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Research Article

Global Behavior of a New Rational Nonlinear Higher-Order Difference Equation

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Let k be a nonnegative integer and c a real number greater than or equal to 1. We present qualitative global behavior of solutions to a rational nonlinear higher-order difference equation $z_{n+1} = (c(z_n + z_{n-k}) + (c-1)z_n z_{n-k})/(z_n z_{n-k} + c)$, $n \geq 0$, with positive initial values $z_{-k}, z_{-k+1}, \cdots, z_0$, and show the global asymptotic stability of its positive equilibrium solution.

1. Introduction

Difference equations have wide applications in biology, computer science, digital signal processing, and economics. A general solution structure exists for linear difference equations [1]. However, in various situations of nonlinear higher-order difference equations, solution properties can only be observed by numerical simulation, and it is often exceedingly difficult to give a full mathematical proof for the properties predicted by numerical simulation and the conclusions formed on the basis of guesswork [2]. It is, therefore, important to make qualitative analysis on nonlinear higher-order difference equations, which is the topic of the current study. There have been some related studies on rational nonlinear difference equations in the literature (see, e.g., [3–8]). Global asymptotic properties of spectral functions are also crucial in determining algebro-geometric solutions of soliton equations (see, e.g., [9, 10]) and scattering data in matrix spectral problems (see, e.g., [11]).

An iterative algorithm to approximate a zero of a given function f reads

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n \geq 0, \tag{1}$$

and an application of this to a quadratic function $f(x) = x^2 - a$, $a > 0$, gives

$$x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}, \quad n \geq 0. \tag{2}$$

Let k be a nonnegative integer and c a real number greater than or equal to 1. We would like to consider a more general rational nonlinear higher-order difference equation

$$z_{n+1} = \frac{c(z_n + z_{n-k}) + (c-1) z_n z_{n-k}}{z_n z_{n-k} + c}, \quad n \geq 0, \tag{3}$$

with positive initial values $z_{-k}, z_{-k+1}, \cdots, z_0$, which engender positive solutions. We take a transformation

$$z_n = \frac{c}{y_n}, \quad n \geq -k, \tag{4}$$

and then obtain another difference equation

$$y_{n+1} = \frac{y_n y_{n-k} + c}{y_n + y_{n-k} + c - 1}, \quad n \geq 0, \tag{5}$$

Obviously, the equilibrium solution of the rational nonlinear difference equation (3), $\overline{z} = c$, becomes the equilibrium solution of the transformed difference equation (5), $\overline{y} = 1$.

If we further take $c = 1$, then we obtain the nonlinear difference equation discussed in [6, 7]:

$$y_{n+1} = \frac{y_n y_{n-k} + 1}{y_n + y_{n-k}}, \quad n \geq 0. \tag{6}$$

Introducing $x_n = \sqrt{a} y_n$ into (6) yields

$$x_{n+1} = \frac{x_n x_{n-k} + a}{x_n + x_{n-k}}, \quad n \geq 0, \tag{7}$$

where $a > 0$. When $k = 1$, this gives the nonlinear difference equation in (2). The equation (7) in the case of $k = 2$ was studied in [5] and its closed-form solution was presented in [6]. In the general case of k, the asymptotic stability of the positive equilibrium solution $\overline{x} = 1$ of the equation (7) was proved in [7].

It is direct to see that the rational nonlinear higher-order difference equation, defined by (3), possesses three equilibria: $\overline{z} = -1, 0, c$. In this article, we would like to explore global behavior of solutions to the rational nonlinear higher-order difference equation (3), show the global asymptotic stability of its positive equilibrium solution $\overline{z} = c$, and present two illustrative examples of positive solutions.

2. Global Behavior

2.1. Classification of Solutions. First of all, based on the rational difference equation (3), one can have

$$z_{n+1} - c = \frac{(c - z_{n-k})(z_n - c)}{z_n z_{n-k} + c}, \quad n \geq 0, \tag{8}$$

$$z_{n+1} - z_n = \frac{z_{n-k}(z_n + 1)(c - z_n)}{z_n z_{n-k} + c}, \quad n \geq 0, \tag{9}$$

$$z_{n+1} - z_{n-k} = \frac{z_n(z_{n-k} + 1)(c - z_{n-k})}{z_n z_{n-k} + c}, \quad n \geq 0. \tag{10}$$

Further from (9) and (10), we can easily derive the following solution properties.

Theorem 1. *If $\{z_n\}_{n=-k}^{\infty}$ is a solution to the rational nonlinear difference equation (3), then one has*

$$\begin{aligned} z_{n+1} &< z_n \quad \text{if } z_n > c, \\ z_{n+1} &> z_n \quad \text{if } z_n < c, \end{aligned} \tag{11}$$

$$\begin{aligned} z_{n+1} &< z_{n-k} \quad \text{if } z_{n-k} > c, \\ z_{n+1} &> z_{n-k} \quad \text{if } z_{n-k} < c, \end{aligned} \tag{12}$$

where $n \geq 0$.

If $k = 0$, the rational difference equation (3) becomes a first-order difference equation

$$z_{n+1} = \frac{2cz_n + (c-1) z_n^2}{z_n^2 + c}, \quad n \geq 0. \tag{13}$$

Then for $n \geq 0$, one has $z_{n+1} \leq c$, because $-z_n^2 + 2cz_n \leq c^2$. For $n \geq 1$, one has $z_{n+1} \geq z_n$, since $(c-1)z_n + c \geq z_n^2$, due to $z_n \leq c$. Therefore, every solution z_n decays to c, when $n \longrightarrow \infty$.

Generally, there are three types of solutions to the rational nonlinear higher-order difference equation (3).

Theorem 2. *Let $k \geq 1$. If $\{z_n\}_{n=-k}^{\infty}$ is a solution to the rational nonlinear higher-order difference equation (3), and then*

(a) it is eventually equal to c, more precisely $z_n = c, n \geq m$, which occurs when $z_m = c$ for some $m \geq 0$;

(b) it is eventually less than c, more precisely $z_n < z_{n+1} < c$, $n \geq m + k$, which occurs when $z_m, z_{m+1}, \cdots, z_{m+k} < c$ for some $m \geq -k$; or

(c) it oscillates about c, possessing at most k consecutive increasing terms less than c and at most k + 1 consecutive decreasing terms greater than c.

Proof. Equality (8) and property (11) directly tell that we have three types of solutions to the rational nonlinear higher-order difference equation (3).

The decreasing and increasing characteristics of oscillatory solutions in the third solution situation (c) can be proved as follows.

Suppose that $n_1, n_2 \geq 0$ are two integers satisfying $n_1 < n_2$. We express

$$\begin{aligned} z_{n_2} - z_{n_1} = {} & \left(z_{n_2} - z_{n_2-1}\right) + \left(z_{n_2-1} - z_{n_2-2}\right) + \cdots \\ & + \left(z_{n_1+1} - z_{n_1}\right) = D, \end{aligned} \tag{14}$$

where D can be written as

$$D = \sum_{j=n_1}^{n_2-1} \frac{z_{j-k}\left(c - z_j\right)\left(z_j + 1\right)}{z_j z_{j-k} + c}, \tag{15}$$

by (9).

If $z_n > c$ for $n_1 \leq n \leq n_2$, then each term in D is less than zero, and so $z_{n_2} < z_{n_1}$, due to (14). If $z_n < c$ for $n_1 \leq n \leq n_2$, then each term in D is greater than zero, and so $z_{n_2} > z_{n_1}$, due to (14). This completes the proof. □

Note that based on (8), we can see that there is no solution situation that a solution of (3) is eventually greater than c.

2.2. Global Asymptotic Stability. When $k = 0$, the equilibrium solution $\overline{z} = c$ of the first-order rational difference equation (13) is globally asymptotically stable, since it is a globally attractive equilibrium solution of a first-order difference equation (see [12] for a general theory).

For a general $k \geq 1$, we can show the same global asymptotic stability of the positive equilibrium solution $\overline{z} = c$ of the rational nonlinear difference equation (3), by establishing the local asymptotic stability and the global attractivity, which imply the global asymptotic stability [2]. Instead, we establish a strong negative feedback property [13] to guarantee the global asymptotic stability of $\overline{z} = c$ (see [14] for details on the strong negative feedback property).

Theorem 3. *The positive equilibrium solution* $\overline{z} = c$ *of the rational nonlinear higher-order difference equation (3) is globally asymptotically stable.*

Proof. Based on the rational nonlinear difference equation (3), one can have

$$\frac{c^2}{z_{n-k}} - z_{n+1} = \frac{(c - z_{n-k})\left[(c-1) z_n z_{n-k} + c z_{n-k} + c^2\right]}{z_{n-k}(z_n z_{n-k} + c)}, \quad n \geq 0. \tag{16}$$

From this equality and the equality in (10), we can obtain

$$(z_{n-k} - z_{n+1})\left(\frac{c^2}{z_{n-k}} - z_{n+1}\right) = -\frac{z_n(z_{n-k}+1)(c - z_{n-k})^2\left[(c-1) z_n z_{n-k} + c z_{n-k} + c^2\right]}{z_{n-k}(z_n z_{n-k} + c)^2}, \quad n \geq 0, \tag{17}$$

which leads to a strong negative feedback property:

$$(z_{n-k} - z_{n+1})\left(\frac{c^2}{z_{n-k}} - z_{n+1}\right) \leq 0, \quad n \geq 0, \tag{18}$$

with equality for all $n \geq 0$ if and only if $z_n = c, n \geq -k$. It, therefore, follows from a stability theorem (Corollary 3 of [14]) that the equilibrium solution $\overline{z} = c$ of the rational nonlinear difference equation (3) is globally asymptotically stable. Thus, the proof is finished. □

The above theorems with $c = 1$ gives the results in [4] ($k = 1$), [5] ($k = 2$) and [7] (a general k). There have also been similar studies on polynomial difference equations (see, for example, [15]) and other studies on positive rational function solutions, called lumps, to partial differential equations (see, e.g., [16]).

2.3. Illustrative Examples and an Open Question. To illustrate the global properties stated in Theorems 2 and 3, here we present two illustrative examples associated with two special cases:

$$c = \frac{3}{2}, \quad k = 3, \quad z_{-3} = \frac{6}{5}, \quad z_{-2} = \frac{16}{9}, \quad z_{-1} = \frac{10}{7}, \quad z_0 = 1;$$

$$c = 2, \quad k = 5, \quad z_{-5} = \frac{5}{4}, \quad z_{-4} = \frac{7}{3}, \quad z_{-3} = \frac{5}{3}, \quad z_{-2} = \frac{9}{4}, \quad z_{-1} = \frac{3}{2}, \quad z_0 = \frac{7}{3}; \tag{19}$$

in Figure 1. From the plot pictures, we see that the convergence is achieved very fast in both cases.

Finally, let $k \geq 1$. For an oscillatory solution $\{z_n\}_{n=-k}^{\infty}$ of the rational nonlinear difference equation (3), we define

$$N_g = \{n \mid z_n > c \text{ and } n \geq 0\}, \quad N_l = \{n \mid z_n < c \text{ and } n \geq 0\}. \tag{20}$$

Since $\{z_n\}_{n=-k}^{\infty}$ is oscillatory, Theorem 2 guarantees that both N_g and N_l have infinitely many numbers. A basic open question that we are very interested in is if z_n is decreasing on N_g and increasing on N_l. We point out that through the above two examples, we failed to find any counterexample to this statement, but found that two cases could occur: either $z_{n-1}, z_{n+1} < c$ but $z_n > c$ or $z_{n-1}, z_{n+1} > c$ but $z_n < c$ for some $n > 1$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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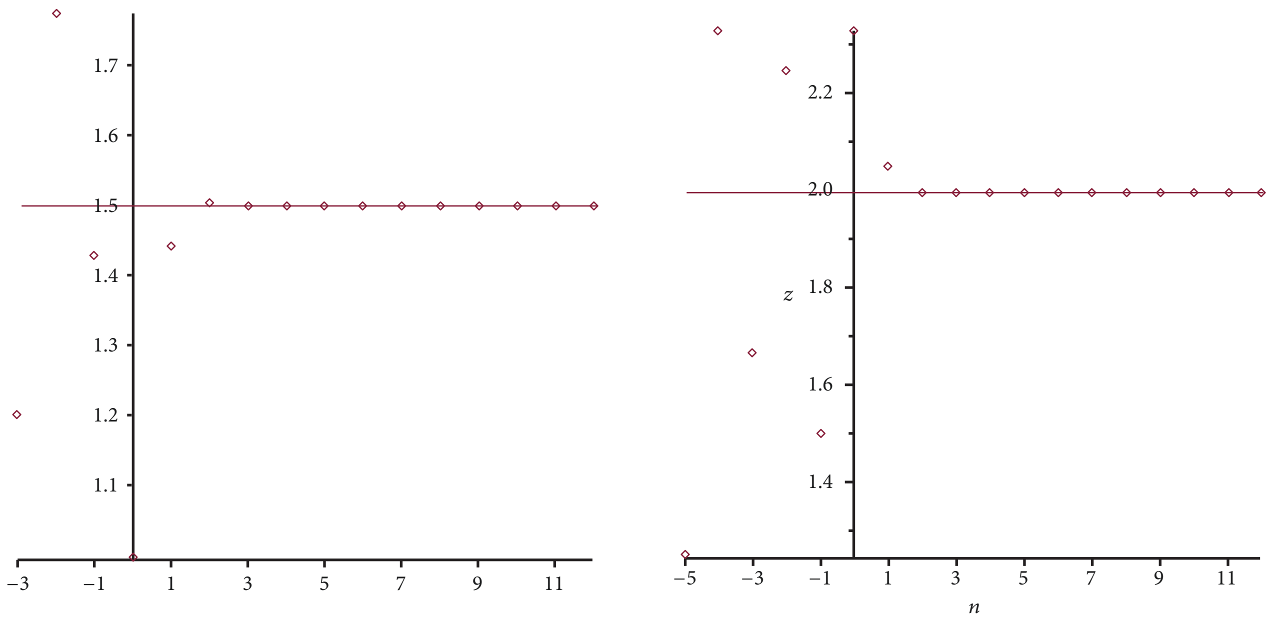

FIGURE 1: Profiles of $\{z_n\}_{n=-k}^{\infty}$ with $c = 3/2,\ k = 3$ (left) and $c = 2,\ k = 5$ (right).

References

[1] P. M. Batchelder, *An Introduction to Linear Difference Equations*, Dover Publications, New York, NY, USA, 1967.

[2] V. L. Kocic and G. Ladas, *Global behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer, Dordrecht, The Netherlands, 1993.

[3] E. Camouzis, G. Ladas, I. W. Rodrigues, and S. Northshield, "The rational recursive sequence $x_{n+1} = \beta x_n^2/(1 + x_{n-1}^2)$," *Computers and Mathematics with Applications*, vol. 28, no. 1-3, pp. 37–43, 1994.

[4] X. Li and D. Zhu, "Global asymptotic stability in a rational equation," *Journal of Difference Equations and Applications*, vol. 9, no. 9, pp. 833–839, 2003.

[5] X. Li and D. Zhu, "Two rational recursive sequences," *Computers & Mathematics with Applications*, vol. 47, no. 10-11, pp. 1487–1494, 2004.

[6] M. Ben H. Rhouma, "The Fibonacci sequence modulo π, chaos and some rational recursive equations," *Journal of Mathematical Analysis and Applications*, vol. 310, no. 2, pp. 506–517, 2005.

[7] R. Abu-Saris, C. Çina, and I. Yalçinkaya, "On the asymptotic stability of $x_{n+1} = (a + x_n x_{n-k})/(x_n + x_{n-k})$," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1172–1175, 2008.

[8] A. Gelişken, C. Çinar, and A. S. Kurbanli, "On the asymptotic behavior and periodic nature of a difference equation with maximum," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 898–902, 2010.

[9] W. X. Ma, "Trigonal curves and algebro-geometric solutions to soliton hierarchies I," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 473, no. 2203, p. 20170232, 2017.

[10] W. X. Ma, "Trigonal curves and algebro-geometric solutions to soliton hierarchies II," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 473, no. 2203, p. 20170233, 2017.

[11] W. X. Ma, "The inverse scattering transform and soliton solutions of a combined modified Korteweg–de Vries equation," *Journal of Mathematical Analysis and Applications*, vol. 471, no. 1-2, pp. 796–811, 2019.

[12] H. Sedaghat, "The impossibility of unstable, globally attracting fixed points for continuous mappings of the line," *The American Mathematical Monthly*, vol. 104, no. 4, pp. 356–358, 1997.

[13] A. Amleh, N. Kruse, and G. Ladas, "On a class of difference equations with strong negative feedback," *Journal of Difference Equations and Applications*, vol. 5, no. 6, pp. 497–515, 1999.

[14] N. Kruse and T. Nesemann, "Global asymptotic stability in some discrete dynamical systems," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 1, pp. 151–158, 1999.

[15] X. Li and D. Zhu, "Global asymptotic stability for two recursive difference equations," *Applied Mathematics and Computation*, vol. 150, no. 2, pp. 481–492, 2004.

[16] W. X. Ma and Y. Zhou, "Lump solutions to nonlinear partial differential equations via Hirota bilinear forms," *Journal of Differential Equations*, vol. 264, no. 4, pp. 2633–2659, 2018.