alpha=(\alpha_1,\cdots,\alpha_p)$ with non-negative integers α_i, $1\le i\le p$, we have*

$$\begin{aligned}&f(D^\alpha g)-((-D)^\alpha f)g=f(\partial_1^{\alpha_1}\cdots\partial_p^{\alpha_p}g)-((-\partial_1)^{\alpha_1}\cdots(-\partial_p)^{\alpha_p}f)g\\ =&\sum_{i=1}^p\partial_i\sum_{\beta_i=0}^{\alpha_i-1}((-\partial_1)^{\alpha_1}\cdots(-\partial_{i-1})^{\alpha_{i-1}}(-\partial_i)^{\beta_i}f)(\partial_i^{\alpha_i-\beta_i-1}\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_p^{\alpha_p}g),\end{aligned} \tag{2.11}$$

where an empty sum is understood to be zero.

Proof. First note that we have

$$a\partial_i^kb-((-\partial_i)^ka)b=\partial_i\sum_{l=0}^{k-1}((-\partial_i)^la)(\partial_i^{k-l-1}b),\ k\ge1,\ 1\le i\le p, \tag{2.12}$$

for any two smooth functions a and b in variables $x^1,\cdots,x^p$, and then we decompose that

$$\begin{aligned}&f(D^\alpha g)-((-D)^\alpha f)g=f(\partial_1^{\alpha_1}\cdots\partial_p^{\alpha_p}g)-((-\partial_1)^{\alpha_1}\cdots(-\partial_p)^{\alpha_p}f)g\\ =&\sum_{i=1}^p[((-\partial_1)^{\alpha_1}\cdots(-\partial_{i-1})^{\alpha_{i-1}}f)(\partial_i^{\alpha_i}\cdots\partial_p^{\alpha_p}g)\\ &\quad-((-\partial_1)^{\alpha_1}\cdots(-\partial_i)^{\alpha_i}f)(\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_p^{\alpha_p}g)].\end{aligned}$$

It follows from (2.12) that each term in the above sum can be computed as follows:

$$\begin{aligned}&((-\partial_1)^{\alpha_1}\cdots(-\partial_{i-1})^{\alpha_{i-1}}f)(\partial_i^{\alpha_i}\cdots\partial_p^{\alpha_p}g\\ &-((-\partial_1)^{\alpha_1}\cdots(-\partial_i)^{\alpha_i}f)(\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_p^{\alpha_p}g)\\ =&((-\partial_1)^{\alpha_1}\cdots(-\partial_{i-1})^{\alpha_{i-1}}f)\partial_i^{\alpha_i}(\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_p^{\alpha_p}g)\\ &-((-\partial_i)^{\alpha_i}((-\partial_1)^{\alpha_1}\cdots(-\partial_{i-1})^{\alpha_{i-1}}f))(\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_p^{\alpha_p}g)\\ =&\partial_i\sum_{\beta_i=0}^{\alpha_i-1}((-\partial_1)^{\alpha_1}\cdots(-\partial_{i-1})^{\alpha_{i-1}}(-\partial_i)^{\beta_i}f)(\partial_i^{\alpha_i-\beta_i-1}\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_p^{\alpha_p}g),\ 1\le i\le p,\end{aligned}$$

where an empty sum is understood to be zero. This allows us to conclude that our equality (2.11) holds for any $\alpha = (\alpha_1, \cdots, \alpha_p)$. The proof is finished. $\square$

This lemma tells us that $fD^\alpha g - ((-D)^\alpha f)g$ is a total divergence function for any two smooth functions f and g.

Theorem 2.1. *Let $\sigma = (\sigma_1, \cdots, \sigma_q)^T \in \mathcal{B}^q$ and $\rho = (\rho_1, \cdots, \rho_l)^T \in \mathcal{B}^l$ be a symmetry and an adjoint symmetry of a system of differential equations (2.6), respectively. Then we have a conservation law for the system (2.6):*

$$\sum_{k=1}^{p} \partial_k \sum_{i=1}^{l} \sum_{j=1}^{q} \sum_{\alpha \geq 0} \sum_{\beta_k=0}^{\alpha_k-1} ((-\partial_1)^{\alpha_1} \cdots (-\partial_{k-1})^{\alpha_{k-1}} (-\partial_k)^{\beta_k} \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j}) \times \\ (\partial_k^{\alpha_k-\beta_k-1} \partial_{k+1}^{\alpha_{k+1}} \cdots \partial_p^{\alpha_p} \sigma_j) = 0, \tag{2.13}$$

where $\alpha = (\alpha_1, \cdots, \alpha_p)$, and an empty sum is understood to be zero.

Proof. Let us compute that

$$\begin{aligned} \rho^T \Delta' \sigma - \sigma^T (\Delta')^* \rho &= \sum_{i=1}^{l} \sum_{j=1}^{q} (\rho_i V_j(\Delta_i) \sigma_j - \sigma_j V_j^*(\Delta_i) \rho_i) \\ &= \sum_{i=1}^{l} \sum_{j=1}^{q} \sum_{\alpha \geq 0} (\rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j} D^\alpha \sigma_j - \sigma_j (-D)^\alpha \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j}). \end{aligned}$$

By using Lemma 2.1, for all $1 \leq i \leq l, 1 \leq j \leq q$ and $\alpha \geq 0$, we have

$$\begin{aligned} & \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j} D^\alpha \sigma_j - \sigma_j (-D)^\alpha \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j} = \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j} (D^\alpha \sigma_j) - ((-D)^\alpha \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j}) \sigma_j \\ = \ & \sum_{k=1}^{p} \partial_k \sum_{\beta_k=0}^{\alpha_k-1} ((-\partial_1)^{\alpha_1} \cdots (-\partial_{k-1})^{\alpha_{k-1}} (-\partial_k)^{\beta_k} \rho_i \frac{\partial \Delta_i}{\partial u_\alpha^j}) \times \\ & (\partial_k^{\alpha_k-\beta_k-1} \partial_{k+1}^{\alpha_{k+1}} \cdots \partial_p^{\alpha_p} \sigma_j), \end{aligned}$$

where an empty sum is understood to be zero. Now, noting that $\Delta' \sigma = 0$ and $(\Delta')^* \rho = 0$ hold for all solutions of (2.6), we see that (2.13) follows. The proof is finished. $\square$

The theorem gives us an explicit formulation of conservation laws for systems of differential equations, regardless of the existence of a Lagrangian. For totally nondegenerate systems of differential equations, we can set

$$\Delta' \sigma = R_\sigma^{\mathrm{sym}} \Delta, \ (\Delta')^* \rho = R_\rho^{\mathrm{asym}} \Delta, \tag{2.14}$$

where R_σ^{sym} and R_ρ^{asym} are two $l \times l$ and $q \times l$ matrix differential operators depending on σ and ρ, respectively [24], and then the following computation

$$\mathrm{Div}\, P = \rho^T \Delta' \sigma - \sigma^T (\Delta')^* \rho = \rho^T R_\sigma^{\mathrm{sym}} \Delta - \sigma^T R_\rho^{\mathrm{asym}} \Delta \sim ((R_\sigma^{\mathrm{sym}})^* \rho - (R_\rho^{\mathrm{asym}})^* \sigma)^T \Delta \tag{2.15}$$

presents the characteristic of an equivalent conservation law:

$$Q = (R_\sigma^{\mathrm{sym}})^* \rho - (R_\rho^{\mathrm{asym}})^* \sigma. \tag{2.16}$$

A direct way to generate more conservation laws from known ones is to use recursion structures. Similar to the definition of recursion operators [25], or hereditary symmetry operators [4], we can also have recursion structures of other kinds for systems of differential equations. Recursion operators transforms symmetries to symmetries, and hereditary symmetry operators provide recursion operators for hierarchies.

Definition 2.3. If an operator $\Psi(u)$ transforms an adjoint symmetry of (2.6) into another adjoint symmetry of (2.6), then $\Psi(u)$ is called an adjoint recursion operator of (2.6).

Definition 2.4. If an operator $\bar{\Phi}(u)$ transforms an adjoint symmetry of (2.6) into a symmetry of (2.6) , then $\bar{\Phi}(u)$ is called a Noether operator of (2.6). Conversely, if an operator $\bar{\Psi}(u)$ transforms a symmetry of (2.6) into an adjoint symmetry of (2.6), then $\bar{\Psi}(u)$ is called an inverse Noether operator of (2.6).

All the above operators are important in establishing more conservation laws, and thus integrability of systems of differential equations. The concepts of Noether operators and inverse Noether operators were also introduced for integrable systems [5].

3. **Application to evolution equations.**

3.1. **Conservation laws.** Let us take a set of independent variables $(t, x^1, \cdots, x^p)$, including a distinguished time variable $t \in \mathbb{R}$, and consider a system of evolution equations

$$u_t = K(u),\ K \in \mathcal{A}^q. \tag{3.1}$$

Obvisouly, we have $\Delta = u_t - K(u)$ with $l = q$, and thus its linearized system and adjoint linearized system read

$$(\sigma(u))_t = K'(u)\sigma(u),\ \sigma \in \mathcal{B}^q, \tag{3.2}$$

$$(\rho(u))_t = -(K^{'})^*(u)\rho(u),\ \rho \in \mathcal{B}^q, \tag{3.3}$$

respectively. Here K' and $(K^{'})^*$ stand for the Gateaux operator of K and its adjoint operator, respectively.

It is easy to see that two vector functions $\sigma, \rho \in \mathcal{A}^q$ are a symmetry and an adjoint symmetry of the system (3.1) if and only if they satisfy

$$\frac{\partial \sigma(u)}{\partial t} = K'(u)\sigma(u) - \sigma'(u)K(u), \tag{3.4}$$

$$\frac{\partial \rho(u)}{\partial t} = -(K^{'})^*(u)\rho(u) - \rho'(u)K(u), \tag{3.5}$$

respectively, when u solves (3.1), where σ' and ρ' are the Gateaux operators of σ and ρ.

A total divergence

$$T_t = \sum_{i=1}^{p} \partial_i X_i,\ T, X_i \in \mathcal{B},\ 1 \le i \le p, \tag{3.6}$$

gives us a conservation law for the system of evolution equations (3.1), and T is a conserved density of (3.1) and $X = (X_1, \cdots, X_p)^T$, a conversed flux vector of (3.1) corresponding to T.

An application of Theorem 2.1 to systems of evolution equations presents the following result on conservation laws for systems of evolution equations.

Theorem 3.1. *Let $\sigma = (\sigma_1, \cdots, \sigma_q)^T \in \mathcal{B}^q$ and $\rho = (\rho_1, \cdots, \rho_q)^T \in \mathcal{B}^q$ be a symmetry and an adjoint symmetry of a system of evolution equations (3.1), respectively. Then we have a dynamical conservation law for the system (3.1):*

$$(\sigma^T \rho)_t = (\sum_{i=1}^{q} \sigma_i \rho_i)_t = \sum_{k=1}^{p} \partial_k X_k, \tag{3.7}$$

where the conserved fluxes are defined by

$$X_k = \sum_{i,j=1}^{q} \sum_{\alpha \geq 0} \sum_{\beta_k=0}^{\alpha_k - 1} ((-\partial_1)^{\alpha_1} \cdots (-\partial_{k-1})^{\alpha_{k-1}} (-\partial_k)^{\beta_k} \rho_i \frac{\partial K_i}{\partial u_\alpha^j}) \times \tag{3.8}$$

$$(\partial_k^{\alpha_k - \beta_k - 1} \partial_{k+1}^{\alpha_{k+1}} \cdots \partial_p^{\alpha_p} \sigma_j), \tag{3.9}$$

where $1 \leq k \leq p$, $\alpha = (\alpha_1, \cdots, \alpha_p)$, and an empty sum is understood to be zero. Therefore, $T = \sigma^T \rho$ is a conserved density of the system (3.1), and $X = (X_1, \cdots, X_p)^T \in \mathcal{B}^p$ is the conserved flux vector of the system (3.1), corresponding to T.

Proof. A simple application of Theorem 2.1 to the case of $\Delta = u_t - K(u)$ with a set of independent variables $(t, x^1, \cdots, x^p)$ presents the result in the above thorem.

Alternatively, we can directly prove the theorem. First, we can have

$$\begin{aligned} & (\sigma^T \rho)_t = \sigma_t^T \rho + \sigma^T \rho_t = \rho^T \sigma_t + \sigma^T \rho_t = \rho^T K' \sigma - \sigma^T (K')^* \rho \\ = & \sum_{i,j=1}^{q} (\rho_i V_j(K_i) \sigma_j - \sigma_j V_j^*(K_i) \rho_i) = \sum_{i,j=1}^{q} \sum_{\alpha \geq 0} (\rho_i \frac{\partial K_i}{\partial u_\alpha^j} D^\alpha \sigma_j - \sigma_j (-D)^\alpha \rho_i \frac{\partial K_i}{\partial u_\alpha^j}). \end{aligned}$$

Then, together with Lemma 2.1 for all $1 \leq i, j \leq q$ and $\alpha \geq 0$, we can see that (3.7) holds for all solutons of (3.1) and the conserved flux vector X is given by (3.9). The proof is finished. □

The theorem gives us a direct formulation of conservation laws for systems of evolution equations, and all expressions are explicitly given for the conserved density and conserved fluxes. The involved conserved density is just a product of a symmetry and an adjoint symmetry, but the conserved fluxes are dependent on the pair of a symmetry and an adjoint symmetry and the system itself.

3.2. **Recursion structures.** For systems of evolution equations, we can easily prove the following theorem, which states sufficient conditions for being recursion operators, adjoint recursion operators, Noether operators or inverse Noether operators.

Theorem 3.2. *The operators $\Phi(x,t,u)$, $\Psi(x,t,u)$, $\bar{\Phi}(x,t,u)$ or $\bar{\Psi}(x,t,u)$ are a recursion operator, an adjoint recursion operator, a Noether operator or an inverse Noether operator of the system (3.1), if they satisfy*

$$\frac{\partial \Phi}{\partial t} + \Phi'[K] + [\Phi, K'] = 0, \tag{3.10}$$

$$\frac{\partial \Psi}{\partial t} + \Psi'[K] + [(K')^*, \Psi] = 0, \tag{3.11}$$

$$\frac{\partial \bar{\Phi}}{\partial t} + \bar{\Phi}'[K] - \bar{\Phi}(K')^* - K' \bar{\Phi} = 0, \tag{3.12}$$

$$\frac{\partial \bar{\Psi}}{\partial t} + \bar{\Psi}'[K] + \bar{\Psi} K' + (K')^* \bar{\Psi} = 0, \tag{3.13}$$

respectively, where K' and $(K^{'})^*$ denote the Gateaux operator of K and its adjoint operator, and the Gateaux operator of an operator A is similarly defined by

$$A'[K] = A'(u)[K] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} A(u+\varepsilon K). \tag{3.14}$$

When the operators Φ, Ψ, $\bar{\Phi}$, $\bar{\Psi}$ don't explicitly depend on t, the above theorem presents the results established in [5]. It is also easy to see that we have two relations

$$\Phi^* = \Psi,\ \bar{\Phi}^{-1} = \bar{\Psi}, \tag{3.15}$$

which means that the adjoint operator of a recursion operator (or an adjoint recursion operator) of (3.1) is an adjoint recursion operator (or a recursion operator) of the same system, and that the inverse operator of a Noether operator (or an inverse Noether operator) of (3.1) is an inverse Noether operator (or a Noether operator) of the same system. It is also easy to prove that the inverse operator Φ^{-1} (or Ψ^{-1}) of a recursion operator Φ (or an adjoint recursion operator Ψ) of a system of evolution equations is still a recursion operator (or an adjoint recursion operator) of the same system, if it exists.

Definition 3.1. If a conserved density $T \in \mathcal{B}$ of a system of evolution equations (3.1) is equivalent to zero function, then T is called trivial, and otherwise T is called nontrivial.

Any spatial divergence function $\sum_{i=1}^p \partial_i f_i$ must be a trivial conserved density of any system of evolution equations. According to the definition (2.3) of adjoint operators, we can have

$$(\Phi^i\sigma)^T(\Psi^j\rho) \sim \sigma^T(\Psi^{i+j}\rho) \sim (\Phi^{i+j}\sigma)^T\rho,\ i,j\ge 0, \tag{3.16}$$

where $\Psi = \Phi^*$. The study of conservation laws aims at presenting nontrivial conserved densities. Thus, we only need to consider $\sigma^T(\Psi^i\rho)$, $(\Phi^i\sigma)^T\rho$, $i \ge 0$, among $(\Phi^i\sigma)^T(\Psi^j\rho)$, $i,j\ge 0$, in order to generate nontrivial and nonequivalent conserved densities, while using recursion operators.

For the Korteweg-de Vries (KdV) equation

$$u_t = K(u) = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x,\ x,t\in\mathbb{R}, \tag{3.17}$$

it is easy to get

$$K' = \frac{3}{2}u_x + \frac{3}{2}u\partial_x + \frac{1}{4}\partial_x^3,\ (K^{'})^* = -\frac{3}{2}u\partial_x - \frac{1}{4}\partial_x^3,\ \partial_x = \frac{\partial}{\partial x}, \tag{3.18}$$

and then a recursion operator, an adjoint recursion operator, a Noether operator, and an inverse Noether operator of the KdV equation (3.17):

$$\Phi = \frac{1}{2}u_x\partial_x^{-1} + u + \frac{1}{4}\partial_x^2,\ \Psi = -\frac{1}{2}\partial_x^{-1}u_x + u + \frac{1}{4}\partial_x^2,\ \bar{\Phi} = \partial_x,\ \bar{\Psi} = \partial_x^{-1}. \tag{3.19}$$

Therefore, the KdV equation (3.17) possesses infinitely many symmetries and adjoint symmetries, $\Phi^i\sigma$ and $\Psi^i\rho$, $i\ge 0$, once we have a pair of a symmetry σ and an adjoint symmetry ρ. We will see later that this really happens.

4. **Illustrative examples.** Now we go on to illustrate by examples rich structures of the conserved densities resulting from symmetries and adjoint symmetries.

4.1. **The heat equation.** Let us consider the heat equation

$$u_t = K(u) = u_{xx},\ \ x, t \in \mathbb{R}, \tag{4.1}$$

which lacks a Lagrangian formulation. Its linearized equation and adjoint linearized equation read as

$$\sigma_t = K'\sigma = \sigma_{xx}, \tag{4.2}$$

$$\rho_t = -(K^{'})^*\rho = -\rho_{xx}, \tag{4.3}$$

respectively.

Among the first-order differential operators, it is easy to obtain two recursion operators of (4.1):

$$\Phi_1 = \partial_x,\ \ \Phi_2 = 2t\partial_x + x, \tag{4.4}$$

and thus two adjoint recursion operators of (4.1):

$$\Psi_1 = \Phi_1^* = -\partial_x,\ \ \Psi_2 = \Phi_2^* = -2t\partial_x + x. \tag{4.5}$$

Obviously, any solution $f = f(x,t)$ to $f_t = f_{xx}$ and the function $K_0 = u$ are symmetries of the heat equation (4.1), and any solution $g = g(x,t)$ to $g_t = -g_{xx}$ and the function $S_0 = u(x,-t)$ are adjoint symmetries of the heat equation (4.1). Here the adjoint symmetry S_0 has a local dependence on u with respect to x, similar to the symmetry K_0. Only for linear equations, adjoint symmetries can have such a local dependence on u.

Now, by the principle in Theorem 3.1, we have infinitely many conserved densities

$$g(x,t)u(x,t),\ \ f(x,t)u(x,-t),\ \ u(x,-t)(\Phi_2^i\Phi_1^j u)(x,t),\ \ i,j \geq 0, \tag{4.6}$$

besides a class of trivial conserved densities given by fg. Noting that we have (3.16) and

$$[\Phi_1, \Phi_2] = \Phi_1\Phi_2 - \Phi_2\Phi_1 = 1, \tag{4.7}$$

we did not list the other equivalent or linear conbination type conserved densities such as

$$(\Psi_1^i g)(\Phi_2^j\Phi_1^k u) \sim (\Psi_1^k\Psi_2^j\Psi_1^i g)u = \bar{g}u,$$

$$u(x,-t)(\Phi_1\Phi_2\Phi_1 u)(x,t) = u(x,-t)(\Phi_2\Phi_1^2 u)(x,t) + u(x,-t)(\Phi_1 u)(x,t),$$

where $\bar{g}$ is a new solution to $g_t = -g_{xx}$. The first class of conserved densities gu in (4.6) presents the ones generated in [1], and the third class of conserved densities contains the following interesting conserved densities:

$$u(x,-t)(\Phi_2 u)(x,t) = u(x,-t)(xu(x,t) + 2tu_x(x,t)), \tag{4.8}$$

$$u(x,-t)(\Phi_2^2 u)(x,t) = u(x,-t)[(x^2+2t)u(x,t) + 4txu_x(x,t) + 4t^2u_{xx}(x,t)]. \tag{4.9}$$

4.2. **Burgers' equation.** Let us now consider Burgers' equation

$$u_t = K(u) = 2uu_x + u_{xx},\ \ x,t \in \mathbb{R}, \tag{4.10}$$

which lacks a Lagrangian formulation as well. Its linearized equation and adjoint linearized equation read

$$\tilde{\sigma}_t = K'\tilde{\sigma} = 2u\tilde{\sigma}_x + 2u_x\tilde{\sigma} + \tilde{\sigma}_{xx}, \tag{4.11}$$

$$\tilde{\rho}_t = -(K^{'})^*\tilde{\rho} = 2u\tilde{\rho}_x - \tilde{\rho}_{xx}, \tag{4.12}$$

respectively.

Since the Cole-Hope transformation

$$v = B(u) = \mathrm{e}^{\partial_x^{-1}u} \text{ or } u = (\ln v)_x \tag{4.13}$$

linearizes Burgers' equation (4.10) to the heat equation (4.1) with a dependent variable v, we can move all above results for the heat equation to Burgers' equation. Note that the Gateaux operator of B and its inverse operator read

$$B' = \mathrm{e}^{\partial_x^{-1} u}\partial_x^{-1},\ (B')^{-1} = \partial_x \mathrm{e}^{-\partial_x^{-1} u}. \tag{4.14}$$

The Cole-Hope transformation gives us two recursion operators for Burgers' equation (4.10):

$$\begin{cases} \tilde{\Phi}_1 = (B')^{-1}\Phi_1 B' = \partial_x \mathrm{e}^{-\partial_x^{-1}u}\partial_x \mathrm{e}^{\partial_x^{-1}u}\partial_x^{-1} = u_x\partial_x^{-1} + u + \partial_x, \\ \tilde{\Phi}_2 = (B')^{-1}\Phi_2 B' = \partial_x \mathrm{e}^{-\partial_x^{-1}u}(2t\partial_x + x)\mathrm{e}^{\partial_x^{-1}u}\partial_x^{-1} = 2t\tilde{\Phi}_1 + x + \partial_x^{-1}, \end{cases} \tag{4.15}$$

and two relations on symmetries and adjoint symmetries between two equations

$$\tilde{\sigma} = (B')^{-1}\sigma,\ \tilde{\rho} = (B')^{\dagger}\rho. \tag{4.16}$$

Now, the conserved densities by the principle in Theorem 3.1 can be computed as follows:

$$\tilde{\sigma}\tilde{\rho} = ((B')^{-1}\sigma)((B')^{\dagger}\rho) = (\partial_x \mathrm{e}^{-\partial_x^{-1}u}\sigma)(-\partial_x^{-1}\mathrm{e}^{\partial_x^{-1}u}\rho) \sim \sigma\rho. \tag{4.17}$$

Therefore, for example, if we choose $\sigma = g$ and $\rho = v$, where g solves $g_t = -g_{xx}$, then we can immediately obtain a class of conserved densities for Burgers' equation (4.10):

$$h_0 = gv = g\mathrm{e}^{\partial_x^{-1}u}, \tag{4.18}$$

which was generated in [1]. In fact, this class of conserved densities corresponds to the following conservation laws

$$h_{0t} = \partial_t(g\mathrm{e}^{\partial_x^{-1}u}) = \partial_x(gu\mathrm{e}^{\partial_x^{-1}u} - g_x\mathrm{e}^{\partial_x^{-1}u}). \tag{4.19}$$

Let us set two basic symmetries of Burgers' equation (4.10):

$$\tilde{K}_0 = (B')^{-1}v_x = (B')^{-1}(u\mathrm{e}^{\partial_x^{-1}u}) = \partial_x(\mathrm{e}^{-\partial_x^{-1}u}u\mathrm{e}^{\partial_x^{-1}u}) = u_x, \tag{4.20}$$

$$\tilde{f}_0 = (B')^{-1}f = \partial_x(\mathrm{e}^{-\partial_x^{-1}u}f) = (f_x - fu)\mathrm{e}^{-\partial_x^{-1}u}, \tag{4.21}$$

where f solves $f_t = f_{xx}$. Since there exist two inverse recursion operators [12]:

$$\tilde{\Phi}_1^{-1} = \partial_x \mathrm{e}^{-\partial_x^{-1}u}\partial_x^{-1}\mathrm{e}^{\partial_x^{-1}u}\partial_x^{-1},\ \tilde{\Phi}_2^{-1} = \partial_x \mathrm{e}^{-\partial_x^{-1}u}(2t\partial_x + x)^{-1}\mathrm{e}^{\partial_x^{-1}u}\partial_x^{-1}, \tag{4.22}$$

where the inverse in the middle of the second formula can be worked out:

$$(2t\partial_x + x)^{-1} = \frac{1}{2t}\mathrm{e}^{-\frac{x^2}{4t}}\partial_x^{-1}\mathrm{e}^{\frac{x^2}{4t}},$$

we can have infinitely many symmetries for Burgers' equation (4.10):

$$\tilde{K}_{ij} = \tilde{\Phi}_2^i\tilde{\Phi}_1^j\tilde{K}_0,\ \tilde{f}_{ij} = \tilde{\Phi}_2^i\tilde{\Phi}_1^j\tilde{f}_0,\ i,j \in \mathbb{Z}, \tag{4.23}$$

where we can not add the other symmetries such as $\tilde{\Phi}_1\tilde{\Phi}_2\tilde{\Phi}_1\tilde{K}_0$ to the algebra spanned by all symmetries in (4.23) because

$$[\tilde{\Phi}_1, \tilde{\Phi}_2] = \tilde{\Phi}_1\tilde{\Phi}_2 - \tilde{\Phi}_2\tilde{\Phi}_1 = 1.$$

The symmetries defined by (4.23) contain all symmetries generated in [27], and in particular, we have a time-dependent symmetry

$$\tilde{K}_{1,-1} = \tilde{\Phi}_2\tilde{\Phi}_1^{-1}u_x = (2t\tilde{\Phi}_1 + x + \partial_x^{-1})\tilde{\Phi}_1^{-1}u_x = 2tu_x + 1. \tag{4.24}$$

Moreover, a direct computation can show that all local adjoint symmetries of (4.10), depending on x, t, u, and derivatives of u with respect to x to some finite order, must be a constant function. By noting that

$$x+\partial_x^{-1}=\partial_x x\partial_x^{-1},\ u_x\partial_x^{-1}+u=\partial_x u\partial_x^{-1},$$

all conserved densities resulted from the products of the above symmetries and a local adjoint symmetry $\tilde\rho_0=1$ must be trivial. For example, the following class of conserved densities

$$\tilde\rho_0\tilde f_0=(f_x-uf)\mathrm{e}^{-\partial_x^{-1}u}=\partial_x(f\mathrm{e}^{-\partial_x^{-1}u}) \tag{4.25}$$

is trivial. This result also provides an evidence why Burgers' equation (4.10) has only one nontrivial conserved density of differential polynomial type [28].

Nevertheless, based on two basic adjoint symmetries of the heat equation, we can obtain two nonlocal basic adjoint symmetries of Burgers' equation (4.10), defined by

$$\tilde g_0=(B')^\dagger g=-\partial_x^{-1}(g\mathrm{e}^{\partial_x^{-1}u}),\ \tilde S_0=(B')^\dagger S_0=-\partial_x^{-1}(S_0\mathrm{e}^{\partial_x^{-1}u}), \tag{4.26}$$

where g solves $g_t=-g_{xx}$ and $S_0(x,t)=u(x,-t)$. Now, by the principle in Theorem 3.1, we have infinitely many conserved densities

$$\tilde g_0\tilde K_{ij},\ \tilde g_0\tilde f_{ij},\ \tilde S_0\tilde K_{ij},\ \tilde S_0\tilde f_{ij},\ i,j\in\mathbb{Z}. \tag{4.27}$$

Several simple classes of conserved densities can be computed as follows:

$$\begin{aligned}
&\tilde g_0\tilde f_0=-\partial_x^{-1}(g\mathrm{e}^{\partial_x^{-1}u})\partial_x(f\mathrm{e}^{\partial_x^{-1}u})\sim fg:=h_1,\\
&\tilde g_0\tilde K_0=-\partial_x^{-1}(g\mathrm{e}^{\partial_x^{-1}u})u_x\sim gu\mathrm{e}^{\partial_x^{-1}u}:=h_2,\\
&\tilde g_0\tilde K_{1,-1}=-\partial_x^{-1}(g\mathrm{e}^{\partial_x^{-1}u})(2tu_x+1)\sim g(2tu+x)\mathrm{e}^{\partial_x^{-1}u}:=h_3,\\
&\tilde g_0\tilde f_{01}=-\partial_x^{-1}(g\mathrm{e}^{\partial_x^{-1}u})[\partial_x^2(f\mathrm{e}^{-\partial_x^{-1}u})+\partial_x(fu\mathrm{e}^{-\partial_x^{-1}u})]\\
&\qquad\quad\sim(g\mathrm{e}^{\partial_x^{-1}u})\partial_x(f\mathrm{e}^{-\partial_x^{-1}u})+fgu=f_xg:=h_4,\\
&\tilde g_0\tilde K_{01}=-\partial_x^{-1}(g\mathrm{e}^{\partial_x^{-1}u})\partial_x(u_x+u^2)\sim g(u_x+u^2)\mathrm{e}^{\partial_x^{-1}u}:=h_5.
\end{aligned}$$

They correspond to the following conservation laws

$$\begin{aligned}
h_{1t}&=\partial_t(fg)=\partial_x(f_xg-fg_x),\\
h_{2t}&=\partial_t(gu\mathrm{e}^{\partial_x^{-1}u})=\partial_x[(gu^2+gu_x-g_xu)\mathrm{e}^{\partial_x^{-1}u}],\\
h_{3t}&=\partial_t[(2tu+x)g\mathrm{e}^{\partial_x^{-1}u}]\\
&=\partial_x\{[g_x(2tu+x)-g(2tu_x+1)-g(2tu+x)u]\mathrm{e}^{\partial_x^{-1}u}\},\\
h_{4t}&=\partial_t(f_xg)=\partial_x(f_{xx}g-f_xg_x),\\
h_{5t}&=\partial_t[g(u_x+u^2)\mathrm{e}^{\partial_x^{-1}u}]=\partial_x\{[-g_x(u_x+u^2)+g(u_{xx}+3uu_x+u^3)]\mathrm{e}^{\partial_x^{-1}u}\},
\end{aligned}$$

the first and the fourth of which are trivial conservation laws of the second kind (see [24] for the definition). Of course, there are the other two classes of conserved densities, generated from the second basic adjoint symmetry $\tilde S_0$.

4.3. **The Korteweg-de Vries equation.** Let us finally consider the Korteweg-de Vries (KdV) equation (3.17), i.e.,

$$u_t=K(u)=\frac{3}{2}uu_x+\frac{1}{4}u_{xxx},\ x,t\in\mathbb{R}.$$

This standard form of the KdV equation also lacks a Lagrangian formulation. Its linearized equation and adjoint linearized equation read

$$\sigma_t = K'\sigma = \frac{3}{2}u\sigma_x + \frac{3}{2}u_x\sigma + \frac{1}{4}\sigma_{xxx}, \tag{4.28}$$

$$\rho_t = -(K^{'})^*\rho = \frac{3}{2}u\rho_x + \frac{1}{4}\rho_{xxx}, \tag{4.29}$$

respectively. It is known that we have infinitely many symmetries

$$K_i = \Phi^i u_x = \partial_x \frac{\delta \tilde{H}_i}{\delta u},\ \tau_i = \Phi^i(\frac{3}{2}tu_x + 1),\ \tilde{H}_i = \int H_i dx,\ H_i \in \mathcal{A},\ i \geq 0, \tag{4.30}$$

with Φ being given by (3.19). They form a Virasoro algebra (see, e.g., [14]):

$$[K_i, K_i] = 0,\ [K_i, \tau_j] = (i + \frac{1}{2})K_{i+j-1},\ [\tau_i, \tau_j] = (i - j)\tau_{i+j-1},\ i, j \geq 0, \tag{4.31}$$

where $K_{-1} = \tau_{-1} = 0$, and the commutator is defined by

$$[Y, Z] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0}(Y(u + \varepsilon Z) - Z(u + \varepsilon Y)),\ Y, Z \in \mathcal{B}^q.$$

In particular, we have

$$\tau_1 = \Phi\tau_0 = t(\frac{3}{8}u_{xxx} + \frac{6}{4}uu_x) + (\frac{1}{2}xu_x + u),$$

but the other time-dependent symmetries $\tau_i,\ i \geq 2$, are all nonlocal.

Since we have an inverse Noether operator ∂_x^{-1}, we can immediately obtain two local adjoint symmetries of the KdV equation (3.17):

$$S_0 = \partial_x^{-1}K_0 = u,\ T_0 = \partial_x^{-1}\tau_0 = \frac{3}{2}tu + x. \tag{4.32}$$

First, note that we have (3.16). Hence, by the principle in Theorem 3.1, all conserved densities of the type $(\Psi^i\rho)(\Phi^j\sigma)$ with $\Psi = \Phi^*$ and with σ and ρ being the above symmetries and adjoint symmetries give us the conserved densities

$$S_0\Phi^i K_0,\ T_0\Phi^i K_0,\ S_0\Phi^i\tau_0,\ T_0\Phi^i\tau_0,\ i \geq 0. \tag{4.33}$$

Second, note that there exist functions $P_i \in \mathcal{A},\ i \geq 0$, such that

$$u\partial_x \frac{\delta \tilde{H}_i}{\delta u} = \partial_x P_i,\ i \geq 0,$$

and we have a relation among H_i:

$$\frac{\delta \tilde{H}_i}{\delta u} = (i + \frac{1}{2})H_{i-1},\ H_{-1} = 2u,\ i \geq 0,$$

which can be found through the Virasoro algebra (4.31). Therefore, we can compute that

$$\begin{aligned}
&S_0\Phi^i K_0 = \partial_x P_i,\ T_0\Phi^i K_0 = -\frac{\delta \tilde{H}_i}{\delta u} + \partial_x(\frac{3}{2}tP_i + x\frac{\delta \tilde{H}_i}{\delta u}) \\
&\quad = -(i + \frac{1}{2})H_{i-1} + \partial_x(\frac{3}{2}tP_i + x\frac{\delta \tilde{H}_i}{\delta u}),\ i \geq 0, \\
&S_0\Phi^i\tau_0 = S_0\Phi^i\partial_x T_0 = S_0\partial_x\Psi^i T_0 \sim (\partial_x S_0)\Psi^i T_0 = K_0\Psi^i T_0 \sim T_0\Phi^i K_0,\ i \geq 0, \\
&T_0\tau_0 = \partial_x(\frac{9}{8}t^2u^2 + \frac{3}{2}txu + \frac{1}{2}x^2), \\
&T_0\tau_1 = \partial_x[\frac{9}{4}t^2P_1 + \frac{1}{4}xu^2 - \frac{1}{4}u_x + x(\frac{3}{4}u^2 + \frac{1}{4}u_{xx}) + \frac{1}{2}x^2u].
\end{aligned}$$

Now it follows that all nontrivial conserved densities defined by (4.33) are infinitely many nontrivial conserved densities $\{H_i\}_{i=-1}^{\infty}$, and infinitely many nonlocal conserved densities $T_0\Phi^i\tau_0$, $i\ge 2$.

Lastly, we would like to show that the Lax pair of the KdV equation can be used to generate nonlocal conserved densities. It is known that there exist many nonlocal symmetries generated from eigenfunctions and adjoint eigenfunctions of the Lax pair [15]. Let us just state the main results. The KdV equation (3.17) has a Lax pair

$$U(u,\lambda)=\begin{pmatrix} 0 & 1\\ \lambda-u & 0\end{pmatrix},\ V(u,\lambda)=\begin{pmatrix} -\frac14 u_x & \lambda+\frac12 u\\ \lambda^2-\frac12 u\lambda-\frac14 u_{xx}-\frac12 u^2 & \frac14 u_x\end{pmatrix}. \tag{4.34}$$

Using this Lax pair, we introduce N replicas of the Lax systems

$$\phi_x^{(s)}=U(u,\lambda_s)\phi^{(s)},\ \phi_t^{(s)}=V(u,\lambda_s)\phi^{(s)},\ \ \phi^{(s)}=(\phi_{1s},\phi_{2s})^T,\ 1\le s\le N,$$

and N replicas of the adjoint Lax systems

$$\psi_x^{(s)}=-U^T(u,\lambda_s)\psi^{(s)},\ \psi_t^{(s)}=-V^T(u,\lambda_s)\psi^{(s)},\ \psi^{(s)}=(\psi_{1s},\psi_{2s})^T,\ 1\le s\le N,$$

where $\lambda_1,\cdots,\lambda_N$ are arbitrary constants. Then, we have an adjoint symmetry and a symmetry

$$\rho_0=P_1^TQ_2,\ \sigma_0=(P_1^TQ_2)_x=P_1^TQ_1-P_2^TQ_2, \tag{4.35}$$

where P_i and Q_i are N dimensional vector functions,

$$P_i=(\phi_{i1},\phi_{i2},\cdots,\phi_{iN})^T,\ Q_i=(\psi_{i1},\psi_{i2},\cdots,\psi_{iN})^T,\ i=1,2.$$

Therefore, we can have infinitely many conserved densities in terms of eigenfunctions and adjoint eigenfunctions:

$$\rho_0\Phi^iK_0,\ \rho_0\Phi^i\tau_0,\ i\ge 0, \tag{4.36}$$

the first two conserved densities of which are the following

$$\rho_0K_0=u_x\rho_0\sim u\sigma_0=u\sum_{s=1}^N(\phi_{1s}\psi_{1s}-\phi_{1s}\psi_{2s}),$$
$$\rho_0\tau_0\sim\frac32 tu\sigma_0+\rho_0=\frac32 tu\sum_{s=1}^N(\phi_{1s}\psi_{1s}-\phi_{1s}\psi_{2s})+\sum_{s=1}^N\phi_{1s}\psi_{2s}.$$

5. **Conclusion and remarks.** We have established a direct formulation of conservation laws for systems of differential equations, regardless of the existence of a Lagrangian, and made an application to systems of evolutions equations, together with three examples of scalar evolution equations. The presented examples include computations of conserved densities for the heat equation, Burgers' equation, and the Korteweg-de Vries (KdV) equation.

We remark that pairs of symmetries and adjoint symmetries have also been used to formulate conservation laws for systems of discrete evolution equations [20], and Theorem 2.1 could be obtained from an application of Noether's theorem to an enlarged Euler-Lagrange system

$$\frac{\delta\mathcal{L}}{\delta v}=0,\ \frac{\delta\mathcal{L}}{\delta u}=0, \tag{5.1}$$

where $\mathcal{L}=\int v^T\Delta(u)\,dx$ with $v=(v^1,\cdots,v^l)^T$. The first subsystem is precisely the original system $\Delta=0$, and the second subsystem $\frac{\delta\mathcal{L}}{\delta u}=(\Delta')^*v=0$ is satisfied if we take v as an adjoint symmetry $\rho(u)$ of the system $\Delta = 0$. When $v = \rho = u$ (or $v = \rho(x, u)$), the system under consideration is called strictly (or nonlinearly) self-adjoint, and the resulting conservation law presents the one in [8, 9]. In the case of self-adjoint systems of differential equations, Theorem 2.1 generates trivial conservation laws of the second kind, since symmetries are adjoint symmetries, too [2]. The idea of constructing conserved densities by symmetries and adjoint symmetries is also similar to that of binary symmetry constraints, which results in a binary nonlinearization theory [16, 10, 11, 29].

There exists a differential geometric formulation for attempting adjoint symmetries of the second-order ordinary differential equations [22], and adjoint symmetries are also used to show separability of Hamiltonian systems of ordinary differential equations [26]. The existence of symmetry algebras are due to Lie algebraic structures associated with Lax operators corresponding to symmetries (see [17, 18, 19] for systems of continuous evolution equations and [6, 13] for systems of discrete evolution equations). How about Lie algebraic structures for adjoint symmetries? What kind of commutators between adjoint symmetries can be introduced? A good candidate for commutators could be taken as

$$[\![\rho_1, \rho_2]\!] = (\rho_1')^* \rho_2 - (\rho_2')^* \rho_1, \tag{5.2}$$

where $(\rho_1')^*$ and $(\rho_2')^*$ denote the adjoint operator of their Gateaux operators. However, this doesn't keep the space of adjoint symmetries closed. The phenomenon can be seen from an example in the case of the KdV equation. The KdV equation (3.17) has two adjoint symmetries $\rho_1 = u$ and $\rho_2 = \frac{3}{2}tu + x$, whose expected commutator reads

$$[\![\rho_1, \rho_2]\!] = [\![u, \frac{3}{2}tu + x]\!] = x. \tag{5.3}$$

But the resulting function x is not an adjoint symmetry of the KdV equation (3.17). The characteristic in (2.16) may be useful in exploring a successful commutator of adjoint symmetries.

Acknowledgments. This work was in part supported by NSF under the grant DMS-664561, NNSFC under the grants 11371326, 11371086 and 11571079, the 111 project of China (B16002), Natural Science Fund for Colleges and Universities of Jiangsu Province under the grant 17KJB110020, and the distinguished professorships of the Shanghai University of Electric Power and Shanghai Second Polytechnic University.

REFERENCES

[1] S. C. Anco and G. Bluman, Direct computation of conservation laws from filed equations, *Phys. Rev. Lett.*, **78** (1997), 2869–2873.

[2] S. C. Anco and G. Bluman, Derivation of conservation laws from nonlocal symmetries of differential equations, *J. Math. Phys.*, **37** (1996), 2361–2375.

[3] G. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York-Berlin, 1989.

[4] B. Fuchsstiener, Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Anal.*, **3** (1979), 849–862.

[5] B. Fuchsstiener and A. C. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D*, **4** (1981), 47–66.

[6] B. Fuchssteiner and W. X. Ma, An approach to master symmetries of lattice equations, in: *Symmetries and Integrability of Difference Equations* (Canterbury, 1996) (ed. P. A. Clarkson and F. W. Nijhoff), London Math. Soc. Lecture Note Ser., 255, Cambridge Univ. Press, Cambridge, 1999, 247–260.

[7] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, Holland, 1985.

[8] N. H. Ibragimov, A new conservation theorem, *J. Math. Anal. Appl.*, **333** (2007), 311–328.

[9] N. H. Ibragimov, Nonlinear self-adjointness and conservation laws, *J. Phys. A: Math. Theor.*, **44** (2011), 432002 (8pp).

[10] X. Y. Li and Q. L. Zhao, A new integrable symplectic map by the binary nonlinearization to the super AKNS system, *J. Geom. Phys.*, **121** (2017), 123–137.

[11] X. Y. Li, Q. L. Zhao, Y. X. Li and H. H. Dong, Binary Bargmann symmetry constraint associated with 3×3 discrete matrix spectral problem, *J. Nonlinear Sci. Appl.*, **8** (2015), 496–506.

[12] Y. S. Li, Some algebraic properties of "C-integrable" nonlinear equations I. Burgers equation and Calogero equation, *Sci. China Ser. A*, **33** (1990), 513–520.

[13] W. X. Ma and B. Fuchssteiner, Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations, *J. Math. Phys.*, **40** (1999), 2400–2418.

[14] W. X. Ma, K-symmetries and τ-symmetries of evolution equations and their Lie algebras, *J. Phys. A: Math. Gen.*, **23** (1990), 2707–2716.

[15] W. X. Ma, New finite-dimensional integrable systems by symmetry constraint of the KdV equations, *J. Phys. Soc. Jpn.*, **64** (1995), 1085–1091.

[16] W. X. Ma and W. Strampp, An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, *Phys. Lett. A*, **185** (1994), 277–286.

[17] W. X. Ma, The algebraic structures of isospectral Lax operators and applications to integrable equations, *J. Phys. A: Math. Gen.*, **25** (1992), 5329–5343.

[18] W. X. Ma, Lax representations and Lax operator algebras of isospectral and nonisospectral hierarchies of evolution equations, *J. Math. Phys.*, **33** (1992), 2464–2476.

[19] W. X. Ma, A method of zero curvature representation for constructing symmetry algebras of integrable systems, in: *Proceedings of the 21st International Conference on the Differential Geometry Methods in Theoretical Physics* (ed. M. L. Ge), World Scientific, Singapore, 1993, 535–538.

[20] W. X. Ma, Conservation laws of discrete evolution equations by symmetries and adjoint symmetries, *Symmetry*, **7** (2015), 714–725.

[21] W. X. Ma and R. G. Zhou, Adjoint symmetry constraints leading to binary nonlinearization, *J. Nonlinear Math. Phys.*, **9** (2002), 106–126.

[22] P. Morando and S. Pasquero, The symmetry in the structure of dynamical and adjoint symmetries of second-order differential equations, *J. Phys. A: Math. Gen.*, **28** (1995), 1943–1955.

[23] C. S. Morawetz, Variations on conservation laws for the wave equation, *Bull. Amer. Math. Soc. (N.S.)*, **37** (2000), 141–154.

[24] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics 107, Springer-Verlag, New York-Berlin, 1986.

[25] P. J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.*, **18** (1977), 1212–1215.

[26] W. Sarlet and A. Ramos, Adjoint symmetries, separability, and volume forms, *J. Math. Phys.*, **41** (2000), 2877–2888.

[27] C. Tian, New strong symmetry, symmetries and Lie algebra of Burgers' equation, *Sci. Sinica Ser. A*, **31** (1988), 141–151.

[28] G. Z. Tu and M. Z. Qin, The invariant groups and conservation laws of nonlinear evolution equations–an approach of symmetric function, *Sci. Sinica*, **24** (1981), 13–26.

[29] X. R. Wang, X. E. Zhang and P. Y. Zhao, Binary nonlinearization for AKNS-KN coupling system, *Abstr. Appl. Anal.*, **2014** (2014), Art. ID 253102, 12 pp.

Received for publication January 2017.

E-mail address: mawx@cas.usf.edu