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Reduced nonlocal integrable mKdV equations of type $(-\lambda, \lambda)$ and their exact soliton solutions

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Abstarct

We conduct two group reductions of the Ablowitz–Kaup–Newell–Segur matrix spectral problems to present a class of novel reduced nonlocal reverse-spacetime integrable modified Korteweg–de Vries equations. One reduction is local, replacing the spectral parameter with its negative and the other is nonlocal, replacing the spectral parameter with itself. Then by taking advantage of distribution of eigenvalues, we generate soliton solutions from the reflectionless Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues.

Keywords: nonlocal integrable equation, soliton solution, Riemann–Hilbert problem

1. Introduction

Group reductions of matrix spectral problems can produce nonlocal integrable equations and keep the corresponding integrable structures that the original integrable equations possess [1–3]. If one group reduction is taken, we can obtain three kinds of nonlocal nonlinear Schrödinger equations and two kinds of nonlocal modified Kortweweg-de Vries (mKdV) equations [1, 4]. Recently, we have shown that a new kind of nonlocal integrable equations could be generated by conducting two group reductions simultaneously. The inverse scattering transform, Darboux transformation and the Hirota bilinear method can be applied to analysis of soliton solutions to nonlocal integrable equations [5–7].

The Riemann–Hilbert technique has been proved to be another powerful method to solve integrable equations, and especially to construct their soliton solutions [8, 9]. Various kinds of integrable equations have been investigated via analyzing the associated Riemann–Hilbert problems and we refer the interested readers to the recent studies [10–12] and [3, 13–15] for details in the local and nonlocal cases, respectively. In this paper, we would like to present a kind of novel reduced nonlocal integrable mKdV equations by taking two group reductions and construct their soliton solutions through the relectionless Riemann–Hilbert problems.

The rest of this paper is structured as follows. In section 2, we make two group reductions of the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems to generate type $(-\lambda, \lambda)$ reduced nonlocal integrable mKdV equations. Two scalar examples are

$$p_{1,t} = p_{1,xxx} - 6\sigma p_1^2 p_{1,x} - 3\sigma p_1(-x, -t)(p_1 p_1(-x, -t))_x,$$

and

$$p_{1,t} = p_{1,xxx} + 6\delta p_1 p_1(-x, -t) p_{1,x} + 3\delta (p_1 p_1(-x, -t))_x p_1,$$

where $\sigma = \delta = \pm 1$. In section 3, based on distribution of eigenvalues, we establish a formulation of solutions to the corresponding reflectionless Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues, and compute soliton solutions to the resulting reduced nonlocal integrable mKdV equations. In the last section, we gives a conclusion, together with a few concluding remarks.

2. Reduced nonlocal integrable mKdV equations

2.1. The matrix AKNS integrable hierarchies revisited

Let us recall the AKNS hierarchies of matrix integrable equations, which will be used in the subsequent analysis. As normal, let λ denote the spectral parameter, and assume that $m, n \geqslant 1$ are two given integers and p, q are two matrix potentials:

$$p = p(x, t) = (p_{jk})_{m\times n}, \quad q = q(x, t) = (q_{kj})_{n\times m}. \tag{1}$$

The matrix AKNS spectral problems are defined as follows:

$$\begin{cases} -\mathrm{i}\phi_x = U\phi = U(u, \lambda)\phi = (\lambda\Lambda + P)\phi, \\ -\mathrm{i}\phi_t = V^{[r]}\phi = V^{[r]}(u, \lambda)\phi = (\lambda^r\Omega + Q^{[r]})\phi, \quad r \geqslant 0. \end{cases} \tag{2}$$

Here the constant square matrices Λ and Ω are defined by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n), \quad \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{3}$$

with I_s being the identity matrix of size s, and α_1, α_2 and β_1, β_2 being two arbitrary pairs of distinct real constants. The other two involved square matrices of size $m + n$ are defined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{4}$$

called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{5}$$

where $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ are defined recursively as follows:

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a^{[0]} = \beta_1 I_m, \quad d^{[0]} = \beta_2 I_n, \tag{6a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-\mathrm{i}b_x^{[s]} - pd^{[s]} + a^{[s]}p), \quad s \geqslant 0, \tag{6b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(\mathrm{i}c_x^{[s]} + qa^{[s]} - d^{[s]}q), \quad s \geqslant 0, \tag{6c}$$

$$a_x^{[s]} = \mathrm{i}(pc^{[s]} - b^{[s]}q), \quad d_x^{[s]} = \mathrm{i}(qb^{[s]} - c^{[s]}p), \quad s \geqslant 1, \tag{6d}$$

with zero constants of integration being taken. Particularly, we can obtain

$$Q^{[1]} = \frac{\beta}{\alpha}P, \quad Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2}I_{m,n}(P^2 + \mathrm{i}P_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + \mathrm{i}P_x) - \frac{\beta}{\alpha^3}(\mathrm{i}[P, P_x] + P_{xx} + 2P^3),$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. The relations in (6) also imply that

$$W = \sum_{s\geqslant 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \tag{7}$$

solves the stationary zero curvature equation

$$W_x = \mathrm{i}[U, W], \tag{8}$$

which is crucial in defining an integrable hierarchy.

The compatibility conditions of the two matrix spectral problems in (2), i.e. the zero curvature equations

$$U_t - V_x^{[r]} + \mathrm{i}[U, V^{[r]}] = 0, \quad r \geqslant 0, \tag{9}$$

generate one so-called matrix AKNS integrable hierarchy (see, e.g. [16]):

$$p_t = \mathrm{i}\alpha b^{[r+1]}, \quad q_t = -\mathrm{i}\alpha c^{[r+1]}, \quad r \geqslant 0, \tag{10}$$

which has a bi-Hamiltonian structure. The second ($r = 3$) nonlinear integrable equations in the hierarchy give us the AKNS matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp),$$
$$q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x), \tag{11}$$

where the two matrix potentials, p and q, are defined by (1).

2.2. Reduced nonlocal integrable mKdV equations

We would like to construct a kind of novel reduced nonlocal integrable mKdV equations by taking two group reductions for the matrix AKNS spectral problems in (2). One reduction is local while the other is nonlocal (see also [17] for the local case).

Let Σ_1, Δ_1 and Σ_2, Δ_2 be two pairs of constant invertible symmetric matrices of sizes m and n, respectively. We consider two group reductions for the spectral matrix U:

$$U^{\mathrm{T}}(x, t, -\lambda) = (U(x, t, -\lambda))^{\mathrm{T}} = -\Sigma U(x, t, \lambda)\Sigma^{-1}, \tag{12}$$

and

$$U^{\mathrm{T}}(-x, -t, \lambda) = (U(-x, -t, \lambda))^{\mathrm{T}} = \Delta U(x, t, \lambda)\Delta^{-1}, \tag{13}$$

where the two constant invertible matrices, Σ and Δ, are defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{14}$$

These two group reductions lead equivalently to

$$P^{\mathrm{T}}(x, t) = -\Sigma P(x, t)\Sigma^{-1}, \tag{15}$$

and

$$P^{\mathrm{T}}(-x, -t) = \Delta P(x, t)\Delta^{-1}, \tag{16}$$

respectively. More precisely, they enable us to make the reductions for the matrix potentials:

$$q(x, t) = -\Sigma_2^{-1}p^{\mathrm{T}}(x, t)\Sigma_1, \tag{17}$$

and

$$q(x, t) = \Delta_2^{-1}p^{\mathrm{T}}(-x, -t)\Delta_1, \tag{18}$$

respectively. It then follows that to satisfy both group reductions in (12) and (13), an additional constraint is required for the matrix potential p:

$$-\Sigma_2^{-1}p^{\mathrm{T}}(x,t)\Sigma_1 = \Delta_2^{-1}p^{\mathrm{T}}(-x,-t)\Delta_1. \tag{19}$$

Moreover, we notice that under the group reductions in (12) and (13), we have that

$$\begin{cases} W^{\mathrm{T}}(x,t,-\lambda) = (W(x,t,-\lambda))^{\mathrm{T}} = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^{\mathrm{T}}(-x,-t,\lambda) = (W(-x,-t,\lambda))^{\mathrm{T}} = \Delta W(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{20}$$

which implies that

$$\begin{cases} V^{[2s+1]\mathrm{T}}(x,t,-\lambda) = (V^{[2s+1]}(x,t,-\lambda))^{\mathrm{T}} \\ \quad = -\Sigma V^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2s+1]\mathrm{T}}(-x,-t,\lambda) = (V^{[2s+1]}(-x,-t,\lambda))^{\mathrm{T}} \\ \quad = \Delta V^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{21}$$

and

$$\begin{cases} Q^{[2s+1]\mathrm{T}}(x,t,-\lambda) = (Q^{[2s+1]}(x,t,-\lambda))^{\mathrm{T}} \\ \quad = -\Sigma Q^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ Q^{[2s+1]\mathrm{T}}(-x,-t,\lambda) = (Q^{[2s+1]}(-x,-t,\lambda))^{\mathrm{T}} \\ \quad = \Delta Q^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{22}$$

where $s \geqslant 0$.

Consequently, we see that under the potential reductions (15) and (16), the integrable matrix AKNS equations in (10) with $r = 2s+1, \quad s \geqslant 0$, reduce to a hierarchy of nonlocal reverse-spacetime integrable matrix mKdV type equations:

$$p_t = \mathrm{i}\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1=\Delta_2^{-1}pT(-x,-t)\Delta_1},\ s \geqslant 0, \tag{23}$$

where p is an $m \times n$ matrix potential which satisfies (19), Σ_1, Δ_1 are a pair of arbitrary invertible symmetric matrices of size m, and Σ_2, Δ_2 are a pair of arbitrary invertible symmetric matrices of size n. Each reduced equation in the hierarchy (23) with a fixed integer $s \geqslant 0$ possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in (2) with $r = 2s+1$, and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (10) with $r = 2s+1$.

If we fix $s = 1$, i.e. $r = 3$, then the reduced matrix integrable mKdV type equations in (23) give a kind of reduced nonlocal integrable matrix mKdV equations:

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p_x - 3p_x\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p) \\ &= -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Delta_2^{-1}p^{\mathrm{T}}(-x,-t)\Delta_1 p_x \\ &\quad + 3p_x\Delta_2^{-1}p^{\mathrm{T}}(-x,-t)\Delta_1 p), \end{aligned} \tag{24}$$

where p is an $m \times n$ matrix potential satisfying (19).

In what follows, we would like to present a few examples of these novel reduced nonlocal integrable matrix mKdV equations, by taking different values for m, n and appropriate choices for Σ, Δ.

Let us first consider $m = 1$ and $n = 2$. We take

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{25}$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$. Then the potential constraint (19) requires

$$p_2 = -\sigma\delta p_1(-x,-t), \tag{26}$$

where $p = (p_1, p_2)$, and thus, the corresponding potential matrix P reads

$$P = \begin{bmatrix} 0 & p_1 & -\sigma\delta p_1(-x,-t) \\ -\sigma p_1 & 0 & 0 \\ \delta p_1(-x,-t) & 0 & 0 \end{bmatrix}. \tag{27}$$

Further, the corresponding novel reduced nonlocal integrable mKdV equations become

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1^2 p_{1,x} \\ &\quad - 3\sigma p_1(-x,-t)(p_1p_1(-x,-t))_x], \end{aligned} \tag{28}$$

where $\sigma = \pm 1$. These two equations are quite different from the ones studied in [1, 18, 19], in which only one nonlocal factor appears. Similarly, if we take

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \tag{29}$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$ again, then we obtain another pair of novel scalar nonlocal integrable mKdV equations:

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\delta p_1 p_1(-x,-t)p_{1,x} \\ &\quad + 3\delta(p_1p_1(-x,-t))_x p_1], \end{aligned} \tag{30}$$

where $\delta = \pm 1$. This pair has a different nonlocality pattern from the one in (28). Moreover, in each of these two equations, there are two nonlocal nonlinear terms, but in each of their counterparts in [1, 18, 19], there is only one nonlocal nonlinear term.

Let us second consider $m = 1$ and $n = 4$. We take

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_1 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & \delta_2 & 0 \end{bmatrix}, \tag{31}$$

where σ_j and δ_j are real constants and satisfy $\sigma_j^2 = \delta_j^2 = 1, \quad j = 1, 2$. Then the potential constraint (19) generates

$$p_2 = -\sigma_1\delta_1 p_1(-x, -t), \quad p_4 = -\sigma_2\delta_2 p_3(-x, -t), \quad (32)$$

where $p = (p_1, p_2, p_3, p_4)$, and so the corresponding potential matrix P becomes

$$P = \begin{bmatrix} 0 & p_1 & -\sigma_1\delta_1 p_1(-x,-t) & p_3 & -\sigma_2\delta_2 p_3(-x,-t) \\ -\sigma_1 p_1 & 0 & 0 & 0 & 0 \\ \delta_1 p_1(-x,-t) & 0 & 0 & 0 & 0 \\ -\sigma_2 p_3 & 0 & 0 & 0 & 0 \\ \delta_2 p_3(-x,-t) & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (33)$$

This enables us to obtain a class of two-component reduced nonlocal integrable mKdV equations:

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma_1 p_1^2 p_{1,x} - 3\sigma_1 p_1(-x,-t)(p_1 p_1(-x,-t))_x \\ \quad -3\sigma_2 p_3(p_1 p_3)_x - 3\sigma_2 p_3(-x,-t)(p_1 p_3(-x,-t))_x], \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}[p_{3,xxx} - 3\sigma_1 p_1(p_1 p_3)_x - 3\sigma_1 p_1(-x,-t)(p_1(-x,-t)p_3)_x \\ \quad -6\sigma_2 p_3^2 p_{3,x} - 3\sigma_2 p_3(-x,-t)(p_3 p_3(-x,-t))_x], \end{cases} \quad (34)$$

where σ_j are real constants and satisfy $\sigma_j^2 = 1, \quad j = 1, 2$.

Let us third consider $m = 2$ and $n = 2$. We take

$$\Sigma_1 = I_2, \quad \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix},$$
$$\Delta_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \quad (35)$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$. Then the potential constraint (19) tells

$$p_{21} = \sigma\delta p_{12}(-x, -t), \quad p_{22} = \sigma\delta p_{11}(-x, -t), \quad (36)$$

and so the corresponding matrix potentials reads

$$p = \begin{bmatrix} p_{11} & p_{12} \\ \sigma\delta p_{12}(-x,-t) & \sigma\delta p_{11}(-x,-t) \end{bmatrix},$$
$$q = \begin{bmatrix} \sigma p_{11} & \delta p_{12}(-x,-t) \\ \sigma p_{12} & \delta p_{11}(-x,-t) \end{bmatrix}. \quad (37)$$

This enables us to get another class of two-component reduced nonlocal integrable mKdV equations:

$$\begin{cases} p_{11,t} = -\dfrac{\beta}{\alpha^3}[p_{11,xxx} + 6\sigma p_{11}^2 p_{11,x} + 3\sigma p_{12}(p_{11}p_{12})_x \\ +3\sigma p_{12}(-x,-t)(p_{11}p_{12}(-x,-t))_x + 3\sigma p_{11}(-x,-t)(p_{12}p_{12}(-x,-t))_x], \\ p_{12,t} = -\dfrac{\beta}{\alpha^3}[p_{12,xxx} + 3\sigma p_{11}(p_{11}p_{12})_x + 6\sigma p_{12}^2 p_{12,x} \\ +3\sigma p_{12}(-x,-t)(p_{11}p_{11}(-x,-t))_x + 3\sigma p_{11}(-x,-t)(p_{12}p_{11}(-x,-t))_x], \end{cases} \quad (38)$$

where $\sigma = \pm 1$. The pattern of the second nonlocal nonlinear terms in these two equations is different from the one in (34).

In the second and third cases, we can also take other similar choices for Σ and Δ as did in the first case, and generate different two-component reduced integrable mKdV equations.

3. Soliton solutions

3.1. Distribution of eigenvalues

Under the group reduction in (12) (or (13)), we can see that λ is an eigenvalue of the matrix spectral problems in (2) if and only if $\hat{\lambda} = -\lambda$ (or $\hat{\lambda} = \lambda$) is an adjoint eigenvalue, i.e. the adjoint matrix spectral problems hold:

$$\mathrm{i}\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u, \hat{\lambda}), \quad \mathrm{i}\tilde{\phi}_t = \tilde{\phi}V^{[r]} = \tilde{\phi}V^{[r]}(u, \hat{\lambda}), \quad (39)$$

where $r = 2s + 1, \quad s \geqslant 0$. Consequently, we can assume to have eigenvalues λ: $\mu, \ -\mu$, and adjoint eigenvalues $\hat{\lambda}$: $-\mu, \ \mu$, where $\mu \in \mathbb{C}$.

Moreover, under the group reduction in (12) (or (13)), if $\phi(\lambda)$ is an eigenfunction of the matrix spectral problems in (2) associated with an eigenvalue λ, then $\phi T(-\lambda)\Sigma$ (or $\phi T(-x, -t, \lambda)\Delta$) presents an adjoint eigenfunction associated with the same eigenvalue λ.

3.2. General solutions to reflectionless Riemann–Hilbert problems

We would like to present a formulation of solutions to the corresponding reflectionless Riemann–Hilbert problems.

Let $N_1, N_2 \geqslant 0$ be two integers such that $N = 2N_1 + N_2 \geqslant 1$. First, we take N eigenvalues λ_k and N adjoint eigenvalues $\hat{\lambda}_k$ as follows:

$$\lambda_k, \quad 1 \leqslant k \leqslant N: \quad \mu_1, \ \cdots, \ \mu_{N_1}, -\mu_1, \ \cdots, \ -\mu_{N_1}, \ \nu_1, \ \cdots, \ \nu_{N_2}, \quad (40)$$

and

$$\hat{\lambda}_k, \quad 1 \leqslant k \leqslant N: \quad -\mu_1, \ \cdots, \ -\mu_{N_1}, \mu_1, \ \cdots, \ \mu_{N_1}, \ -\nu_1, \ \cdots, \ -\nu_{N_2}, \quad (41)$$

where $\mu_k \in \mathbb{C}$, $1 \leqslant k \leqslant N_1$, and $\nu_k \in \mathbb{C}$, $1 \leqslant k \leqslant N_2$, and assume that their corresponding eigenfunctions and adjoint eigenfunctions are given by

$$v_k, \quad 1 \leqslant k \leqslant N, \quad \text{and} \ \hat{v}_k, \quad 1 \leqslant k \leqslant N, \quad (42)$$

respectively. We point out that in the current nonlocal case, we do not have the property

$$\{\lambda_k \mid 1 \leqslant k \leqslant N\} \bigcap \{\hat{\lambda}_k \mid 1 \leqslant k \leqslant N\} = \varnothing,$$

and thus, we need generalized solutions to reflectionless Riemann–Hilbert problems. Such solutions are provided by

$$G^+(\lambda) = I_{m+n} - \sum_{k,l=1}^{N} \frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda - \hat{\lambda}_l},$$
$$(G^-)^{-1}(\lambda) = I_{m+n} + \sum_{k,l=1}^{N} \frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda - \lambda_k}, \quad (43)$$

where M is a square matrix $M = (m_{kl})_{N\times N}$ with its entries defined by

$$m_{kl} = \begin{cases} \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if} \lambda_l \neq \hat{\lambda}_k, \\ 0, & \text{if} \lambda_l = \hat{\lambda}_k, \end{cases} \quad \textit{where} \quad 1 \leqslant k, l \leqslant N. \quad (44)$$

As shown in [14], these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the reflectionless Riemann–Hilbert problem:

$$(G^-)^{-1}(\lambda)G^+(\lambda) = I_{m+n}, \quad \lambda \in \mathbb{R}, \tag{45}$$

when the orthogonal condition:

$$\hat{v}_k v_l = 0 \text{ if } \lambda_l = \hat{\lambda}_k, \tag{46}$$

is satisfied.

As a consequence of the matrix spectral problems in (2) with zero potentials, we can derive

$$v_k = v_k(x, t, \lambda_k) = \mathrm{e}^{\mathrm{i}\lambda_k \Lambda x + \mathrm{i}\lambda_k^{2s+1}\Omega t} w_k, \quad 1 \leqslant k \leqslant N, \tag{47}$$

and based on the preceding analysis, we can take

$$\hat{v}_k = \hat{v}_k(x, t, \hat{\lambda}_k) = v_k^{\mathrm{T}}(x, t, -\lambda_k)\Sigma = \hat{w}_k e^{-\mathrm{i}\hat{\lambda}_k \Lambda x - \mathrm{i}\hat{\lambda}_k^{2s+1}\Omega t},$$
$$\hat{w}_k = w_k^{\mathrm{T}}\Sigma, \quad 1 \leqslant k \leqslant N, \tag{48}$$

where w_k, $1 \leqslant k \leqslant N$, are constant column vectors. In this way, the orthogonal condition (46) becomes

$$w_k^{\mathrm{T}}\Sigma w_l = 0 \text{ if } \lambda_l = \hat{\lambda}_k, \tag{49}$$

where $1 \leqslant k, l \leqslant N$.

Now, making an asymptotic expansion

$$G^+(\lambda) = I_{m+n} + \frac{1}{\lambda}G_1^+ + O\left(\frac{1}{\lambda^2}\right), \tag{50}$$

as $\lambda \to \infty$, we obtain

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl}\hat{v}_l, \tag{51}$$

and further, substituting this into the matrix spatial spectral problems, we obtain

$$P = -[\Lambda, G_1^+] = \lim_{\lambda \to \infty} [G^+(\lambda), \Lambda]. \tag{52}$$

This give rise to the N-soliton solutions to the matrix AKNS equation (13):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl}\hat{v}_l^2, \quad q = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl}\hat{v}_l^1. \tag{53}$$

Here for each $1 \leqslant k \leqslant N$, we have made the splittings, $v_k = ((v_k^1)^{\mathrm{T}}, (v_k^2)^{\mathrm{T}})^{\mathrm{T}}$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, where v_k^1 and $\hat{v}_k^1$ are column and row vectors of dimension m, respectively, while v_k^2 and $\hat{v}_k^2$ are column and row vectors of of dimension n, respectively.

To present N-soliton solutions for the reduced nonlocal integrable mKdV equation (23), we need to check if G_1^+ defined by (51) satisfies the involution properties:

$$(G_1^+)^{\mathrm{T}} = \Sigma G_1^+ \Sigma^{-1}, \quad (G_1^+)^{\mathrm{T}}(-x, -t) = -\Delta G_1^+ \Delta^{-1}. \tag{54}$$

These mean that the resulting potential matrix P given by (52) will satisfy the two group reduction conditions in (15) and (16). Therefore, the above N-soliton solutions to the matrix AKNS equation (10) reduce to the following class of N-soliton solutions:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl}\hat{v}_l^2, \tag{55}$$

to the reduced nonlocal integrable mKdV equation (23).

3.3. Realization

Let us now check how to realize the involution properties in (54).

First, following the preceding analysis in section 3.1, all adjoint eigenfunctions $\hat{v}_k$, $1 \leqslant k \leqslant 2N_1$, can be determined by

$$\hat{v}_k = \hat{v}_k(x, t, \hat{\lambda}_k) = v_k^{\mathrm{T}}(-\lambda_k)\Sigma$$
$$= v_{N_1+k}^{\mathrm{T}}(-x, -t, \lambda_k)\Delta, \quad 1 \leqslant k \leqslant N_1, \tag{56}$$

and

$$\hat{v}_{N_1+k} = \hat{v}_{N_1+k}(x, t, \hat{\lambda}_{N_1+k}) = v_{N_1+k}^{\mathrm{T}}(-\lambda_{N_1+k})\Sigma$$
$$= v_k^{\mathrm{T}}(-x, -t, \lambda_k)\Delta, \quad 1 \leqslant k \leqslant N_1. \tag{57}$$

These choices in (56) (or (57)) engender the selections on w_k, $1 \leqslant k \leqslant N$:

$$\begin{cases} (\Sigma\Delta^{-1}\Sigma - \Sigma^{-1}\Delta)w_k = 0, & 1 \leqslant k \leqslant N_1, \\ w_k = \Sigma^{-1}\Delta w_{k-N_1}, & N_1 + 1 \leqslant k \leqslant 2N_1. \end{cases} \tag{58}$$

We emphasize that all these selections aim to satisfy the reduction conditions in (15) and (16).

Now, note that when the solutions to the reflectionless Riemann–Hilbert problems, defined by (43) and (44), possess the involution properties in (54), the corresponding relevant matrix G_1^+ will satisfy the involution properties in (54), which are consequences of the group reductions in (12) and (13). Therefore, when the selections in (58) are made and the orthogonal condition for w_k in (49) is satisfied, the formula (55), together with (43), (44), (47) and (48), gives rise to N-soliton solutions to the reduced nonlocal matrix integrable mKdV equation (23).

Finally, let us consider the case of $m = n/2 = s = N = 1$. We take $\lambda_1 = \nu$, $\hat{\lambda}_1 = -\nu$, $\nu \in \mathbb{C}$, and choose

$$w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^{\mathrm{T}}, \tag{59}$$

where $w_{1,1}, w_{1,2}, w_{1,3}$ are arbitrary complex numbers and $w_{1,3}^2 = w_{1,2}^2$. Such a situation leads to a class of one-soliton solutions to the reduced nonlocal integrable mKdV equation (28):

$$p_1 = \frac{2\sigma\nu(\alpha_1 - \alpha_2)w_{1,1}w_{1,2}}{w_{1,1}^2 \mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)\nu x + \mathrm{i}(\beta_1-\beta_2)\nu^3 t} + 2\sigma w_{1,2}^2 \mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)\nu x - \mathrm{i}(\beta_1-\beta_2)\nu^3 t}}, \tag{60}$$

where $\nu \in \mathbb{C}$ is arbitrary and $w_{1,1}, w_{1,2} \in \mathbb{C}$ are arbitrary but need to satisfy $w_{1,1}^2 = \pm 2w_{1,2}^2$, which is a consequence of the involution properties in (54).

4. Concluding remarks

Type $(-\lambda, \lambda)$ reduced nonlocal reverse-spacetime integrable mKdV hierarchies and their soliton solutions were presented. The analysis is based on two group reductions, one of which is local while the other is nonlocal. The resulting nonlocal integrable mKdV hierarchies are different from the existing ones in the literature.

We remark that it would also be interesting to search for other kinds of reduced nonlocal integrable equations from different kinds of Lax pairs [20], integrable couplings [21] and variable coefficient integrable equations [22]. In the pair of the considered two group reductions, we can also take

$$U^{\mathrm{T}}(x + x_0, t + t_0, -\lambda) = (U(x + x_0, t + t_0, -\lambda))^{\mathrm{T}} = -\Sigma U(x, t, ,\lambda)\Sigma^{-1},$$

and

$$U^{\mathrm{T}}(-x + x_{0'}, -t + t_{0'}, \lambda) = (U(-x + x_{0'}, -t + t_{0'}, \lambda))^{\mathrm{T}} = \Delta U(x, t, ,\lambda)\Delta^{-1},$$

with the shifted potentials, where x_0, $x_{0'}$, t_0, $t_{0'}$ are arbitrary constants (see, e.g. [23]). Another interesting topic is to study dynamical properties of exact solutions, including lump solutions [24], soliton solutions [25–27], rogue wave solutions [28, 29], solitonless solutions [30] and algebro-geometric solutions [31, 32], from a perspective of Riemann–Hilbert problems. All this will greatly enrich the mathematical theory of nonlocal integrable equations.

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