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Reduced matrix integrable hierarchies via group reduction involving off-diagonal block matrices

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Abstract
This paper proposes an innovative form of group reduction or similarity transformation involving off-diagonal block matrices. The proposed method is applied to the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problem, leading to the generation of reduced matrix AKNS integrable hierarchies. As a result, a variety of reduced multiple-component integrable nonlinear Schrödinger and modified Korteweg–de Vries models are derived from the analysis of the reduced AKNS matrix spectral problem.

Keywords: Lax pair, integrable model, zero-curvature equation, soliton hierarchy, group reduction

1. Introduction

Hamiltonian hierarchies of integrable structures are constructed from the Lax pair formulation of matrix spectral problems [1], with the initial step being the selection of an appropriate matrix spatial spectral problem. The inverse scattering transform has emerged as a powerful technique for solving initial value problems in nonlinear equations, especially in the context of nonlinear integrable models [2, 3]. It is well-established that key integrable models, for example, the nonlinear Schrödinger (NLS) and modified Korteweg–de Vries (mKdV) integrable structures, can be derived from the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problem through an individual group constraint [4–6]. Furthermore, the application of multiple group constraints allows for the generation of a broader family of integrable structures with specific properties [7]. A significant challenge arises in ensuring compatibility between the various reductions imposed on the system and the resulting potentials. These constraints introduce additional conditions that must be satisfied while preserving the invariance of the corresponding zero-curvature equations under the imposed reductions [8].

In recent years, group constraints have been increasingly applied as effective tools in the study of integrable structures. In particular, nonlocal integrable equations exhibiting reflection-type symmetries have emerged through such methods [9, 10]. A detailed taxonomy of integrable structures of lower order associated with the AKNS matrix spectral problem has identified three types of nonlocal NLS equations and two types of nonlocal mKdV equations [11]. Alongside these developments, various effective techniques have been established for analyzing reduced novel integrable structures, especially in constructing soliton solutions and formulating their associated Riemann–Hilbert problems. A nonlinear analog of the Fourier transform, known as the inverse scattering transform, has been developed and further extended to address initial value problems associated with nonlocal integrable structures (see, e.g. [12–14]). Additional powerful methods include Darboux and Bäcklund transformations, the Riemann–Hilbert approach and the Hirota bilinear method, all of which have been successfully employed in the study of nonlocal integrable models. Furthermore, the broader mathematical framework of integrable structures has been significantly generalized to incorporate various types of nonlocal scenarios (see [11, 15–18]).

In this paper, we introduce a novel form of group reduction, or similarity transformation, that leads to new sorts of reduced integrable structures. The central contribution lies in formulating a distinct similarity transformation based on off-diagonal block matrix structures. To establish the foundation for our analysis, section 2 revisits the matrix AKNS spectral problems and their corresponding hierarchies of integrable structures. We then present a group reduction or similarity transformation that gives rise to reduced matrix NLS and mKdV integrable hierarchies, with particular attention to the matrix NLS and mKdV systems. Section 3 provides four illustrative examples, each employing specific pairs of submatrices to define the reduction. These case studies underscore the variety of reduced spectral problems and the nonlinear integrable structures that arise from them. Lastly, we conclude with a summary of the main results and their implications.

2. Deriving matrix integrable models from group reduction

2.1. The AKNS integrable hierarchies revisited

Let m, n be two arbitrarily given natural numbers. We introduce two matrix potentials p and q:

$$p = p(x, t) = (p_{jk})_{m\times n}, \quad q = q(x, t) = (q_{kj})_{n\times m}, \tag{2.1}$$

and let the dependent variable vector, building from p and q, be denoted by u. Let $r \geqslant 0$ be arbitrarily given. The associated standard matrix AKNS spectral problems are described as follows:

$$-\mathrm{i}\phi_x = U\phi, \quad -\mathrm{i}\phi_t = V^{[r]}\phi, \tag{2.2}$$

where the Lax pairs are determined by

$$\begin{cases} U = U(u, \lambda) = \lambda\Lambda + P, \quad \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \quad P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \\ V^{[r]} = V^{[r]}(u, \lambda) = \lambda^r\Omega + Q^{[r]}, \quad \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \quad Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \end{cases} \tag{2.3}$$

In the above Lax pairs, λ denotes the spectral parameter, I_k is the identity matrix of size k, α_1, α_2 and β_1, β_2 are two sets of distinct constants, each chosen arbitrarily, $Q^{[0]}$ is the zero matrix of order $m + n$, and

$$W = \sum_{s\geqslant 0} \lambda^{-s} W^{[s]} = \sum_{s\geqslant 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \tag{2.4}$$

solves the stationary zero-curvature equation

$$W_x + \mathrm{i}[W, U] = 0, \tag{2.5}$$

provided with the initial selection $W^{[0]} = \Omega$. This formal series solution plays a key role in generating integrable hierarchies (see, e.g. [19, 20]).

A common key object are the zero-curvature conditions:

$$U_t - V_x^{[r]} + \mathrm{i}[U, V^{[r]}] = 0, \quad r \geqslant 0, \tag{2.6}$$

which are the compatibility conditions of the two matrix spectral problems in equation (2.2). Together, they produce a matrix AKNS hierarchy of integrable structures:

$$p_t = \mathrm{i}\alpha b^{[r+1]}, \quad q_t = -\mathrm{i}\alpha c^{[r+1]}, \quad r \geqslant 0, \tag{2.7}$$

where $\alpha = \alpha_1 - \alpha_2$. In the basic case where m and n each equal one, the construction reduces to the integrable hierarchy characterized by two scalar dependent variables [21]. An integrable characteristic is that the above class of matrix models possesses a bi-Hamiltonian formulation, along with infinitely many symmetries and conserved quantities (see, e.g. [22–24]).

The integrable models in equation (2.7) naturally separate into two classes: those corresponding to even–even values of r and those to odd values. These form the matrix hierarchies associated with the NLS and mKdV integrable structures, respectively. The two first nonlinear (which corresponds to the case when s is equal to one) integrable structures in the resulted matrix NLS and mKdV integrable sequences present the matrix NLS and mKdV models:

$$\begin{cases} p_t = -\frac{\beta}{\alpha^2}\mathrm{i}(p_{xx} + 2pqp), \\ q_t = \frac{\beta}{\alpha^2}\mathrm{i}(q_{xx} + 2qpq), \end{cases} \tag{2.8}$$

and

$$\begin{cases} p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp), \\ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x), \end{cases} \tag{2.9}$$

respectively, where $\beta = \beta_1 - \beta_2$. The corresponding Lax matrices $V^{[2]}$ and $V^{[3]}$ are given by

$$\begin{aligned} V^{[2]} &= \lambda^2\Omega + \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + \mathrm{i}P_x), \\ V^{[3]} &= \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + \mathrm{i}P_x) \\ &\quad - \frac{\beta}{\alpha^3}(\mathrm{i}[P, P_x] + P_{xx} + 2P^3), \end{aligned} \tag{2.10}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. Illustrative scenarios leading to higher-order matrix integrable structures can likewise be developed (see, for example, [25]).

The following analysis addresses a specific sort of the above matrix spectral problems characterized by a particular form of the potential matrices. Assume that

$$m = n, \quad \alpha_1 = -\alpha_2 = 1, \quad \beta_1 = -\beta_2 = 2, \qquad (2.11)$$

where n is an arbitrary natural number. That is, we restrict our attention to integrable reductions of the above matrix NLS and mKdV models under the condition $m = n$, which yields two square matrix potentials p and q of the same dimension.

2.2. Integrable reductions via similarity transformations

We propose a novel sort of group constraint by considering two constant invertible nth order matrix blocks, denoted by Δ_1 and Δ_2, and define an invertible constant square matrix of order $2n$ as follows:

$$\Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}. \qquad (2.12)$$

Here, Δ is a block matrix with off-diagonal elements. Noting equation (2.11) and

$$\Delta^{-1} = \begin{bmatrix} 0 & \Delta_2^{-1} \\ \Delta_1^{-1} & 0 \end{bmatrix}. \qquad (2.13)$$

The matrix Δ is found to satisfy key similarity transformation properties:

$$\Delta\Lambda\Delta^{-1} + \Lambda = 0, \quad \Delta\Omega\Delta^{-1} + \Omega = 0. \qquad (2.14)$$

With these structures established, we proceed to introduce the following group constraint, referred to as a group reduction or similarity transformation:

$$\Delta U(\lambda)\Delta^{-1} = -(U(\lambda))^{\mathrm{T}} = -U^{\mathrm{T}}(\lambda). \qquad (2.15)$$

Here, A^{T} denotes the transposed matrix. We will demonstrate that this group reduction or similarity transformation will uphold the invariance property of the original zero-curvature equations. Given the specific structure of the spectral matrix U, the group reduction or similarity transformation imposes the following constraint on P:

$$\Delta P\Delta^{-1} = -P^{\mathrm{T}}. \qquad (2.16)$$

As a result of this reduction, the matrix potentials p and q need to satisfy:

$$p^{\mathrm{T}} = -\Delta_2 p\Delta_1^{-1}, \quad q^{\mathrm{T}} = -\Delta_1 q\Delta_2^{-1}. \qquad (2.17)$$

In summary, the group reduction or similarity transformation given by equation (2.15) produces a sort of reduced spectral problems:

$$-\mathrm{i}\phi_x = U\phi, \quad U = \begin{bmatrix} \lambda I_n & p \\ q & -\lambda I_n \end{bmatrix}, \qquad (2.18)$$

where p and q are constrained as stated in equation (2.17).

2.3. Matrix AKNS integrable hierarchies arising from group reduction

Let us examine how the imposed group reduction or similarity transformation in equation (2.15) affect the Laurent series matrix W determined by equation (2.4), given the initial data

$$W^{[0]} = \Omega = \begin{bmatrix} 2I_n & 0 \\ 0 & -2I_n \end{bmatrix}. \qquad (2.19)$$

First, we can readily check

$$\Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty} = -(W(\lambda))^{\mathrm{T}}|_{\lambda=\infty} = -W^{\mathrm{T}}(\lambda)|_{\lambda=\infty}. \qquad (2.20)$$

Consequently, the uniqueness of solutions to the stationary zero-curvature equation implies that

$$\Delta W(\lambda)\Delta^{-1} = -(W(\lambda))^{\mathrm{T}} = -W^{\mathrm{T}}(\lambda). \qquad (2.21)$$

Furthermore, for all $r \geqslant 0$, we can show that

$$\Delta V^{[r]}(\lambda)\Delta^{-1} = -(V^{[r]}(\lambda))^{\mathrm{T}} = -V^{[r]\mathrm{T}}(\lambda). \qquad (2.22)$$

As a consequence of the group reduction or similarity transformation in equation (2.15), it is found that

$$\begin{aligned} &\Delta(U_t - V_x^{[r]} + \mathrm{i}[U, V^{[r]}])(\lambda)\Delta^{-1} \\ &= (-U^{\mathrm{T}}(\lambda))_t - (-V^{[r]\mathrm{T}}(\lambda))_x + \mathrm{i}[-U^{T}(\lambda), -V^{[r]\mathrm{T}}(\lambda)] \\ &= -((U_t - V_x^{[r]} + \mathrm{i}[U, V^{[r]}])(\lambda))^{\mathrm{T}}, \end{aligned} \qquad (2.23)$$

and therefore, the matrix AKNS integrable models in equation (2.7) form a reduced hierarchy of integrable models:

$$p_t = 2\mathrm{i}b^{[r+1]}|_{(2.17)}, \quad q_t = -2\mathrm{i}c^{[r+1]}|_{(2.17)}, \quad r \geqslant 0. \quad (2.24)$$

The matrix spectral problems, comprising (2.18) and

$$-\mathrm{i}\phi_t = V^{[r]}|_{(2.17)}\phi, \quad r \geqslant 0, \qquad (2.25)$$

present a corresponding pair of matrix spectral problems associated with every member of the reduced hierarchy of integrable structures given in equation (2.24).

The integrability and mutual commutativity of the reduced models in each hierarchy stem from the underlying algebraic composition of the pertinent Lax matrix algebras (see, e.g. [26]). We emphasize that the two invertible square matrices, Δ_1 and Δ_2, are independently and arbitrarily selected. Once Δ_1 and Δ_2 are appropriately chosen, various corresponding hierarchies of reduced integrable models can be generated. It is also important to mention that integrable NLS- and mKdV-type models can be derived in a similar manner, based on symmetric spaces, which are special reductions of the general linear algebra (see, e.g. [27, 28]).

3. Case studies

We proceed in this section to exemplify the general framework through four specific cases, each yielding reduced matrix AKNS spectral problems and associated NLS and mKdV integrable structures. We consider eight distinct combinations of the parameters, assuming that:

$$\sigma = \pm 1, \quad \delta = \pm 1, \quad \gamma = \pm 1, \tag{3.1}$$

which results in eight possible scenarios to explore.

Case Study 3.a: We begin our analysis with the case in which n is equal to two. We select the following specific values for the pair of matrices:

$$\Delta_1 = \begin{bmatrix} 0 & 1 \\ \delta & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & -\delta \\ -1 & 0 \end{bmatrix}. \tag{3.2}$$

Then, the group reduction or similarity transformation in equation (2.15) yields

$$U = U(u, \lambda) = \begin{bmatrix} \lambda I_2 & p \\ q & -\lambda I_2 \end{bmatrix}$$
$$\text{with } p = \begin{bmatrix} p_2 & p_1 \\ p_3 & \delta p_2 \end{bmatrix}, q = \begin{bmatrix} q_2 & q_1 \\ q_3 & \delta q_2 \end{bmatrix}, \tag{3.3}$$

where $u = (p_1, p_2, p_3, q_1, q_2, q_3)^{\mathrm{T}}$. As a result, the corresponding reduced novel integrable models are formulated as:

$$\begin{cases} \mathrm{i}p_{1,t} = p_{1,xx} + 2(p_1^2 q_3 + 2p_1 p_2 q_2 + \delta p_2^2 q_1), \\ \mathrm{i}p_{2,t} = p_{2,xx} + 2(p_1 p_2 q_3 + \delta p_1 p_3 q_2 + p_2^2 q_2 + p_2 p_3 q_1), \\ \mathrm{i}p_{3,t} = p_{3,xx} + 2(\delta p_2^2 q_3 + 2p_2 p_3 q_2 + p_3^2 q_1), \end{cases} \tag{3.4}$$

$$\begin{cases} -\mathrm{i}q_{1,t} = q_{1,xx} + 2(p_3 q_1^2 + 2p_2 q_1 q_2 + \delta p_1 q_2^2), \\ -\mathrm{i}q_{2,t} = q_{2,xx} + 2(p_1 q_2 q_3 + \delta p_2 q_1 q_3 + p_2 q_2^2 + p_3 q_1 q_2), \\ -\mathrm{i}q_{3,t} = p_{3,xx} + 2(p_1 q_3^2 + 2p_2 q_2 q_3 + \delta p_3 q_2^2); \end{cases} \tag{3.5}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2}p_{1,xxx} - 3[(p_1 q_3 + p_2 q_2)p_{1,x} + (p_1 q_2 + \delta p_2 q_1)p_{2,x}], \\ p_{2,t} = -\frac{1}{2}p_{2,xxx} - \frac{3}{2}[(p_2 q_3 + \delta p_3 q_2)p_{1,x} + (p_1 q_3 + 2p_2 q_2 + p_3 q_1)p_{2,x} + (\delta p_1 q_2 + p_2 q_1)p_{3,x}], \\ p_{3,t} = -\frac{1}{2}p_{3,xxx} - 3[(\delta p_2 q_3 + p_3 q_2)p_{2,x} + (p_2 q_2 + p_3 q_1)p_{3,x}], \end{cases} \tag{3.6}$$

$$\begin{cases} q_{1,t} = -\frac{1}{2}q_{1,xxx} - 3[(p_2 q_2 + p_3 q_1)q_{1,x} + (\delta p_1 q_2 + p_2 q_1)q_{2,x}], \\ q_{2,t} = -\frac{1}{2}q_{2,xxx} - \frac{3}{2}[(\delta p_2 q_3 + p_3 q_2)q_{1,x} + (p_1 q_3 + 2p_2 q_2 + p_3 q_1)q_{2,x} + (p_1 q_2 + \delta p_2 q_1)q_{3,x}], \\ q_{3,t} = -\frac{1}{2}q_{3,xxx} - 3[(p_2 q_3 + \delta p_3 q_2)q_{2,x} + (p_1 q_3 + p_2 q_2)q_{3,x}]; \end{cases} \tag{3.7}$$

respectively.

Case Study 3.b: We now examine the case with n being two and set the pair of matrices to the following specific values:

$$\Delta_1 = \begin{bmatrix} 1 & \sigma \\ 0 & \delta \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} -1 & 0 \\ -\sigma & -\delta \end{bmatrix}. \tag{3.8}$$

Then, the group reduction or similarity transformation in equation (2.15) leads to

$$U = U(u, \lambda) = \begin{bmatrix} \lambda I_2 & p \\ q & -\lambda I_2 \end{bmatrix}$$
$$\text{with } p = \begin{bmatrix} p_1 & \sigma p_1 + \delta p_3 \\ p_3 & p_2 \end{bmatrix}, q = \begin{bmatrix} q_2 & \delta q_3 - \sigma q_1 \\ q_3 & q_1 \end{bmatrix}, \tag{3.9}$$

where u is a six-dimensional vector defined as $u = (p_1, p_2, p_3, q_1, q_2, q_3)^{\mathrm{T}}$. Furthermore, the corresponding reduced matrix NLS and mKdV equations take the following form:

$$\begin{cases} \mathrm{i}p_{1,t} = p_{1,xx} + 2[p_1^2(q_2 + \sigma q_3) + 2\delta p_1 p_3 q_3 + \delta p_3^2 q_1], \\ \mathrm{i}p_{2,t} = p_{2,xx} + 2[\delta p_3^2 q_2 + p_2 p_3(2\delta q_3 - \sigma q_1) + \sigma p_1 p_3 q_2 + p_2(\sigma p_1 q_3 + p_2 q_1)], \\ \mathrm{i}p_{3,t} = p_{3,xx} + 2[p_3^2(\delta q_3 - \sigma q_1) + p_1 p_2 q_3 + p_1 p_3 q_2 + p_2 p_3 q_1], \end{cases} \tag{3.10}$$

$$\begin{cases} -\mathrm{i}q_{1,t} = q_{1,xx} + 2[(p_2 - \sigma p_3)q_1^2 + 2\delta p_3 q_1 q_3 + \delta p_1 q_3^2], \\ -\mathrm{i}q_{2,t} = q_{2,xx} + 2[p_1 q_2^2 + (\sigma p_1 + 2\delta p_3)q_2 q_3 - \sigma p_3 q_1 q_2 + p_2(\delta q_3 - \sigma q_1)q_3], \\ -\mathrm{i}q_{3,t} = p_{3,xx} + 2[(\sigma p_1 + \delta p_3)q_3^2 + p_1 q_2 q_3 + p_2 q_1 q_3 + p_3 q_1 q_2]; \end{cases} \tag{3.11}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2}p_{1,xxx} - 3[(p_1 q_2 + \sigma p_1 q_3 + \delta p_3 q_3)p_{1,x} + \delta(p_1 q_3 + p_3 q_1)p_{3,x}], \\ p_{2,t} = -\frac{1}{2}p_{2,xxx} - \frac{3}{2}[\sigma(p_2 q_3 + p_3 q_2)p_{1,x} + (\sigma p_1 q_3 + 2p_2 q_1 - \sigma p_3 q_1 + 2\delta p_3 q_3)p_{2,x} \\ \quad +(\sigma p_1 q_2 - \sigma p_2 q_1 + 2\delta p_2 q_3 + 2\delta p_3 q_2)p_{3,x}], \\ p_{3,t} = -\frac{1}{2}p_{3,xxx} - \frac{3}{2}[(p_2 q_3 + p_3 q_2)p_{1,x} + (p_1 q_3 + p_3 q_1)p_{2,x} \\ \quad +(p_1 q_2 + p_2 q_1 - 2\sigma p_3 q_1 + 2\delta p_3 q_3)p_{3,x}], \end{cases} \tag{3.12}$$

$$\begin{cases} q_{1,t} = -\frac{1}{2}q_{1,xxx} - 3[(p_2 q_2 - \sigma p_3 q_2 + \delta p_3 q_3)q_{2,x} + \delta(p_1 q_3 + p_3 q_2)q_{3,x}], \\ q_{2,t} = -\frac{1}{2}q_{2,xxx} - \frac{3}{2}[(2p_1 q_1 + \sigma p_1 q_3 - \sigma p_3 q_2 + 2\delta p_3 q_3)q_{1,x} - \sigma(p_2 q_3 + p_3 q_1)q_{2,x} \\ \quad +(\sigma p_1 q_1 - \sigma p_2 q_2 + 2\delta p_2 q_3 + 2\delta p_3 q_1)q_{3,x}], \\ q_{3,t} = -\frac{1}{2}q_{3,xxx} - \frac{3}{2}[(p_1 q_3 + p_3 q_2)q_{1,x} + (p_2 q_3 + p_3 q_1)q_{2,x} \\ \quad +(p_1 q_1 + 2\sigma p_1 q_3 + p_2 q_2 + 2\delta p_3 q_3)q_{3,x}]; \end{cases} \tag{3.13}$$

respectively.

Case Study 3.c: In the case $n = 3$, we adopt the following pair of matrix selections:

$$\Delta_1 = \Delta_2 = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \gamma \end{bmatrix}. \tag{3.14}$$

Now, the group reduction or similarity transformation in equation (2.15) engenders

$$U = U(u, \lambda) = \begin{bmatrix} \lambda I_3 & p \\ q & -\lambda I_3 \end{bmatrix}$$
$$\text{with} \quad p = \begin{bmatrix} 0 & p_1 & p_2 \\ -\sigma\delta p_1 & 0 & p_3 \\ -\sigma\gamma p_2 & -\delta\gamma p_3 & 0 \end{bmatrix},$$
$$q = \begin{bmatrix} 0 & q_1 & q_2 \\ -\sigma\delta q_1 & 0 & q_3 \\ -\sigma\gamma q_2 & -\delta\gamma q_3 & 0 \end{bmatrix}, \tag{3.15}$$

where u is again the six-dimensional vector given by $u = (p_1, p_2, p_3, q_1, q_2, q_3)^{\mathrm{T}}$. The system of equations for $n = 3$ exhibits interwoven interaction mechanisms between the six components of the potential matrix, encompassing nonlinear effects as well as differential operators. These nontrivial coupling mechanisms capture the structural features of the reduced linear eigenvalue problems under consideration, along with the corresponding NLS and mKdV integrable structures. The corresponding reduced matrix NLS and mKdV integrable structures are determined by:

$$\begin{cases} \mathrm{i}p_{1,t} = p_{1,xx} - 2p_1(\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3), \\ \mathrm{i}p_{2,t} = p_{2,xx} - 2p_2(\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3), \\ \mathrm{i}p_{3,t} = p_{3,xx} - 2p_3(\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3), \end{cases} \tag{3.16}$$

$$\begin{cases} -\mathrm{i}q_{1,t} = q_{1,xx} - 2q_1(\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3), \\ -\mathrm{i}q_{2,t} = q_{2,xx} - 2q_2(\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3), \\ -\mathrm{i}q_{3,t} = p_{3,xx} - 2q_3(\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3); \end{cases} \tag{3.17}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2}p_{1,xxx} + \frac{3}{2}[(2\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3)p_{1,x} + \sigma\gamma p_1 q_2 p_{2,x} + \delta\gamma p_1 q_3 p_{3,x}], \\ p_{2,t} = -\frac{1}{2}p_{2,xxx} + \frac{3}{2}[\sigma\delta p_2 q_1 p_{1,x} + (\sigma\delta p_1 q_1 + 2\sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3)p_{2,x} + \delta\gamma p_2 q_3 p_{3,x}], \\ p_{3,t} = -\frac{1}{2}p_{3,xxx} + \frac{3}{2}[\sigma\delta p_3 q_1 p_{1,x} + \sigma\gamma p_3 q_2 p_{2,x} + (\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + 2\delta\gamma p_3 q_3)p_{3,x}], \end{cases} \tag{3.18}$$

$$\begin{cases} q_{1,t} = -\frac{1}{2}q_{1,xxx} + \frac{3}{2}[(2\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3)q_{1,x} + \sigma\gamma p_2 q_1 q_{2,x} + \delta\gamma p_3 q_1 q_{3,x}], \\ q_{2,t} = -\frac{1}{2}q_{2,xxx} + \frac{3}{2}[\sigma\delta p_1 q_2 q_{1,x} + (\sigma\delta p_1 q_1 + 2\sigma\gamma p_2 q_2 + \delta\gamma p_3 q_3)q_{2,x} + \delta\gamma p_3 q_2 q_{3,x}], \\ q_{3,t} = -\frac{1}{2}q_{3,xxx} + \frac{3}{2}[\sigma\delta p_1 q_3 q_{1,x} + \sigma\gamma p_2 q_3 q_{2,x} + (\sigma\delta p_1 q_1 + \sigma\gamma p_2 q_2 + 2\delta\gamma p_3 q_3)q_{3,x}]; \end{cases} \tag{3.19}$$

respectively.

Case Study 3.d: For $n = 3$, let us consider another pair of matrix choices:

$$\Delta_1 = \begin{bmatrix} 0 & 0 & \gamma \\ 0 & \delta & 0 \\ \sigma & 0 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & 0 & \sigma \\ 0 & \delta & 0 \\ \gamma & 0 & 0 \end{bmatrix}. \tag{3.20}$$

In this case, the group reduction or similarity transformation in equation (2.15) generates

$$U = U(u, \lambda) = \begin{bmatrix} \lambda I_3 & p \\ q & -\lambda I_3 \end{bmatrix} \text{with} \quad p = \begin{bmatrix} p_2 & p_1 & 0 \\ p_3 & 0 & -\delta\gamma p_1 \\ 0 & -\sigma\delta p_3 & -\sigma\gamma p_2 \end{bmatrix}, q = \begin{bmatrix} q_2 & q_1 & 0 \\ q_3 & 0 & -\sigma\delta q_1 \\ 0 & -\delta\gamma q_3 & -\sigma\gamma q_2 \end{bmatrix}, \tag{3.21}$$

where $u = (p_1, p_2, p_3, q_1, q_2, q_3)^{\mathrm{T}}$. Note that p and q differ slightly from each other due to the distinction between Δ_1 and Δ_2. This shows that by choosing a structurally different set of matrices, interesting nonlinear interactions arises in the reduced AKNS spectral matrix, as well as in the corresponding NLS and mKdV equations. While the overall structure of the equations is analogous to that of Example 3.c, significant differences appear in the nonlinear interaction terms, attributed to the variations in the matrices Δ_1 and Δ_2. The novel reduced matrix integrable models are therefore expressed as follows:

$$\begin{cases} \mathrm{i}p_{1,t} = p_{1,xx} + 2p_1(p_1 q_3 + p_2 q_2 + p_3 q_1), \\ \mathrm{i}p_{2,t} = p_{2,xx} + 2p_2(p_1 q_3 + p_2 q_2 + p_3 q_1), \\ \mathrm{i}p_{3,t} = p_{3,xx} + 2p_3(p_1 q_3 + p_2 q_2 + p_3 q_1), \end{cases} \tag{3.22}$$

$$\begin{cases} -\mathrm{i}q_{1,t} = q_{1,xx} + 2q_1(p_1 q_3 + p_2 q_2 + p_3 q_1), \\ -\mathrm{i}q_{2,t} = q_{2,xx} + 2q_2(p_1 q_3 + p_2 q_2 + p_3 q_1), \\ -\mathrm{i}q_{3,t} = p_{3,xx} + 2q_3(p_1 q_3 + p_2 q_2 + p_3 q_1); \end{cases} \tag{3.23}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2}p_{1,xxx} - \frac{3}{2}[(2p_1 q_3 + p_2 q_2 + p_3 q_1)p_{1,x} + p_1 q_2 p_{2,x} + p_1 q_1 p_{3,x}], \\ p_{2,t} = -\frac{1}{2}p_{2,xxx} - \frac{3}{2}[p_2 q_3 p_{1,x} + (p_1 q_3 + 2p_2 q_2 + p_3 q_1)p_{2,x} + p_2 q_1 p_{3,x}], \\ p_{3,t} = -\frac{1}{2}p_{3,xxx} - \frac{3}{2}[p_3 q_3 p_{1,x} + p_3 q_2 p_{2,x} + (p_1 q_3 + p_2 q_2 + 2p_3 q_1)p_{3,x}], \end{cases} \tag{3.24}$$

$$\begin{cases} q_{1,t} = -\frac{1}{2}q_{1,xxx} - \frac{3}{2}[(p_1 q_3 + p_2 q_2 + 2p_3 q_1)q_{1,x} + p_2 q_1 q_{2,x} + p_1 q_1 q_{3,x}], \\ q_{2,t} = -\frac{1}{2}q_{2,xxx} - \frac{3}{2}[p_3 q_2 q_{1,x} + (p_1 q_3 + 2p_2 q_2 + p_3 q_1)q_{2,x} + p_1 q_2 q_{3,x}], \\ q_{3,t} = -\frac{1}{2}q_{3,xxx} - \frac{3}{2}[p_3 q_3 q_{1,x} + p_2 q_3 q_{2,x} + (2p_1 q_3 + p_2 q_2 + p_3 q_1)q_{3,x}]; \end{cases} \tag{3.25}$$

respectively.

In each of the examples presented, incorporating the spectral matrix exposes nonlinear interactions crucial to the integrable structure of the multiple-component NLS and mKdV models. The varying parameters σ, δ and γ serve as key factors in modulating the system's dynamics and controlling the nature of interactions between the components.

These examples demonstrate the applicability and fundamental role of the linear spectral problem formulation in constructing integrable models. This newly adopted approach combines multiple group constraints, facilitating the generation of a broad family of reduced integrable structures, each exhibiting distinct characteristics (see, e.g. [29–32]). The combination of distinct constraints enable the exploration of multiple nonlinear dispersive wave behaviors, with promising applications in a wide range of scientific disciplines. In addition, these results contribute valuable insights to the field of integrable structures, associated with the 4×4 matrix spectral problems, as discussed in [33–37].

4. Concluding remarks

This paper presents a novel local group reduction or similarity transformation and applies it to a specific sort of linear spectral problems, which yields reduced hierarchies of matrix integrable structures. Several model scenarios of these reduced linear eigenvalue problems and their associated NLS and mKdV integrable structures are provided. A central contribution of this work is the introduction of a new group reduction or similarity transformation that involves off-diagonal block matrices. This approach offers a distinct perspective compared to prior studies [8, 37, 38], where similarity matrices were restricted to diagonal block forms.

These configurations showcase the adaptability of the linear spectral problem formulation in investigating integrable structures, showing how various group constraints can yield an extensive family of reduced integrable structures, each characterized by distinct nonlinear interactions. Together with the choice of parameters, the imposed group constraints fundamentally shape the symmetry properties of these integrable structures. The inherent adaptability of linear spectral problems allows for the development of tailored models, making it a versatile tool bridging theoretical insights and practical implementation.

By further pursuing this approach and investigating various forms of group reductions and similarity transformations, a broader spectrum of complex structures and special features inherent to integrable models can be uncovered. Such explorations may bring to light rich and diverse nonlinear wave dynamics, including soliton, positon, negaton and complexiton waves, breathers, lump waves, and rogue waves (see, e.g. [39–43])—as well as their connections to Bäcklund and Darboux transformations (see, e.g. [44, 45]). This line of investigation offers promising prospects for advancing research on a wide range of integrable structures, with potential applications across various areas of mathematical and physical sciences.

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