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A soliton hierarchy derived from a fourth-order matrix spectral problem possessing four fields

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ABSTRACT

This paper is dedicated to the construction of integrable commuting flows starting from a fourth-order matrix spectral problem involving four fields, which is derived from a specialized matrix Lie algebra over the real domain. This work includes the development of an explicit bi-Hamiltonian formulation and a hereditary recursion structure, which confirms the hierarchy's integrability in the Liouville sense. Furthermore, we examine two second-order and third-order integrable models, along with their reduced, uncombined forms.

1. Introduction

Integrable models comes in hierarchies that possess hereditary recursion operators [1,2] and are often derived from Lax pairs linked to matrix eigenvalue problems [3]. These matrix eigenvalue problems facilitate the development of Hamiltonian formulations, connecting symmetries with conservation laws. Integrable models find widespread applications across the domains of engineering, the natural sciences, and the physical sciences, such as nonlinear optics, plasma physics, water waves, fluid dynamics, and quantum mechanics [4].

Some of the most prominent integrable models include the Ablowitz–Kaup–Newell–Segur (AKNS) fundamental ones [5], and its various integrable couplings, based on matrix Lie algebras that are non-semisimple (see, e.g., [6]). Matrix Lie algebras serve as a robust framework for constructing Liouville integrable systems and their corresponding Lax pairs, which are derived from matrix eigenvalue problems [7,8]. The exploration of various types of Lax pairs capable of generating integrable models has been a longstanding area of study. In this paper, employing the Lax pair (or zero curvature) formalism, we aim to introduce a new 4th-order eigenvalue problem and build a combined integrable hierarchy related to the problem, derived from a particular matrix Lie algebra.

The zero curvature formulation offers an effective method for generating integrable models (see, for instance, [8,9] for more details). As normal, a potential vector of dimension s is expressed as $r = (r_1, \dots, r_s)^T$ and the spectral variable by ξ. Beginning with a given loop matrix algebra $\tilde{g}$ with the loop variable ξ, we take s linearly independent matrices $A_1, \dots, A_s$ to propose the following spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(r, \xi) = r_1 A_1(\xi) + \cdots + r_s A_s(\xi) + A_0(\xi), \tag{1.1}$$

where the following pseudo-regular property holds for the last matrix element A_0:

$$\mathrm{Im}\,\mathrm{ad}_{A_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{A_0} = \tilde{g}, \ \ [\mathrm{Ker}\,\mathrm{ad}_{A_0}, \mathrm{Ker}\,\mathrm{ad}_{A_0}] = 0,$$

in which ad_{A_0} is defined by $\mathrm{ad}_{A_0} B = [A_0, B]$, where B is an arbitrary matrix. Then, within the foundational loop algebra $\tilde{g}$, we need to solve the following matrix equation

$$Y_x - [\mathcal{M}, Y] = 0, \tag{1.2}$$

and specifically, we search for an infinite series solution of the Laurent form $Y = \sum_{n\geq 0} Y^{[n]} \xi^{-n}$.

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The second step is that based on the constructed matrix Y, we identify an infinite set of Lax operators (or matrices)

$$\mathcal{N}^{[m]} = \mathcal{N}^{[m]}(r,\xi) = Y^{[0]}\xi^m + Y^{[1]}\xi^{m-1} + \cdots + Y^{[m]} + \Pi_m(\xi),\ \ \Pi_m \in \tilde{g},\ \ m \geq 0, \tag{1.3}$$

so that the associated compatibility conditions:

$$\mathcal{M}_{t_m} = \mathcal{N}^{[m]}_x + [\mathcal{N}^{[m]}, \mathcal{M}],\ \ m \geq 0, \tag{1.4}$$

generates an infinite sequence of commuting integrable models:

$$r_{t_m} = X^{[m]}(r, r_x, \ldots),\ \ m \geq 0. \tag{1.5}$$

The compatibility conditions in (1.4) essentially correspond to the solvability criteria of the Lax pairs:

$$\begin{cases} \varphi_x = \mathcal{M}(r,\xi)\varphi, \\ \varphi_{t_m} = \mathcal{N}^{[m]}(r,\xi)\varphi, \end{cases} \quad m \geq 0. \tag{1.6}$$

The subsequent step is that we need to find a Magri's geometric structure for the constructed hierarchy (1.5). This will be achieved by exploring a recursion structure being hereditary and utilizing the trace identity:

$$\frac{\delta}{\delta r}\int \operatorname{tr}\big(Y\frac{\partial \mathcal{M}}{\partial \xi}\big)\,dx = \xi^{-\kappa}\frac{\partial}{\partial \xi}\xi^{\kappa}\operatorname{tr}\big(Y\frac{\partial \mathcal{M}}{\partial r}\big), \tag{1.7}$$

where $\frac{\delta}{\delta r}$ and $\frac{\partial}{\partial r}$ denote the variational derivative and partial derivative with respect to the potential vector r, respectively, and the constant κ does not depend on the spectral variable ξ, which can be computed by

$$\kappa = -\frac{\xi}{2}\frac{\partial}{\partial \xi}\ln|\operatorname{tr}(Y^2)|. \tag{1.8}$$

In this way, we see that each model in the constructed sequence possesses a Magri's geometric formulation, which ensures its Liouville integrability (see, for instance, [8–10]).

There are various hierarchies of Liouville integrable Hamiltonian models, presented in the literature [5–20]. The two-component case is widely studied, with several well-known examples, including the AKNS models [5], the Kaup–Newell models [21], the Heisenberg models [22], and the Wadati–Konno–Ichikawa models [23]. These four soliton hierarchies are linked to the following matrix spectral matrices:

$$\mathcal{M} = \begin{bmatrix} -\xi & r_1 \\ r_2 & \xi \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} -\xi^2 & \xi r_1 \\ \xi r_2 & \xi^2 \end{bmatrix}, \tag{1.9}$$

$$\mathcal{M} = \begin{bmatrix} -\xi r_3 & \xi r_1 \\ \xi r_2 & \xi r_3 \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} -\xi & \xi r_1 \\ \xi r_2 & \xi \end{bmatrix}, \tag{1.10}$$

where $r_1 r_2 = 1 - r_3^2$, respectively. These spectral matrices are rare and specific objects in soliton theory, and formulating such spectral matrices, which can generate soliton hierarchies, presents a significant challenge in the field.

This paper proposes a specific 4th-order spectral matrix with four potentials, derived using the trial-and-error method combined with symbolic computation, and grounded in a distinct matrix Lie algebra within the real domain. It then develops a commuting hierarchy of integrable models in the Liouville sense, employing the Lax pair formulation. To demonstrate the Liouville integrability of the constructed models, we explore a hereditary recursion structure and a Magri's geometric structure, Two examples are provided, including generalized combined 2nd-order and 3th-order integrable models, along with their special reductions. The main contribution is the formulation of a 4th-order spectral matrix possessing 4 potentials, which generates an integrable hierarchy. The concluding section provides a summary and additional remarks.

2. Combined commuting flows with four fields

Let ζ be an arbitrary scalar, and Θ an invertible matrix of order $s \in \mathbb{N}$, such that its inverse is equal to itself. It is straightforward to observe that a collection $\tilde{g}$ of 2×2 block matrices

$$G = \left[\begin{array}{c|c} G_1 & G_2 \\ \hline G_3 & G_4 \end{array}\right]_{2s\times 2s}, \tag{2.1}$$

where

$$G_4 = \Theta G_1 \Theta^{-1},\ G_3 = \zeta \Theta G_2 \Theta^{-1}, \tag{2.2}$$

forms a matrix Lie algebra under the matrix commutator $[G, G'] = GG' - G'G$. In what follows, an example of the above matrix algebra with $s = 2$ and

$$\Theta = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.3}$$

will be used to introduce a specific spectral matrix to create a related integrable sequence.

We denote the four-dimensional vector of dependent variables by

$$r = r(x,t) = (r_1, r_2, r_3, r_4)^T, \tag{2.4}$$

take two pairs of arbitrary constants, α_1, α_2 and η_1, η_2, and assume that

$$\alpha \neq 0, \tag{2.5}$$

where $\alpha = \alpha_1 - \alpha_2$. Inspired by recent research works on matrix eigenvalue problems containing 4 fields, particularly, two examples of arbitrary-order in [24,25] and two examples of fourth-order in [26,27], we aim to examine an eigenvalue problem defined by the following matrix form:

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(r,\xi)\varphi,\ \ \mathcal{M} = \left[\begin{array}{cc|cc} \alpha_1\xi & \eta_1 r_1 & r_2 & 0 \\ \eta_1 r_3 & \alpha_2\xi & 0 & r_4 \\ \hline \eta_1\eta_2 r_4 & 0 & \alpha_2\xi & \eta_1 u_3 \\ 0 & \eta_1\eta_2 r_2 & \eta_1 r_1 & \alpha_1\xi \end{array}\right] \tag{2.6}$$

where ξ continues to function as the eigenvalue coordinate. This spectral matrix $\mathcal{M}$ was formulated using a trial-and-error method combined with symbolic computation in Maple. The key aspect to note is the existence of solutions to the stationary zero curvature equation. Obviously, the spectral matrix $\mathcal{M}$ belongs to the matrix Lie algebra $\tilde{g}$ mentioned previously. The eigenvalue problem cannot be any reduction of the matrix AKNS eigenvalue problem (see, e.g., [28]). We aim to develop a corresponding commuting sequence of four-component Hamiltonian models with specific combined structures. Interestingly, when $\eta_2 = 0$, it gives rise to non-perturbation type integrable couplings.

We now proceed with the construction of the integrable models associated the above spectral matrix described above. To begin, we solve the corresponding stationary zero curvature Eq. (1.2) by starting with

$$Y = \left[\begin{array}{cc|cc} \eta_1 a & \eta_1 b & e & f \\ \eta_1 c & -\eta_1 a & -f & g \\ \hline \eta_1\eta_2 g & -\eta_1\eta_2 f & -\eta_1 a & \eta_1 c \\ \eta_1\eta_2 f & \eta_1\eta_2 e & \eta_1 b & \eta_1 a \end{array}\right] = \sum_{n\geq 0} Y^{[n]}\xi^{-n}, \tag{2.7}$$

where the undetermined coefficient matrices are assumed to be as follows:

$$Y^{[n]} = \left[\begin{array}{cc|cc} \eta_1 a^{[n]} & \eta_1 b^{[n]} & e^{[n]} & f^{[n]} \\ \eta_1 c^{[n]} & -\eta_1 a^{[n]} & -f^{[n]} & g^{[n]} \\ \hline \eta_1\eta_2 g^{[n]} & -\eta_1\eta_2 f^{[n]} & -\eta_1 a^{[n]} & \eta_1 c^{[n]} \\ \eta_1\eta_2 f^{[n]} & \eta_1\eta_2 e^{[n]} & \eta_1 b^{[n]} & \eta_1 a^{[n]} \end{array}\right],\ n \geq 0. \tag{2.8}$$

The reason to take this form is that with $\mathcal{M}$ in (2.6), an arbitrary matrix in $\tilde{g}$ will engender a commutator matrix of the above form. In this way, we can find that the corresponding stationary zero curvature Eq. (1.2) gives

$$\begin{cases} a_x = \eta_1 c r_1 + \eta_2\, g r_2 - \eta_1 b r_3 - \eta_2 e r_4, \\ b_x = \alpha\xi b - 2\eta_1 a r_1 - 2\eta_2 f r_2, \\ c_x = -\alpha\xi c + 2\eta_1 a r_3 + 2\eta_2 f r_4, \end{cases} \tag{2.9}$$

$$\begin{cases} e_x = \alpha\xi e - 2\eta_1 a r_2 - 2\eta_1 f r_1, \\ g_x = -\alpha\xi g + 2\eta_1 a r_4 + 2\eta_1 f r_3, \\ f_x = \eta_1(c r_2 + g r_1 - b r_4 - e r_3). \end{cases} \tag{2.10}$$

These equations leads equivalently to the initial conditions:

$$a_x^{[0]} = 0,\ b^{[0]} = c^{[0]} = e^{[0]} = g^{[0]} = 0,\ f_x^{[0]} = 0, \tag{2.11}$$

and the recursion relations for determining the Laurent series solution:

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha}(b_x^{\langle n\rangle} + 2\eta_1 a^{[n]} r_1 + 2\eta_2 f^{[n]} r_2), \\ c^{[n+1]} = -\frac{1}{\alpha}(c_x^{\langle n\rangle} - 2\eta_1 a^{[n]} r_3 - 2\eta_2 f^{[n]} r_4), \end{cases} \tag{2.12}$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha}(e_x^{\langle n\rangle} + 2\eta_1 f^{[n]} r_1 + 2\eta_1 a^{[n]} r_2), \\ g^{[n+1]} = -\frac{1}{\alpha}(g_x^{[n]} - 2\eta_1 f^{[n]} r_3 - 2\eta_1 a^{[n]} r_4), \end{cases} \tag{2.13}$$

$$\begin{cases} a_x^{[n+1]} = -\eta_1 b^{[n+1]} r_3 + \eta_1 c^{[n+1]} r_1 - \eta_2 e^{[n+1]} r_4 + \eta_2 g^{[n+1]} r_2, \\ f_x^{[n+1]} = \eta_1(c^{[n+1]} r_2 + g^{[n+1]} r_1 - b^{[n+1]} r_4 - e^{[n+1]} r_3), \end{cases} \tag{2.14}$$

where $n \geq 0$. To compute the solution concretely, we take the initial data,

$$a^{[0]} = \frac{1}{2}\beta,\ f^{[0]} = \frac{1}{2}\gamma, \tag{2.15}$$

where the constants β and γ are arbitrarily chosen, and we assume

$$a^{[n]}|_{u=0} = 0,\ f^{[n]}|_{u=0} = 0,\ n \geq 1, \tag{2.16}$$

which implies that the constants of integration are taken as zero. Under those conditions, one can work out that

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}(\eta_1\beta r_1 + \eta_2\gamma r_2),\ c^{[1]} = \frac{1}{\alpha}(\eta_1\beta r_3 + \eta_2\gamma r_4), \\ e^{[1]} = \frac{\eta_1}{\alpha}(\gamma r_1 + \beta r_2),\ g^{[1]} = \frac{\eta_1}{\alpha}(\gamma r_3 + \beta r_4), \\ a^{[1]} = 0,\ f^{[1]} = 0; \end{cases}$$

$$\begin{cases} b^{[2]} = \frac{1}{\alpha^2}(\eta_1\beta r_{1,x} + \eta_2\gamma r_{2,x}),\ c^{[2]} = -\frac{1}{\alpha^2}(\eta_1\beta r_{3,x} + \eta_2\gamma r_{4,x}), \\ e^{[2]} = \frac{\eta_1}{\alpha^2}(\gamma r_{1,x} + \beta r_{2,x}),\ g^{[2]} = -\frac{\eta_1}{\alpha^2}(\gamma r_{3,x} + \beta r_{4,x}), \end{cases}$$

$$\begin{cases} a^{[2]} = -\frac{\eta_1}{\alpha^2}[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2], \\ f^{[2]} = -\frac{\eta_1}{\alpha^2}[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]; \end{cases}$$

$$\begin{cases} b^{[3]} = \frac{1}{\alpha^3}[\eta_1\beta r_{1,xx} + \eta_2\gamma r_{2,xx} - 2\eta_1(\eta_1\beta r_3 + \eta_2\gamma r_4)(\eta_1 r_1^2 + \eta_2 r_2^2) - 4\eta_1^2\eta_2(\gamma r_3 + \beta r_4)r_1 r_2], \\ c^{[3]} = \frac{1}{\alpha^3}[\eta_1\beta r_{3,xx} + \eta_2\gamma r_{4,xx} - 2\eta_1(\eta_1\beta r_1 + \eta_2\gamma r_2)(\eta_1 r_3^2 + \eta_2 r_4^2) - 4\eta_1^2\eta_2(\gamma r_1 + \beta r_2)r_3 r_4], \end{cases}$$

$$\begin{cases} e^{[3]} = \frac{1}{\alpha^3}[\eta_1\gamma r_{1,xx} + \eta_1\beta r_{2,xx} - 2\eta_1^2(\gamma r_3 + \beta r_4)(\eta_1 r_1^2 + \eta_2 r_2^2) - 4\eta_1^2(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 r_2], \\ g^{[3]} = \frac{1}{\alpha^3}[\eta_1\gamma r_{3,xx} + \eta_1\beta r_{4,xx} - 2\eta_1^2(\gamma r_1 + \beta r_2)(\eta_1 r_3^2 + \eta_2 r_4^2) - 4\eta_1^2(\eta_1\beta r_1 + \eta_2\gamma r_2)r_3 r_4], \end{cases}$$

$$\begin{cases} a^{[3]} = \frac{\eta_1}{\alpha^3}[-(\eta_1\beta r_3 + \eta_2\gamma r_4)r_{1,x} - \eta_2(\gamma r_3 + \beta r_4)r_{2,x} + (\eta_1\beta r_1 + \eta_2\gamma r_2)r_{3,x} + \eta_2(\gamma r_1 + \beta r_2)r_{4,x}], \\ f^{[3]} = \frac{\eta_1}{\alpha^3}[-\eta_1(\gamma r_3 + \beta r_4)r_{1,x} - (\eta_1\beta r_3 + \eta_2\gamma r_4)r_{2,x} + \eta_1(\gamma r_1 + \beta r_2)r_{3,x} + (\eta_1\beta r_1 + \eta_2\gamma r_2)r_{4,x}]; \end{cases}$$

and

$$\begin{cases} b^{[4]} = \frac{1}{\alpha^4}\{\eta_1\beta r_{1,xxx} + \eta_2\gamma r_{2,xxx} - 6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{1,x} \\ \quad -6\eta_1\eta_2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{2,x}\}, \\ c^{[4]} = -\frac{1}{\alpha^4}\{\eta_1\beta r_{3,xxx} + \eta_2\gamma r_{4,xxx} - 6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{3,x} \\ \quad -6\eta_1\eta_2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{4,x}\}, \end{cases}$$

$$\begin{cases} e^{[4]} = \frac{1}{\alpha^4}\{\eta_1\gamma r_{1,xxx} + \eta_1\beta r_{2,xxx} - 6\eta_1^2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{1,x} \\ \quad -6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{2,x}\}, \\ g^{[4]} = -\frac{1}{\alpha^4}\{\eta_1\gamma r_{3,xxx} + \eta_1\beta r_{4,xxx} - 6\eta_1^2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{3,x} \\ \quad -6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{4,x}\}, \end{cases}$$

$$\begin{aligned} a^{[4]} = \frac{\eta_1}{\alpha^4}[&-(\eta_1\beta r_3 + \eta_2\gamma r_4)r_{1,xx} - \eta_2(\gamma r_3 + \beta r_4)r_{2,xx} - (\eta_1\beta r_1 + \eta_2\gamma r_2)r_{3,xx} \\ &-\eta_2(\gamma r_1 + \beta r_2)r_{4,xx} + (\eta_1\beta r_{1,x} + \eta_2\gamma r_{2,x})r_{3,x} + \eta_2(\gamma r_{1,x} + \beta r_{2,x})r_{4,x} \\ &+3\eta_1^2(\eta_1\beta r_3^2 + 2\eta_2\gamma r_3 r_4 + \eta_2\beta r_4^2)r_1^2 + 6\eta_1\eta_2(\gamma r_3^2 + 2\beta r_3 r_4 + \eta\gamma r_4^2)r_1 r_2 \\ &+3\eta_1\eta_2(\eta_1\beta r_3^2 + 2\eta_2\gamma r_3 r_4 + \eta_2\beta r_4^2)r_2^2], \end{aligned}$$

$$\begin{aligned} f^{[4]} = \frac{\eta_1}{\alpha^4}[&-\eta_1(\gamma r_3 + \beta r_4)r_{1,xx} - (\eta_1\beta r_3 + \eta_2\gamma r_4)r_{2,xx} - \eta_1(\gamma r_1 + \beta r_2)r_{3,xx} \\ &-(\eta_1\beta r_1 + \eta_2\gamma r_2)r_{4,xx} + \eta_1(\gamma r_{3,x} + \beta r_{4,x})r_{1,x} + (\eta_1\beta r_{3,x} + \eta_2\gamma r_{4,x})r_{2,x} \\ &+3\eta_1^2(\eta_1\gamma r_3^2 + 2\eta_1\beta r_3 r_4 + \eta_2\gamma r_4^2)r_1^2 + 6\eta_1^2(\eta_1\beta r_3^2 + 2\eta_2\gamma r_3 r_4 + \eta_2\beta r_4^2)r_1 r_2 \\ &+3\eta_1\eta_2(\eta_1\gamma r_3^2 + 2\eta_1\beta r_3 r_4 + \eta_2\gamma r_4^2)r_2^2]. \end{aligned}$$

Upon observing the above computations, we can take Π_m as zero for each positive integer m, and treat

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(r,\xi)\varphi, \ m \geq 0, \tag{2.17}$$

where

$$\mathcal{N}^{[m]} = Y^{[m]} + Y^{[m-1]}\xi + \cdots + Y^{[0]}\xi^m, \ m \geq 0, \tag{2.18}$$

as the time evolution parts of the Lax pairs within the Lax pair formulation. We examine the conditions for the solvability of (2.6) and (2.17). Those equations generate an infinite sequence of combined nonlinear models with four fields:

$$r_{t_m} = X^{[m]}(r, r_x, \ldots) = (X_1^{[m]}, X_2^{[m]}, X_3^{[m]}, X_4^{[m]})^T, \tag{2.19}$$

where

$$X_1^{[m]} = \alpha b^{[m+1]}, \ X_2^{[m]} = \alpha e^{[m+1]}, \ X_3^{[m]} = -\alpha c^{[m+1]}, \ X_4^{[m]} = -\alpha g^{[m+1]}, \ m \geq 0. \tag{2.20}$$

More concretely, we have

$$r_{1,t_m} = \alpha b^{[m+1]}, \ r_{2,t_m} = \alpha e^{[m+1]}, \ r_{3,t_m} = -\alpha c^{[m+1]}, \ r_{4,t_m} = -\alpha g^{[m+1]}, \ m \geq 0. \tag{2.21}$$

We can work out the first and second nonlinear models in the aforementioned hierarchy. The first example involves the combined 2nd-order integrable equations:

$$\begin{cases} r_{1,t_2} = \frac{1}{\alpha^2}[\eta_1\beta r_{1,xx} + \eta_2\gamma r_{2,xx} - 2\eta_1(\eta_1\beta r_3 + \eta_2\gamma r_4)(\eta_1 r_1^2 + \eta_2 r_2^2) - 4\eta_1^2\eta_2(\gamma r_3 + \beta r_4)r_1 r_2], \\ r_{2,t_2} = \frac{1}{\alpha^2}[\eta_1\gamma r_{1,xx} + \eta_1\beta r_{2,xx} - 2\eta_1^2(\gamma r_3 + \beta r_4)(\eta_1 r_1^2 + \eta_2 r_2^2) - 4\eta_1^2 r_1 r_2(\eta_1\beta r_3 + \eta_2\gamma r_4)], \\ r_{3,t_2} = -\frac{1}{\alpha^2}[\eta_1\beta r_{3,xx} + \eta_2\gamma r_{4,xx} - 2\eta_1(\eta_1\beta r_1 + \eta_2\gamma r_2)r_3^2 - 4\eta_1^2\eta_2(\gamma r_1 + \beta r_2)r_3 r_4], \\ r_{4,t_2} = -\frac{1}{\alpha^2}[\eta_1\gamma r_{3,xx} + \eta_1\beta r_{4,xx} - 2\eta_1^2(\gamma r_1 + \beta r_2)(\eta_1 r_3^2 + \eta_2 r_4^2) - 4\eta_1^2(\eta_1\beta r_1 + \eta_2\gamma r_2)r_3 r_4], \end{cases} \tag{2.22}$$

and the second example consists of the combined 3rd-order integrable equations:

$$\begin{cases} r_{1,t_3} = \frac{1}{\alpha^3}\{\eta_1\beta r_{1,xxx} + \eta_2\gamma r_{2,xxx} - 6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{1,x} \\ \quad -6\eta_1\eta_2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{2,x}\}, \\ r_{2,t_3} = \frac{1}{\alpha^3}\{\eta_1\gamma r_{1,xxx} + \eta_1\beta r_{2,xxx} - 6\eta_1^2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{1,x} \\ \quad -6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{2,x}\}, \\ r_{3,t_3} = -\frac{1}{\alpha^3}\{-\eta_1\beta r_{3,xxx} - \eta_2\gamma r_{4,xxx} + 6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{3,x} \\ \quad +6\eta_1\eta_2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{4,x}\}, \\ r_{4,t_3} = -\frac{1}{\alpha^3}\{-\eta_1\gamma r_{3,xxx} - \eta_1\beta r_{4,xxx} + 6\eta_1^2[\eta_1(\gamma r_3 + \beta r_4)r_1 + (\eta_1\beta r_3 + \eta_2\gamma r_4)r_2]r_{3,x} \\ \quad +6\eta_1^2[(\eta_1\beta r_3 + \eta_2\gamma r_4)r_1 + \eta_2(\gamma r_3 + \beta r_4)r_2]r_{4,x}\}. \end{cases} \tag{2.23}$$

The above systems give rise to two integrable models, each with four fields and they expand the family of coupled second-order and third-order integrable models (see, for instance, [29,30]). A notable feature of these coupled models is that each includes two linear highest-order derivative terms, which is why we designate these systems as combined models. Notably, most previously existing soliton hierarchies involve only a single dispersive term.

Three special cases of $\eta_2 = 0$, $\beta = 0$ and $\gamma = 0$ in the resulting hierarchy are interesting. The first case produces novel integrable couplings of the AKNS hierarchy, which are not of perturbation type. The other two cases correspond to simplified hierarchies of decoupled integrable models.

A choice of $\alpha = \beta = -\eta_1 = \eta_2 = 1$ and $\gamma = 0$ reduces the model (2.22) to an uncombined 2nd-order integrable model:

$$\begin{cases} r_{1,t_2} + r_{1,xx} + 4r_1r_2r_4 - 2r_3(r_1^2 - r_2^2) = 0, \\ r_{2,t_2} + r_{2,xx} - 4r_1r_2r_3 - 2r_4(r_1^2 - r_2^2) = 0, \\ r_{3,t_2} + r_{3,xx} - 4r_2r_3r_4 + 2r_1(r_3^2 - r_4^2) = 0, \\ r_{4,t_2} - r_{4,xx} + 4r_1r_3r_4 + 2r_2(r_3^2 - r_4^2) = 0. \end{cases} \tag{2.24}$$

A choice of $\alpha = \gamma = -\eta_1 = \eta_2 = 1$ and $\beta = 0$ reduces the model (2.22) to another uncombined 2nd-order integrable model:

$$\begin{cases} r_{1,t_2} - r_{2,xx} + 4r_1r_2r_3 + 2r_4(r_1^2 - r_2^2) = 0, \\ r_{2,t_2} + r_{1,xx} + 4r_1r_2r_4 - 2r_3(r_1^2 - r_2^2) = 0, \\ r_{3,t_2} + r_{4,xx} - 4r_1r_3r_4 - 2r_2(r_3^2 - r_4^2) = 0, \\ r_{4,t_2} - r_{3,xx} - 4r_2r_3r_4 + 2r_1(r_3^2 - r_4^2) = 0. \end{cases} \tag{2.25}$$

The selection of $\alpha = \beta = -\eta_1 = \eta_2 = 1$ and $\gamma = 0$ in the model (2.23), leads to an uncombined 3rd-order integrable model:

$$\begin{cases} r_{1,t_3} + r_{1,xxx} - 6(r_1r_3 - r_2r_4)r_{1,x} + 6(r_1r_4 + r_2r_3)r_{2,x} = 0, \\ r_{2,t_3} + r_{2,xxx} - 6(r_1r_4 + r_2r_3)r_{1,x} - 6(r_1r_3 - r_2r_4)r_{2,x} = 0, \\ r_{3,t_3} + r_{3,xxx} - 6(r_1r_3 - r_2r_4)r_{3,x} + 6(r_1r_4 + r_2r_3)r_{4,x} = 0, \\ r_{4,t_3} + r_{4,xxx} - 6(r_1r_4 + r_2r_3)r_{3,x} - 6(r_1r_3 - r_2r_4)r_{4,x} = 0. \end{cases} \tag{2.26}$$

The selection of $\alpha = \gamma = -\eta_1 = \eta_2 = 1$ and $\beta = 0$ in the model (2.23) yields another uncombined 3rd-order integrable model:

$$\begin{cases} r_{1,t_3} - r_{2,xxx} + 6(r_1r_4 + r_2r_3)r_{1,x} + 6(r_1r_3 - r_2r_4)r_{2,x} = 0, \\ r_{2,t_3} + r_{1,xxx} - 6(r_1r_3 - r_2r_4)r_{1,x} + 6(r_1r_4 + r_2r_3)r_{2,x} = 0, \\ r_{3,t_3} - r_{4,xxx} + 6(r_1r_4 + r_2r_3)r_{3,x} + 6(r_1r_3 - r_2r_4)r_{4,x} = 0, \\ r_{4,t_3} + r_{3,xxx} - 6(r_1r_3 - r_2r_4)r_{3,x} + 6(r_1r_4 + r_2r_3)r_{4,x} = 0. \end{cases} \tag{2.27}$$

These models are different from the vector AKNS integrable models. The first class of integrable models contain the ones, presented earlier in [26,27]. An intriguing phenomenon arises where, in each pair of models, the first and second components are swapped, as are the third and fourth components, and additionally, one of the components undergoes a sign inversion in the right-hand side vector fields. Furthermore, all those 4 models remain mutually commutative, and thus, they exhibit mutual symmetries. Such diversity in the presented integrable models not only showcases the deep algebraic and geometric structures of integrable models but also enhances their potential practical applications across a broad range of scientific fields, including water waves, nonlinear optics, and plasma physics.

3. Bi-Hamiltonian formalism

Let us assume $\eta_1\eta_2 \neq 0$ now. We aim to investigate the complete integrability for the obtained hierarchy in the Liouville sense (2.21). To the end, we establish a bi-Hamiltonian formalism [2,31] in the framework of the spatial problem (2.6). Observing that the Laurent series solution Y is determined by (2.7), we can then easily compute

$$\mathrm{tr}\big(Y\frac{\partial \mathcal{M}}{\partial \xi}\big) = 2\eta_1\alpha a, \ \mathrm{tr}\big(Y\frac{\partial \mathcal{M}}{\partial r}\big) = (2\eta_1^2 c, 2\eta_1\eta_2\ g, 2\eta_1^2 b, 2\eta_1\eta_2 e)^T, \tag{3.1}$$

and accordingly, the trace identity generates

$$\alpha\frac{\delta}{\delta r}\big(\int a^{[n+1]}\,dx\big)\,\xi^{-(n+1)} = \xi^{-\kappa}\frac{\partial}{\partial \xi}\xi^{\kappa-n}(\eta_1 c^{[n]}, \eta_2 g^{[n]}, \eta_1 b^{[n]}, \eta_2 e^{[n]})^T, \ n \geq 0. \tag{3.2}$$

This identity, when n is taken as two, yields $\kappa = 0$, and consequently, we obtain

$$\frac{\delta\mathcal{H}^{[n]}}{\delta r} = (\eta_1 c^{[n+1]}, \eta_2 g^{[n+1]}, \eta_1 b^{[n+1]}, \eta_2 e^{[n+1]})^T, \ n \geq 0, \tag{3.3}$$

where the Hamiltonian functionals are given by

$$\mathcal{H}^{[n]} = -\frac{\alpha}{n+1}\int a^{[n+2]}\,dx, \ n \geq 0. \tag{3.4}$$

These formulas provide the Hamiltonian functionals for the hierarchy (2.21), enabling us to furnish its Hamiltonian formulations:

$$r_{t_m} = X^{[m]} = J_1\frac{\delta\mathcal{H}^{[m]}}{\delta r}, \ m \geq 0, \tag{3.5}$$

where the Hamiltonian operator J_1 is defined by

$$J_1 = \left[\begin{array}{c|c} 0 & J_{12} \\ \hline J_{21} & 0 \end{array}\right], \ J_{12} = \begin{bmatrix} \frac{\alpha}{\eta_1} & 0 \\ 0 & \frac{\alpha}{\eta_2} \end{bmatrix}, \ J_{21} = \begin{bmatrix} -\frac{\alpha}{\eta_1} & 0 \\ 0 & -\frac{\alpha}{\eta_2} \end{bmatrix}, \tag{3.6}$$

and the Hamiltonian quantities $\mathcal{H}^{[m]}$ are stated in (3.4). From the Hamiltonian formulations, we can derive a symmetry $J_1\frac{\delta\mathcal{H}}{\delta r}$, which originates from a conserved quantity $\mathcal{H}$ corresponding to each member of the integrable Hamiltonian sequence.

The characteristic commutative property for the constructed fields $X^{[n]}$ is given by

$$[\![X^{[p]},X^{[q]}]\!]\equiv X^{[p]\prime}(r)[X^{[q]}]-X^{[q]\prime}(r)[X^{[p]}]=0,\ \ p,q\geq 0. \tag{3.7}$$

More fundamentally, there exists a Lax algebra, expressed as

$$[\![\mathcal{N}^{[p]},\mathcal{N}^{[q]}]\!]\equiv \mathcal{N}^{[p]\prime}(r)[X^{[q]}]-\mathcal{N}^{[q]\prime}(r)[X^{[p]}]-[\mathcal{N}^{[q]},\mathcal{N}^{[p]}]=0,\ \ p,q\geq 0, \tag{3.8}$$

which ensures the commutative property of the vector fields. This can directly be verified through an analysis of the algebraic structures underlying the Lax pairs (see [32] for details). Furthermore, this commuting characteristic of vector fields is preserved under reciprocal transformations [33].

Additionally, the recursive structure of $X^{[p+1]}=\Phi X^{[p]}$ determines a hereditary recursion operator $\Phi=(\Phi_{jk})_{4\times 4}$ [31] for the hierarchy in Eq. (2.21). It reads as follows:

$$\begin{cases}\Phi_{11}=\frac{1}{\alpha}(\partial_x-2\eta_1^2r_1\partial^{-1}r_3-2\eta_1\eta_2r_2\partial^{-1}r_4),\ \Phi_{12}=\frac{1}{\alpha}(-2\eta_1\eta_2r_1\partial^{-1}r_4-2\eta_1\eta_2r_2\partial^{-1}r_3),\\ \Phi_{13}=\frac{1}{\alpha}(-2\eta_1^2r_1\partial^{-1}r_1-2\eta_1\eta_2r_2\partial^{-1}r_2),\ \Phi_{14}=\frac{1}{\alpha}(-2\eta_1\eta_2r_1\partial^{-1}r_2-2\eta_1\eta_2r_2\partial^{-1}r_1);\end{cases} \tag{3.9}$$

$$\begin{cases}\Phi_{21}=\frac{1}{\alpha}(-2\eta_1^2r_1\partial^{-1}r_4-2\eta_1^2r_2\partial^{-1}r_3),\ \Phi_{22}=\frac{1}{\alpha}(\partial_x-2\eta_1^2r_1\partial^{-1}r_3-2\eta_1\eta_2r_2\partial^{-1}r_4),\\ \Phi_{23}=\frac{1}{\alpha}(-2\eta_1^2r_1\partial^{-1}r_2-2\eta_1^2r_2\partial^{-1}r_1),\ \Phi_{24}=\frac{1}{\alpha}(-2\eta_1^2r_1\partial^{-1}r_1-2\eta_1\eta_2r_2\partial^{-1}r_2);\end{cases} \tag{3.10}$$

$$\begin{cases}\Phi_{31}=\frac{1}{\alpha}(2\eta_1^2r_3\partial^{-1}r_3+2\eta_1\eta_2r_4\partial^{-1}r_4),\ \Phi_{32}=\frac{1}{\alpha}(2\eta_1\eta_2r_3\partial^{-1}r_4+2\eta_1\eta_2r_4\partial^{-1}r_3),\\ \Phi_{33}=\frac{1}{\alpha}(-\partial_x+2\eta_1^2r_3\partial^{-1}r_1+2\eta_1\eta_2r_4\partial^{-1}r_2),\ \Phi_{34}=\frac{1}{\alpha}(2\eta_1\eta_2r_3\partial^{-1}r_2+2\eta_1\eta_2r_4\partial^{-1}r_1);\end{cases} \tag{3.11}$$

$$\begin{cases}\Phi_{41}=\frac{1}{\alpha}(2\eta_1^2r_3\partial^{-1}r_4+2\eta_1^2r_4\partial^{-1}r_3),\ \Phi_{42}=\frac{1}{\alpha}(2\eta_1^2r_3\partial^{-1}r_3+2\eta_1\eta_2r_4\partial^{-1}r_4),\\ \Phi_{43}=\frac{1}{\alpha}(2\eta_1^2r_3\partial^{-1}r_2+2\eta_1^2r_4\partial^{-1}r_1),\ \Phi_{44}=\frac{1}{\alpha}(-\partial_x+2\eta_1^2r_3\partial^{-1}r_1+2\eta_1\eta_2r_4\partial^{-1}r_2).\end{cases} \tag{3.12}$$

The concept of hereditariness [34] means that the operator Φ satisfies the condition

$$L_{\Phi X}\Phi=\Phi L_X\Phi, \tag{3.13}$$

where $L_Y\Phi$ denotes the Lie derivative of Φ along the direction of Y, given by

$$(L_Y\Phi)Z=\Phi[\![Y,Z]\!]-[\![Y,\Phi Z]\!], \tag{3.14}$$

with Z being a vector field. For an evolution equation $r_t=X(r)$, an operator $\Psi=\Psi(x,t,r,r_x,\ldots)$ presents a recursion operator [35] if it satisfies the condition

$$\frac{\partial\Psi}{\partial t}+L_X\Psi=0. \tag{3.15}$$

It is worth noting that Φ is autonomous. A simple verification shows that the operator Φ constructed above acts as a recursion operator for the first model $r_{t_0}=X^{[0]}(r)$ in the hierarchy, as it satisfies $L_{X^{[0]}}\Phi=0$. In view of these two facts, we can compute that

$$L_{X^{[p]}}\Phi=L_{\Phi X^{[p-1]}}\Phi=\Phi L_{X^{[p-1]}}\Phi=\cdots=\Phi^pL_{X^{[0]}}\Phi=0,\ \ p\geq 1. \tag{3.16}$$

This implies that the operator Φ functions as a universal recursion operator for the entire constructed hierarchy (2.21).

With further analysis, we can establish that any linear combination of the two operators J_1 and $J_2=\Phi J_1$ is also Hamiltonian, meaning that J_1 and J_2 form a Hamiltonian pair. An operator J is considered Hamiltonian if it satisfies the condition

$$\langle (Z^{[1]})^TJ'(r)[JZ^{[2]}],Z^{[3]}\rangle+\mathrm{cycle}(Z^{[1]},Z^{[2]},Z^{[3]})=0, \tag{3.17}$$

where J' denotes the Gateaux derivative and the inner product is given by

$$\langle Z^{[1]},Z^{[2]}\rangle=\int (Z^{[1]})^TZ^{[2]}\,dx, \tag{3.18}$$

with $Z^{[1]},Z^{[2]}$ and $Z^{[3]}$ being column vector fields. As a consequence, the hierarchy (2.21) possesses a bi-Hamiltonian formulation [2]:

$$r_{t_m}=X^{[m]}=J_1\frac{\delta\mathcal{H}^{[m]}}{\delta r}=J_2\frac{\delta\mathcal{H}^{[m-1]}}{\delta r},\ \ \ m\geq 1. \tag{3.19}$$

It immediately follows that the corresponding Hamiltonian functionals are mutually compatible, under the following Poisson brackets [8]:

$$\{\mathcal{H}^{[p]},\mathcal{H}^{[q]}\}_{J_1}=\langle\frac{\delta\mathcal{H}^{[p]}}{\delta r},J_1\frac{\delta\mathcal{H}^{[q]}}{\delta r}\rangle=0,\ \ p,q\geq 0, \tag{3.20}$$

and

$$\{\mathcal{H}^{[p]},\mathcal{H}^{[q]}\}_{J_2}=\langle\frac{\delta\mathcal{H}^{[p]}}{\delta r},J_2\frac{\delta\mathcal{H}^{[q]}}{\delta r}\rangle=0,\ \ p,q\geq 0, \tag{3.21}$$

associated with the two Hamiltonian operators J_1 and J_2.

In conclusion, the models in the hierarchy (2.21) are Liouville integrable, possessing a continuous series of commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and an infinite sequence of commuting conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. Two particular illustrative models in the hierarchy are the systems in (2.22) and (2.23), and they add further contributions to the established class of combined coupled integrable models with four fields, which are encompassed by Magri's geometric formalism.

4. Concluding remarks

We derived a class of combined coupled bi-Hamiltonian models with four fields, being integrable in the Liouville sense, by beginning with a specific 4th-order matrix eigenvalue problem and employing the Lax pair formulation. A key step in this derivation involved finding a series solution of Laurent form to the corresponding stationary Lax equation within the Lax pair formalism. Our analysis shows that these integrable models possess hereditary recursive structures which lead to bi-Hamiltonian geometric formulations, ensuring their Liouville integrability. These results were obtained through the implementation of the trace variational identity associated with the matrix eigenvalue problem.

We note that when $\eta_2 = 0$, integrable couplings are obtained, and in this case, we need to utilize the variational identity in deriving a Magri's geometric formulation. A particularly interesting direction for future exploration is the algebraic and geometric structure of soliton solutions of those novel combined integrable models. Several powerful methods can be employed for this purpose, including the Zakharov–Shabat dressing technique [36], the Riemann–Hilbert problem approach [37], the Darboux transformation method [38–42], and the determinant-based method [43,44]. In addition to soliton solutions, other types of solutions, such as lumps, kinks, breathers, and rogue waves – especially their interaction solutions (see, for instance, [45–53]) – are likewise of considerable importance. These solutions are frequently obtained from specific wave number reductions of general soliton solutions. Furthermore, similarity transformations of spectral matrices, which induce nonlocal group reductions, can give rise to nonlocal integrable models. The solitons and their associated dynamics in these models are significant in both mathematical theory and physical applications (see, for instance, [42]). It is worth noting that nonlocal models of differential equations display notably varied solution patterns, as demonstrated in [54,55].

Integrable models hold great importance because of their significant ties to diverse areas of mathematics, including Hamiltonian dynamics, algebraic geometry, the theory of Lie groups and algebras, and orthogonal polynomials. Studying these models shed light on the mechanisms driving physical system dynamics and bridges the gap between theoretical concepts and experimental observations, thereby enhancing our understanding of the fundamental principles that govern natural phenomena.

Declaration of competing interest

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

All data generated or analyzed during this study are available within the published article.

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