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Linear superposition of Wronskian rational solutions to the KdV equation

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Abstract

A linear superposition is studied for Wronskian rational solutions to the KdV equation, which include rogue wave solutions. It is proved that it is equivalent to a polynomial identity that an arbitrary linear combination of two Wronskian polynomial solutions with a difference two between the Wronskian orders is again a solution to the bilinear KdV equation. It is also conjectured that there is no other rational solutions among general linear superpositions of Wronskian rational solutions.

Keywords: soliton equation, Wronskian solution, rational solution, rogue wave, the KdV equation

1. Introduction

The Wronskian formulation is one of powerful approaches to soliton solutions [1, 2]. It has been generalized to present more diverse solutions [3, 4], particularly rational solutions and complexiton solutions [5, 6]. There are plenty of recent studies on nonlinear dispersive waves [7–9], including optical solitons and rogue waves [10–12]. The corresponding soliton equations and even hierarchies of soliton equations can be solved through the inverse scattering transform [13–15] and Riemann–Hilbert problems (see, e.g. [16]).

It is shown [5] that when the functions ϕ_i, $0 \leqslant i \leqslant N-1$, satisfy

$$-\phi_{i,xx} = \sum_{j=1}^{N-1} \lambda_{ij}\phi_j,\ 0 \leqslant i \leqslant N-1, \tag{1}$$

and

$$\phi_{i,t} = -4\phi_{i,xxx},\ 0 \leqslant i \leqslant N-1, \tag{2}$$

where λ_{ij} are arbitrary constants, the Wronskian $f = W(\phi_0, \phi_1, \cdots, \phi_{N-1})$ yields a solution $u = -2(\ln f)_{xx}$ to the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{3}$$

Particularly, rational, soliton, negaton and complexiton solutions correspond to the cases of zero, positive, negative and complex eigenvalues of the coefficient matrix $\Lambda = (\lambda_{ij})_{0\leqslant i,j\leqslant N-1}$, respectively [5].

One of the resulting rational solutions is given by $u = \frac{2}{x^2}$, associated with $\phi_0 = x$. Through the x-translational and t-translational invariance: $\tilde{u}(x, t) = u(x + a, t + b)$ and the Galilean invariance: $\tilde{u}(x, t) = u(x + 6ct, t) + c$, where a, b and c are arbitrary constants, we can generate rogue wave solutions from the Wronskian rational solutions for the KdV equation. Obviously, one such rogue wave solution is

$$\tilde{u}(x, t) = \frac{2}{(x + 6ct + b + ai)^2} + c, \tag{4}$$

where $a \neq 0$, b and c are arbitrary real constants, and a special case with $a = 1/2$, $b = 0$ and $c = -1$ leads to the rogue wave solution presented recently in [17]:

$$\tilde{u}(x, t) = \frac{8}{(2x - 12t + i)^2} - 1. \tag{5}$$

Generally, since the KdV equation is nonlinear, the linear superposition principle can not be applied to its solutions. However, a special linear superposition can exist among Wronskian rational solutions to the KdV equation. This will be the main topic of our discussion in this paper. More specifically, we would like to explore a linear superposition principle for two Wronskian rational solutions to the KdV equation, which have a difference two between the orders of the two involved Wronskian determinants.

The rest of the paper is organized as follows. In section 2, we will set up Wronskian rational solutions. In section 3, we will present a polynomial identity, originated from a linear superposition of Wronskian rational solutions, and in section 4, we will show that it is equivalent to the polynomial identity that a sum of two Wronskian polynomial solutions with a difference two between the Wronskian orders is again a solution to the bilinear KdV equation. A few concluding remarks will be given in the final section, together with a conjecture on general linear superpositions of Wronskian rational solutions.

2. Wronskian rational solutions

Let us recall that the KdV equation (3) is transformed into a Hirota bilinear form

$$(D_x^4+D_xD_t)f\cdot f=2(f_{xt}f-f_tf_x +f_{xxxx}f-4f_{xxx}f_x+3f_{xx}^2)=0, \tag{6}$$

under $u=-2(\ln f)_{xx}$ [5]. Actually, we have

$$u_t-6uu_x+u_{xxx}=-\left[\frac{(D_x^4+D_xD_t)f\cdot f}{f^2}\right]_x. \tag{7}$$

Obviously, a polynomial solution f to the bilinear KdV equation (6) will lead to a rational solution to the KdV equation (3) by the indicated transformation.

Let $N\geqslant 0$ be an arbitrary integer. Assume that f_N is a polynomial solution, defined by the Wronskian [5]:

$$f_N=(\widehat{N-1})=W(\phi_0,\phi_1,\cdots,\phi_{N-1}), \tag{8}$$

where ϕ_i, $i\geqslant 0$, are polynomial functions of x and t, determined by (1) and (2) with

$$\Lambda=(\lambda_{ij})_{0\leqslant i,j\leqslant N-1}=\begin{bmatrix}0 & & & & 0\\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0\end{bmatrix}. \tag{9}$$

We only consider the above case of the coefficient matrix Λ, since similar transformations of Λ generate the same Wronskian solutions to the KdV equation.

Some examples of such Wronskian rational solutions are determined by [5]:

$$\psi(\eta)=2\sinh(\eta x-4\eta^3t)=\sum_{i=0}^{\infty}\phi_i\eta^{2i+1}. \tag{10}$$

More examples can be generated from the Adler–Moser polynomials introduced in [19]. The Wronskian rational solutions can also yield rogue wave solutions through using the x-translational and t-translational invariance and the Galilean invariance of the KdV equation, as illustrated earlier.

3. A polynomial identity

For a sequence of smooth functions of x: $\phi_i=\phi_i(x)$, $i\geqslant 0$, we define

$$\phi_i^{(j)}=\frac{\mathrm{d}^j\phi_i}{\mathrm{d}x^j},\ \Phi_i^{(j)}=(\phi_0^{(j)},\phi_1^{(j)},\cdots,\phi_{i-1}^{(j)})^T,\ i,j\geqslant 0. \tag{11}$$

Assume that m, $n\geqslant 1$ are two integers. Note that a Wronskian of order m is denoted by

$$(\widehat{m-1})=W(\phi_0,\phi_1,\cdots,\phi_{m-1}) =\det(\Phi_m^{(0)},\Phi_m^{(1)},\cdots,\Phi_m^{(m-1)}). \tag{12}$$

Similarly, let us denote

$$(\widehat{m-1},i_1,i_2,\cdots,i_n)=\det(\Phi_{m+n}^{(0)},\Phi_{m+n}^{(1)},\cdots, \Phi_{m+n}^{(m-1)},\Phi_{m+n}^{(i_1)},\Phi_{m+n}^{(i_2)},\cdots,\Phi_{m+n}^{(i_n)}), \tag{13}$$

where $m\leqslant i_1<i_2<\cdots<i_n$, which is called a generalized Wronskian of order $m+n$. We point out that a Wronskian of order m involves a square matrix of size m, and a generalized Wronskian of order $m+n$ involves a square matrix of size $m+n$.

We will discuss about Wronskian rational solutions, and so let us focus on the sequence of polynomials ϕ_i, $i\geqslant 0$, determined by

$$\phi_{0,xx}=0,\ \phi_{i+1,xx}=\phi_i,\ i\geqslant 0, \tag{14}$$

which follows from the assumption for Λ in (9).

A polynomial identity: *Let $N\geqslant 3$ be an arbitrary integer. Then we conjecture [18] that under (14), the following equality holds for generalized Wronskians:*

$$\begin{aligned}&(\widehat{N-3},N-2,N-1)(\widehat{N-1},N+2,N+3)\\ &+(\widehat{N-3},N,N+1)(\widehat{N-1},N,N+1)\\ &-(\widehat{N-3},N-2,N)(\widehat{N-1},N+1,N+3)\\ &-(\widehat{N-3},N-1,N+1)(\widehat{N-1},N,N+2)\\ &+(\widehat{N-3},N-2,N+1)(\widehat{N-1},N+1,N+2)\\ &+(\widehat{N-3},N-1,N)(\widehat{N-1},N,N+3)=0.\end{aligned} \tag{15}$$

More compactly, the identity (15) can be expressed as

$$\begin{aligned}&(\widehat{N-1})(\widehat{N-1},N+2,N+3)\\ &+(\widehat{N-3},N,N+1)(\widehat{N+1})\\ &-(\widehat{N-2},N)(\widehat{N-1},N+1,N+3)\\ &-(\widehat{N-3},N-1,N+1)(\widehat{N},N+2)\\ &+(\widehat{N-2},N+1)(\widehat{N-1},N+1,N+2)\\ &+(\widehat{N-3},N-1,N)(\widehat{N},N+3)=0,\end{aligned} \tag{16}$$

or rewritten in an even more compact form:

$$\sum_{\substack{a+b+c+d=2\\ -2\leqslant a<b\leqslant 1,0\leqslant c<d\leqslant 3}} (-1)^{a+b+1}(\widehat{N-3}, N+a, N+b)(\widehat{N-1}, N+c, N+d) = 0. \quad (17)$$

By the Laplace expansion around the first $N+2$ rows, we can also put (15) simply in a determinant form:

$$\begin{vmatrix} [\widehat{N-3}] & 0 & 0 & 0 & N-2 & N-1 & N & N+1 & \chi_1 & \chi_2 \\ 0 & [\widehat{N-3}] & N-2 & N-1 & N & N+1 & N+2 & N+3 & 0 & 0 \end{vmatrix} = 0, \quad (18)$$

without any sign change in the six terms, where

$$[\widehat{N-3}] = (\Phi_{N+2}^{(0)}, \Phi_{N+2}^{(1)}, \cdots, \Phi_{N+2}^{(N-3)}),$$
$$N+i = \Phi_{N+2}^{(N+i)}, -2 \leqslant i \leqslant 3,$$

and

$$\chi_1 = (\underbrace{0, \cdots, 0}_N, 1, 0)^{\mathrm{T}}, \ \chi_2 = (\underbrace{0, \cdots, 0}_N, 0, 1)^{\mathrm{T}}.$$

The identity (15) is very similar to the simplest case of the Plücker relations [7], but we do not know what kind of mathematical property it reflects really. It might be helpful in proving the identity (15) if we apply the Laplace expansion for determinants and Jacobi's identity for Wronskians:

$$\begin{aligned}&(W(\psi_1, \psi_2, \cdots, \psi_{m-1}, \chi))_x W(\psi_1, \psi_2, \cdots, \psi_m)\\ &-W(\psi_1, \psi_2, \cdots, \psi_{m-1}, \chi)(W(\psi_1, \psi_2, \cdots, \psi_m))_x\\ &= W(\psi_1, \psi_2, \cdots, \psi_{m-1})W(\psi_1, \psi_2, \cdots, \psi_m, \chi), \ m \geqslant 1, \end{aligned} \quad (19)$$

where $\psi_1, \psi_2, \cdots, \psi_m, \chi$ are sufficiently differentiable functions and $W(\psi_1, \cdots, \psi_{m-1}) = 1$ when $m = 1$.

The identity (15) will be used to show a linear superposition principle for Wronskian rational solutions, including rogue wave solutions, to the KdV equation.

4. Linear superposition

It is common sense that there is no linear superposition principle for nonlinear equations. However, among the Wronskian rational solutions to the KdV equation, we would like to explore a special kind of linear superpositions of solutions.

Let f and g be two Wronskian solutions to the bilinear KdV equation (6). Then, we have

$$\begin{aligned}&(D_x^4 + D_x D_t)(f + cg) \cdot (f + cg)\\ &= 2c[(f_{xt} + f_{xxxx})g + f(g_{xt} + g_{xxxx})\\ &\quad -(f_t + 4f_{xxx})g - f(g_t + 4g_{xxx}) + 6f_{xx}g_{xx}], \end{aligned} \quad (20)$$

where c is an arbitrary constant. Therefore, for two Wronskian solutions f and g, we see that an arbitrary linear combination of f and g solves (6) iff so does $f+g$.

It can be readily seen that $f_1 + f_3$ and $f_2 + f_4$ solve the bilinear KdV equation (3), where f_m's are the Wronskian solutions defined by (8). In the general case, we would like to show that the identity in (15) is equivalent to say that $f_N + f_{N+2}$ is again a polynomial solution to the bilinear KdV equation (6). We refer the reader to [19, 20] for more illustrative examples of such solutions generated from linear combinations.

Theorem 4.1. *Let $N \geqslant 3$ be an arbitrary integer and the polynomial functions ϕ_i, $i \geqslant 0$, determined by (14) and*

$$\phi_{i,t} = -4\phi_{i,x,x,x}, \ i \geqslant 0. \quad (21)$$

Then for the bilinear KdV equation (6), $f_N + f_{N+2} = (\widehat{N-1}) + (\widehat{N+1})$, a sum of two Wronskian solutions, again presents a solution iff the equality (15) holds.Proof: Note that $\lambda_{ii} = 0$, $i \geqslant 0$. By (2.5) of lemma 2.2 in [5], we have

$$(\widehat{N-3}, N-1, N) = (\widehat{N-2}, N+1), \quad (22)$$

and computing its derivative with respect to x leads to

$$(\widehat{N-4}, N-2, N-1, N) = (\widehat{N-2}, N+2). \quad (23)$$

By (2.7) of lemma 2.2 in [5], we have

$$\begin{aligned}&(\widehat{N-5}, N-3, N-2, N-1, N)\\ &= -(\widehat{N-3}, N, N+1) + (\widehat{N-3}, N-1, N+2), \end{aligned} \quad (24)$$

and combining (2.7) and (2.8) of lemma 2.2 in [5] tells

$$\begin{aligned}&(\widehat{N-4}, N-2, N-1, N+1)\\ &=(\widehat{N-3}, N, N+1) + (\widehat{N-2}, N+3). \end{aligned} \quad (25)$$

Let $f = f_N = (\widehat{N-1})$. Then, we can compute that

$$f_x = (\widehat{N-2}, N), \quad (26)$$

$$\begin{aligned} f_{xx} &= (\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1)\\ &= 2(\widehat{N-3}, N-1, N) = 2(\widehat{N-2}, N+1), \end{aligned} \quad (27)$$

$$f_{xxx} = 2\frac{\partial}{\partial x}(\widehat{N-2}, N+1) = 2[(\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2)], \quad (28)$$

$$
\begin{aligned}
f_{xxxx} = &\ 2[(\widehat{N-4}, N-2, N-1, N+1) \\
&+ (\widehat{N-3}, N, N+1) \\
&+ 2(\widehat{N-3}, N-1, N+2) + (\widehat{N-2}, N+3)] \\
= &\ 4[(\widehat{N-3}, N, N+1) + (\widehat{N-3}, N-1, N+2) \\
&+ (\widehat{N-2}, N+3)],
\end{aligned} \tag{29}
$$

where we have used (22) and (25) in (27) and (29), respectively. Moreover, by using the conditions in (21), we have

$$
\begin{aligned}
f_t = -4[&(\widehat{N-4}, N-2, N-1, N) \\
&- (\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2)],
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
f_{xt} = -4[&(\widehat{N-5}, N-3, N-2, N-1, N) \\
&- (\widehat{N-3}, N, N+1) + (\widehat{N-2}, N+3)];
\end{aligned} \tag{31}
$$

and then, upon using (23) and (24) in (30) and (31), respectively, we obtain

$$
f_t = 4[(\widehat{N-3}, N-1, N+1) - 2(\widehat{N-2}, N+2)], \tag{32}
$$

$$
\begin{aligned}
f_{xt} = 4[&2(\widehat{N-3}, N, N+1) \\
&- (\widehat{N-3}, N-1, N+2) - (\widehat{N-2}, N+3)].
\end{aligned} \tag{33}
$$

Now, it is direct to see that

$$
f_t + 4f_{xxx} = 12(\widehat{N-3}, N-1, N+1), \tag{34}
$$

and

$$
f_{xt} + f_{xxxx} = 12(\widehat{N-3}, N, N+1). \tag{35}
$$

Further, we take two Wronskian polynomial solutions $f = f_N = (\widehat{N-1})$ and $g = f_{N+2} = (\widehat{N+1})$ satisfying the conditions in (14) and (21). For $g = (\widehat{N+1})$, we just need to change N into $N+2$ in all computations for $f = (\widehat{N-1})$. Then, based on (27), (34) and (35), we can see from (20) that the sum $f + g$ solves the bilinear KdV equation (6) if and only if the equality (15) holds. □

It is direct to show that the Boussinesq equation does not have such a linear superposition for Wronskian rational solutions, which are given in [6]. We can also directly see that among more general linear combinations

$$
f_N + c_1 f_{N+1} + c_2 f_{N+2} + c_3 f_{N+3} + c_4 f_{N+4} + c_5 f_{N+5},
$$

there is only one solution $f_N + c_2 f_{N+2}$, where c_i, $1 \leqslant i \leqslant 5$, are arbitrary constants.

5. Concluding remarks

We have discussed about a specific linear superposition of Wronskian rational solutions to the KdV equation. It has been explored that it is equivalent to a polynomial identity that a linear combination of two Wronskian polynomial solutions with a difference two between the Wronskian orders is again a solution to the bilinear KdV equation.

It is easy to see that there is only the linear combination solution

$$
f_N + c_2 f_{N+2}
$$

among

$$
f_N + c_1 f_{N+1} + c_2 f_{N+2} + c_3 f_{N+3} + c_4 f_{N+4} + c_5 f_{N+5},
$$

where f_m is the Wronskian of order m defined by (8) and c_i, $1 \leqslant i \leqslant 5$, are arbitrary constants. We conjecture that such a statement in a general case will be true, i.e. there is no other solution among the linear combinations

$$
f_N + \sum_{i=1}^{n} c_i f_{N+i},
$$

where $n \geqslant 1$ is an arbitrary integer and c_i, $1 \leqslant i \leqslant n$, are arbitrary constants, besides $f_N + c_2 f_{N+2}$.

Recently, there have been various studies on a kind of simple but important rational solutions, called lump solutions, to nonlinear dispersive wave equations (see, e.g. [21, 22]) and different nonlinearities can go together to engender nonlinearity-managed lump solutions [23, 24]. It is known that for local integrable equations, soliton solutions can be derived from the τ-function [25] and Hirota bilinear forms (see, e.g. [26]), and lump solutions can be obtained by taking long-wave limits of soliton solutions [27]. How about generating lump solutions for nonlocal integrable equations? Can we apply the Riemann–Hilbert technique (see, e.g. [28, 29] for solitons) to exploring their phase interaction characteristics? Certainly, any investigation in this research area would be helpful in understanding complex dynamical phenomena (see, e.g. [30, 31]) in dispersive wave theories.

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