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An extended AKNS eigenvalue problem and its affiliated integrable Hamiltonian hierarchies

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ABSTRACT

This paper aims to study an extended 4×4 AKNS eigenvalue problem and construct its affiliated integrable hierarchies of bi-Hamiltonian models over the real field. The Lax pair framework serves as the fundamental tool, ensuring integrability through bi-Hamiltonian structures with hereditary recursion operators. We compute illustrative examples of lower-order equations to demonstrate the affiliated integrable hierarchies.

1. Introduction

Within soliton theory, the concept of Lax pairs [1] stands as a cornerstone, pivotal in unraveling the intricacies of integrable models. At its essence, a Lax pair entails framing a linear eigenvalue dilemma linked to a specific nonlinear partial differential equation. Through adept construction of such pairs, we unfurl a cohesive ensemble of model equations endowed with extraordinary integrable characteristics. These equations unveil soliton solutions, rendering them receptive to analytical methodologies and affording profound glimpses into their dynamic behavior [2,3].

In the realm of constructing integrable models using Lax pairs, the process typically initiates by defining a suitable spectral matrix $\mathcal{P}(u,\lambda)$, contingent upon a column potential vector u and a spectral parameter denoted by λ. At the heart of this methodology lies the creation of a pair of eigenvalue problems, termed the Lax pair, intricately linked through a compatibility condition. This condition serves to ensure that the nonlinear equations affiliated with the Lax pair are integrable. The Lax pair comprises two differential equations:

$$\phi_x = \mathcal{P}(u,\lambda)\phi,\ \ \phi_1 = \mathcal{Q}(u,\lambda)\phi, \tag{1}$$

where ϕ represents the eigenfunction and $\mathcal{Q}(u,\lambda)$ represents the temporal spectral matrix. The zero curvature condition, or compatibility condition, is articulated as:

$$\mathcal{P}_t - \mathcal{Q}_x + [\mathcal{P},\mathcal{Q}] = 0, \tag{2}$$

where $[\mathcal{P},\mathcal{Q}]$ denotes the commutator of $\mathcal{P}$ and $\mathcal{Q}$, ensuring the coherent evolution of the eigenfunction ϕ across both spatial and temporal dimensions, thereby establishing the integrability of the system.

To exemplify, let us delve into the AKNS (Ablowitz–Kaup–Newell–Segur) system, renowned for its capacity to engender integrable equations. Within this framework, the definitions of $\mathcal{P}$ and $\mathcal{Q}$ are elucidated as follows:

$$\mathcal{P}(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},\ \mathcal{Q}(u,\lambda) = \begin{bmatrix} -A & B \\ C & A \end{bmatrix}, \tag{3}$$

where p and q represent components of the potential vector u, while A, B, and C are functions contingent upon u and λ. The specific expressions of A, B, and C are contingent upon the particular integrable model being scrutinized. The zero curvature condition (2) then engenders a set of nonlinear partial differential equations for p and q. For instance, in the domain of the integrable nonlinear Schrödinger (NLS) model, this condition yields:

$$\begin{cases} p_t = -p_{xx} + 2p^2 q, \\ q_t = q_{xx} - 2pq^2, \end{cases} \tag{4}$$

under the choice of

$$A = 2\lambda^2 - pq,\ \ B = 2\lambda p - p_x,\ \ C = 2\lambda q + q_x, \tag{5}$$

and in the domain of the integrable modified Kortweg–de Vries (mKdV) model, the condition leads to:

$$\begin{cases} p_t = p_{xxx} - 6pqp_x, \\ q_t = q_{xxx} - 6pqq_x, \end{cases} \tag{6}$$

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under the choice of

$$A = 4\lambda^3 - 2pq\lambda + p_x q - pq_x,\ \ B = 4\lambda^2 p - 2\lambda p_x + p_{xx} - 2p^2 q,\\ C = 4\lambda^2 q + 2\lambda q_x + q_{xx} - 2pq^2. \tag{7}$$

Solving these equations reveals the integrable structure of the AKNS eigenvalue system, characterized by soliton solutions, infinite symmetries, and conserved quantities.

By judiciously selecting $\mathcal{P}$ and $\mathcal{Q}$, a plethora of integrable models such as the Korteweg–de Vries (KdV) equation, the sine-Gordon equation, and others can be derived. These models exhibit extraordinary integrable properties and are amenable to potent analytical techniques like the inverse scattering transform.

Hamiltonian structures serve as foundational elements in the examination of integrable systems, furnishing a framework to probe the Liouville integrability of resultant models. One approach to engendering Hamiltonian structures involves the application of the classical trace identity or the variational identity. Notably, the trace identity emerges as a formidable tool in this domain.

The trace identity is articulated as (see [4,5] for details):

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(W\frac{\partial \mathcal{P}}{\partial \lambda}\big)\,dx = \lambda^{-\kappa}\frac{\partial}{\partial \lambda}\lambda^{\kappa}\mathrm{tr}\big(W\frac{\partial \mathcal{P}}{\partial u}\big), \tag{8}$$

where $\frac{\delta}{\delta u}$ signifies the variational derivative with respect to u, “tr” symbolizes the trace of a matrix, and κ denotes a constant independent of the spectral parameter λ. In this context, W satisfies the equation:

$$W_x = [\mathcal{P}, W], \tag{9}$$

where $\mathcal{P}$ represents the spectral matrix. The trace identity establishes a connection between the variational derivative of an integral involving the spectral parameter λ and the trace of a matrix expression. This linkage serves as a conduit between the eigenvalue problem and the Hamiltonian structure of the system, facilitating deeper insights into its dynamics.

Indeed, the Lax pair formulation outlined previously facilitates the generation of copious Liouville integrable hierarchies of soliton Hamiltonian models. This process harnesses the inherent loop algebras derived from special linear algebras (see, e.g., [4–15]) and special orthogonal algebras (see, e.g., [16–18]). These hierarchies assume paramount importance in the realm of integrable models as they furnish a meticulously organized framework for delving into the solutions and properties of zero curvature equations.

This paper endeavors to introduce a distinct 4×4 spectral matrix and subsequently derive a Liouville integrable hierarchy of bi-Hamiltonian equations with four dependent variables utilizing the Lax pair framework over the real field. The associated bi-Hamiltonian structures for this hierarchy of soliton equations are provided by employing the trace identity. Concrete examples are elucidated, encompassing four-component coupled integrable NLS and mKdV equations. Finally, the paper concludes with a conclusive section, offering closing remarks on the discussed findings.

2. An eigenvalue problem and its affiliated soliton hierarchies

Motivated by recent research on four-component integrable models through the Lax pair formulation (see, e.g., [19,20]), we propose a matrix eigenvalue problem of the following form:

$$\phi_x = \mathcal{P}\phi = \mathcal{P}(u,\lambda)\phi,\ \ \mathcal{P} = \begin{bmatrix} \alpha_1\lambda & u_1 & \epsilon\beta_1\lambda & \epsilon u_3 \\ u_2 & \alpha_2\lambda & \epsilon u_4 & \epsilon\beta_2\lambda \\ \beta_1\lambda & u_3 & \alpha_1\lambda & u_1 \\ u_4 & \beta_2\lambda & u_2 & \alpha_2\lambda \end{bmatrix}, \tag{10}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ϵ stand for arbitrary real constant parameters, λ is the spectral parameter again, and u represents the dependent variable comprising four components:

$$u = u(x,t) = (u_1, u_2, u_3, u_4)^T. \tag{11}$$

By setting the top-right 2×2 block to zero (i.e., choosing $\epsilon = 0$) and taking $\alpha_1 = -\alpha_2 = -1$ and $\beta_1 = -\beta_2 = 1$, this eigenvalue problem aligns with non-perturbation type integrable couplings of the AKNS integrable hierarchy [21]. To ensure the generation of an integrable hierarchy via the Lax pair formation from this new spectral problem, we need to enforce a necessary and sufficient condition:

$$\alpha^2 - \epsilon\beta^2 \neq 0, \tag{12}$$

where α and β are given by

$$\alpha = \alpha_1 - \alpha_2,\ \ \beta = \beta_1 - \beta_2. \tag{13}$$

When $\beta_1 = \beta_2 = 0$ and $u_3 = u_4 = 0$, the eigenvalue problem simplifies to two identical copies of the standard AKNS eigenvalue problem [6], thus providing a generalized version of the AKNS eigenvalue problem.

To present an affiliated Liouville integrable hierarchy with four dependent variables, we first solve the affiliated stationary zero curvature Eq. (9) by seeking a Laurent series matrix solution:

$$W = \begin{bmatrix} a & b & \epsilon e & \epsilon f \\ c & -a & \epsilon g & -\epsilon e \\ e & f & a & b \\ g & -e & c & -a \end{bmatrix} = \sum_{n\geq 0}\lambda^{-n}W^{\{n\}}, \tag{14}$$

where the basic objects are assumed to be expanded in Laurent series of the spectral parameter λ:

$$a = \textstyle\sum_{n\geq 0}\lambda^{-n}a^{\{n\}},\ b = \sum_{n\geq 0}\lambda^{-n}b^{\{n\}},\ c = \sum_{n\geq 0}\lambda^{-n}c^{\{n\}},\\ e = \textstyle\sum_{n\geq 0}\lambda^{-n}e^{\{n\}},\ f = \sum_{n\geq 0}\lambda^{-n}f^{\{n\}},\ g = \sum_{n\geq 0}\lambda^{-n}g^{\{n\}}. \tag{15}$$

It is obvious that the corresponding affiliated stationary zero curvature Eq. (9) engenders the following relations:

$$\begin{cases} a_x = u_1 c - u_2 b + \epsilon u_3 g - \epsilon u_4 f, \\ b_x = \alpha\lambda b + \epsilon\beta\lambda f - 2u_1 a - 2\epsilon u_3 e, \\ c_x = -\alpha\lambda c - \epsilon\beta\lambda g + 2u_2 a + 2\epsilon u_4 e, \\ e_x = u_1 g - u_2 f + u_3 c - u_4 b, \\ f_x = \beta\lambda b + \alpha\lambda f - 2u_1 e - 2u_3 a, \\ g_x = -\beta\lambda c - \alpha\lambda g + 2u_2 e + 2u_4 a. \end{cases} \tag{16}$$

This leads to

$$\lambda\begin{bmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} b \\ f \end{bmatrix} = \begin{bmatrix} b_x + 2u_1 a + 2\epsilon u_3 e \\ f_x + 2u_1 e + 2u_3 a \end{bmatrix} \tag{17}$$

and

$$\lambda\begin{bmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} -c_x + 2u_2 a + 2\epsilon u_4 e \\ -g_x + 2u_2 e + 2u_4 a \end{bmatrix}. \tag{18}$$

Therefore, the condition (12) ensuring the invertibility of the coefficient matrix in the above two systems is necessary and sufficient to guarantee that we can determine a Laurent series matrix solution W recursively. In light of (17) and (18) and noting that

$$\begin{bmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{bmatrix}^{-1} = \frac{1}{\alpha^2 - \epsilon\beta^2}\begin{bmatrix} \alpha & -\epsilon\beta \\ -\beta & \alpha \end{bmatrix},$$

we observe that the system (16) yields the initial requirements:

$$b^{\{0\}} = c^{\{0\}} = f^{\{0\}} = g^{\{0\}} = 0,\ a_x^{\{0\}} = e_x^{\{0\}} = 0, \tag{19}$$

and the recursion relations for computing the Laurent series matrix solution:

$$\begin{cases} b^{\{n+1\}} = \frac{\alpha}{\alpha^2-\epsilon\beta^2}(b_x^{\{n\}} + 2u_1 a^{\{n\}} + 2\epsilon u_3 e^{\{n\}}) \\ \qquad\quad - \frac{\epsilon\beta}{\alpha^2-\epsilon\beta^2}(f_x^{\{n\}} + 2u_1 e^{\{n\}} + 2u_3 a^{\{n\}}), \\ f^{\{n+1\}} = -\frac{\beta}{\alpha^2-\epsilon\beta^2}(b_x^{\{n\}} + 2u_1 a^{\{n\}} + 2\epsilon u_3 e^{\{n\}}) \\ \qquad\quad + \frac{\alpha}{\alpha^2-\epsilon\beta^2}(f_x^{\{n\}} + 2u_1 e^{\{n\}} + 2u_3 a^{\{n\}}), \end{cases} \tag{20}$$

$$\left\{\begin{aligned} c^{\{n+1\}} &= \tfrac{\alpha}{\alpha^2-\epsilon\beta^2}(-c_x^{\{n\}}+2u_2a^{\{n\}}+2\epsilon u_4e^{\{n\}})\\ &\quad -\tfrac{\epsilon\beta}{\alpha^2-\epsilon\beta^2}(-g_x^{\{n\}}+2u_2e^{\{n\}}+2u_4a^{\{n\}}),\\ g^{\{n+1\}} &= -\tfrac{\beta}{\alpha^2-\epsilon\beta^2}(-c_x^{\{n\}}+2u_2a^{\{n\}}+2\epsilon u_4e^{\{n\}})\\ &\quad +\tfrac{\alpha}{\alpha^2-\epsilon\beta^2}(-g_x^{\{n\}}+2u_2e^{\{n\}}+2u_4a^{\{n\}}), \end{aligned}\right. \tag{21}$$

$$\left\{\begin{aligned} a_x^{\{n+1\}} &= u_1c^{\{n+1\}}-u_2b^{\{n+1\}}+\epsilon u_3g^{\{n+1\}}-\epsilon u_4f^{\{n+1\}},\\ e_x^{\{n+1\}} &= u_1g^{\{n+1\}}-u_2f^{\{n+1\}}+u_3c^{\{n+1\}}-u_4b^{\{n+1\}}, \end{aligned}\right. \tag{22}$$

where $n\geq 0$. As normal, to determine a particular Laurent series matrix solution W, we proceed with the arbitrary constant initial data,

$$a^{\{0\}}=\frac{1}{2}\gamma,\ e^{\{0\}}=\frac{1}{2}\delta, \tag{23}$$

and consider the case of zero constants of integration,

$$a^{\{n\}}|_{u=0}=0,\ e^{\{n\}}|_{u=0}=0,\ n\geq 1. \tag{24}$$

Using these initial data and integration conditions, one can derive all sequences of $\{a^{\{n\}},b^{\{n\}},c^{\{n\}},e^{\{n\}},f^{\{n\}},g^{\{n\}}\}$ for $n\geq 1$. The first sequence appears as

$$\left\{\begin{aligned} b^{\{1\}} &= \tfrac{1}{\alpha^2-\epsilon\beta^2}[\alpha(\gamma u_1+\epsilon\delta u_3)-\epsilon\beta(\delta u_1+\gamma u_3)],\\ f^{\{1\}} &= \tfrac{1}{\alpha^2-\epsilon\beta^2}[\alpha(\delta u_1+\gamma u_3)-\beta(\gamma u_1+\epsilon\delta u_3)],\\ c^{\{1\}} &= \tfrac{1}{\alpha^2-\epsilon\beta^2}[\alpha(\gamma u_2+\epsilon\delta u_4)-\epsilon\beta(\delta u_2+\gamma u_4)],\\ g^{\{1\}} &= \tfrac{1}{\alpha^2-\epsilon\beta^2}[\alpha(\delta u_2+\gamma u_4)-\beta(\gamma u_2+\epsilon\delta u_4)],\\ a^{\{1\}} &= e^{\{1\}}=0. \end{aligned}\right.$$

The second sequence is as follows:

$$\left\{\begin{aligned} b^{\{2\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^2}(p_{2,1}u_{1,x}+\epsilon p_{2,2}u_{3,x}),\\ f^{\{2\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^2}(p_{2,2}u_{1,x}+p_{2,1}u_{3,x}),\\ c^{\{2\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^2}(-p_{2,1}u_{2,x}-\epsilon p_{2,2}u_{4,x}),\\ g^{\{2\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^2}(-p_{2,2}u_{2,x}-p_{2,1}u_{4,x}), \end{aligned}\right.$$

$$\left\{\begin{aligned} a^{\{2\}} &= -\tfrac{1}{(\alpha^2-\epsilon\beta^2)^2}[(p_{2,1}u_2+\epsilon p_{2,2}u_4)u_1+\epsilon(p_{2,2}u_2+p_{2,1}u_4)u_3],\\ e^{\{2\}} &= -\tfrac{1}{(\alpha^2-\epsilon\beta^2)^2}[(p_{2,2}u_2+p_{2,1}u_4)u_1+(p_{2,1}u_2+\epsilon p_{2,2}u_4)u_3], \end{aligned}\right.$$

where $p_{2,1}$ and $p_{2,2}$ are two specific polynomials of second-order in terms of α and β:

$$p_{2,1}=\alpha^2\gamma-2\epsilon\alpha\beta\delta+\epsilon\beta^2\gamma,\ p_{2,2}=\alpha^2\delta-2\alpha\beta\gamma+\epsilon\beta^2\delta. \tag{25}$$

The third sequence appears as

$$\left\{\begin{aligned} b^{\{3\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^3}[p_{3,1}u_{1,xx}+\epsilon p_{3,2}u_{3,xx}-2(p_{3,1}u_2+p_{3,2}u_4)u_1^2\\ &\quad -4\epsilon(p_{3,2}u_2+p_{3,1}u_4)u_1u_3-2\epsilon(p_{3,1}u_2+\epsilon p_{3,2}u_4)u_3^2],\\ f^{\{3\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^3}[p_{3,2}u_{1,xx}+p_{3,1}u_{3,xx}-2(p_{3,2}u_2+\epsilon p_{3,1}u_4)u_1^2\\ &\quad -4(p_{3,1}u_2+\epsilon p_{3,2}u_4)u_1u_3-2\epsilon(p_{3,2}u_2+p_{3,1}u_4)u_3^2], \end{aligned}\right.$$

$$\left\{\begin{aligned} c^{\{3\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^3}[p_{3,1}u_{2,xx}+\epsilon p_{3,2}u_{4,xx}-2(p_{3,1}u_1+\epsilon p_{3,2}u_3)u_2^2\\ &\quad -4\epsilon(p_{3,2}u_1+p_{3,1}u_3)u_2u_4-2\epsilon(p_{3,1}u_1+\epsilon p_{3,2}u_3)u_4^2],\\ g^{\{3\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^3}[p_{3,2}u_{2,xx}+p_{3,1}u_{4,xx}-2(p_{3,2}u_1+p_{3,1}u_3)u_2^2\\ &\quad -4(p_{3,1}u_1+\epsilon p_{3,2}u_3)u_2u_4-2\epsilon(p_{3,2}u_1+p_{3,1}u_3)u_4^2], \end{aligned}\right.$$

$$\left\{\begin{aligned} a^{\{3\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^3}[(-p_{3,1}u_2-\epsilon p_{3,2}u_4)u_{1,x}+(p_{3,1}u_1+\epsilon p_{3,2}u_3)u_{2,x}\\ &\quad -\epsilon(p_{3,2}u_2+p_{3,1}u_4)u_{3,x}+\epsilon(p_{3,2}u_1+p_{3,1}u_3)u_{4,x}],\\ e^{\{3\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^3}[(-p_{3,2}u_2-p_{3,1}u_4)u_{1,x}+(p_{3,2}u_1+p_{3,1}u_3)u_{2,x}\\ &\quad -(p_{3,1}u_2+\epsilon p_{3,2}u_4)u_{3,x}+(p_{3,1}u_1+\epsilon p_{3,2}u_3)u_{4,x}], \end{aligned}\right.$$

where $p_{3,1}$ and $p_{3,2}$ are two specific polynomials of third-order in terms of α and β:

$$p_{3,1}=\alpha^3\gamma-3\epsilon\alpha^2\beta\delta+3\epsilon\alpha\beta^2\gamma-\epsilon^2\beta^3\delta,\ p_{3,2}=\alpha^3\delta-3\alpha^2\beta\gamma+3\epsilon\alpha\beta^2\delta-\epsilon\beta^3\gamma. \tag{26}$$

The fourth sequence is as follows:

$$\left\{\begin{aligned} b^{\{4\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^4}[p_{4,1}u_{1,xxx}+p_{4,2}u_{3,xxx}\\ &\quad -6(p_{4,1}u_1u_2+\epsilon p_{4,2}u_1u_4+\epsilon p_{4,2}u_2u_3+\epsilon p_{4,1}u_3u_4)u_{1,x}\\ &\quad -6\epsilon(p_{4,2}u_1u_2+p_{4,1}u_1u_4+p_{4,1}u_2u_3+\epsilon p_{4,2}u_3u_4)u_{3,x}],\\ f^{\{4\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^4}[p_{4,2}u_{1,xxx}+p_{4,1}u_{3,xxx}\\ &\quad -6(p_{4,2}u_1u_2+p_{4,1}u_1u_4+p_{4,1}u_2u_3+\epsilon p_{4,2}u_3u_4)u_{1,x}\\ &\quad -6(p_{4,1}u_1u_2+\epsilon p_{4,2}u_1u_4+\epsilon p_{4,2}u_2u_3+\epsilon p_{4,1}u_3u_4)u_{3,x}], \end{aligned}\right.$$

$$\left\{\begin{aligned} c^{\{4\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^4}[-p_{4,1}u_{2,xxx}-\epsilon p_{4,2}u_{4,xxx}\\ &\quad +6(p_{4,1}u_1u_2+\epsilon p_{4,2}u_1u_4+\epsilon p_{4,2}u_2u_3+\epsilon p_{4,1}u_3u_4)u_{2,x}\\ &\quad +6\epsilon(p_{4,2}u_1u_2+p_{4,1}u_1u_4+p_{4,1}u_2u_3+\epsilon p_{4,2}u_3u_4)u_{4,x}],\\ g^{\{4\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^4}[-p_{4,2}u_{2,xxx}-p_{4,1}u_{4,xxx}\\ &\quad +6(p_{4,2}u_1u_2+p_{4,1}u_1u_4+p_{4,1}u_2u_3+\epsilon p_{4,2}u_3u_4)u_{2,x}\\ &\quad +6(p_{4,1}u_1u_2+\epsilon p_{4,2}u_1u_4+\epsilon p_{4,2}u_2u_3+\epsilon p_{4,1}u_3u_4)u_{4,x}], \end{aligned}\right.$$

$$\begin{aligned} a^{\{4\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^4}[-(p_{4,1}u_2+\epsilon p_{4,2}u_4)u_{1,xx}-(p_{4,1}u_1+\epsilon p_{4,2}u_3)u_{2,xx}\\ &\quad -\epsilon(p_{4,2}u_2+p_{4,1}u_4)u_{3,xx}-\epsilon(p_{4,2}u_1+p_{4,1}u_3)u_{4,xx}\\ &\quad +p_{4,1}u_{1,x}u_{2,x}+\epsilon p_{4,2}u_{1,x}u_{4,x}+\epsilon p_{4,2}u_{2,x}u_{3,x}+\epsilon p_{4,1}u_{3,x}u_{4,x}\\ &\quad +3(p_{4,1}u_2^2+2\epsilon p_{4,2}u_2u_4+\epsilon p_{4,1}u_4^2)u_1^2\\ &\quad +6\epsilon(p_{4,2}u_2^2+2p_{4,1}u_2u_4+\epsilon p_{4,2}u_4^2)u_1u_3\\ &\quad -3\epsilon(p_{4,1}u_2^2-2\epsilon p_{4,2}u_2u_4-\epsilon p_{4,1}u_4^2)u_3^2],\\ e^{\{4\}} &= \tfrac{1}{(\alpha^2-\epsilon\beta^2)^4}[(-p_{4,2}u_2-p_{4,1}u_4)u_{1,xx}-(p_{4,2}u_1+p_{4,1}u_3)u_{2,xx}\\ &\quad -(p_{4,1}u_2+\epsilon p_{4,2}u_4)u_{3,xx}-(p_{4,1}u_1+\epsilon p_{4,2}u_3)u_{4,xx}\\ &\quad +p_{4,2}u_{1,x}u_{2,x}+p_{4,1}u_{1,x}u_{4,x}+p_{4,1}u_{2,x}u_{3,x}+\epsilon p_{4,2}u_{3,x}u_{4,x}\\ &\quad +3(p_{4,2}u_2^2+2p_{4,1}u_2u_4+\epsilon p_{4,2}u_4^2)u_1^2\\ &\quad +6(p_{4,1}u_2^2+2\epsilon p_{4,2}u_2u_4+\epsilon p_{4,1}u_4^2)u_1u_3\\ &\quad +3\epsilon(p_{4,2}u_2^2+2p_{4,1}u_2u_4+\epsilon p_{4,2}u_4^2)u_3^2], \end{aligned}$$

where $p_{4,1}$ and $p_{4,2}$ are two specific polynomials of fourth-order in terms of α and β:

$$\left\{\begin{aligned} p_{4,1} &= \alpha^4\gamma-4\epsilon\alpha^3\beta\delta+6\epsilon\alpha^2\beta^2\gamma-4\epsilon^2\alpha\beta^3\delta+\epsilon^2\beta^4\gamma,\\ p_{4,2} &= \alpha^4\delta-4\alpha^3\beta\gamma+6\epsilon\alpha^2\beta^2\delta-4\epsilon\alpha\beta^3\gamma+\epsilon^2\beta^4\delta. \end{aligned}\right. \tag{27}$$

Taking these computations into account, we can choose the zero modification terms in the temporal spectral matrices, to propose

$$\phi_{t_m}=Q^{\{m\}}\phi=Q^{\{m\}}(u,\lambda)\phi,\ Q^{\{m\}}=(\lambda^mW)_+=\sum_{n=0}^{m}\lambda^nW^{\{m-n\}},\ m\geq 0, \tag{28}$$

which form the temporal matrix eigenvalue problems within the Lax pair framework. The conditions that ensure the compatibility of two kinds of matrix eigenvalue problems, i.e., the spatial and temporal eigenvalue problems, in (10) and (28) are exactly the zero curvature equations

$$\mathcal{P}_{t_r}-Q_x^{\{m\}}+[\mathcal{P},Q^{\{m\}}]=0,\ m\geq 0. \tag{29}$$

These equations engender a four-component integrable hierarchy:

$$\begin{aligned} u_{t_m} &= S^{\{m\}}=(S_1^{\{m\}},S_2^{\{m\}},S_3^{\{m\}},S_4^{\{m\}})^T\\ &= (\alpha b^{[m+1]}+\epsilon\beta f^{\{m+1\}},-\alpha c^{[m+1]}-\epsilon\beta g^{\{m+1\}},\\ &\quad \beta b^{[m+1]}+\alpha f^{\{m+1\}},-\beta c^{[m+1]}-\alpha g^{\{m+1\}})^T, \end{aligned} \tag{30}$$

or in a more concrete way,

$$\begin{cases} u_{1,t_m} = \alpha b^{[m+1]} + \epsilon\beta f^{\{m+1\}},\ u_{2,t_m} = -\alpha c^{[m+1]} - \epsilon\beta g^{\{m+1\}}, \\ u_{3,t_m} = \beta b^{[m+1]} + \alpha f^{\{m+1\}},\ u_{4,t_m} = -\beta c^{[m+1]} - \alpha g^{\{m+1\}}, \end{cases} \tag{31}$$

where $m \geq 0$.

As particular examples, this soliton hierarchy includes abundant coupled systems of integrable NLS equations and coupled systems of integrable mKdV equations. Upon taking

$$\epsilon = -2,\ \alpha = 1,\ \beta = 0,\ \gamma = 1,\ \delta = 0, \tag{32}$$

we immediately arrive at a coupled system of integrable NLS models:

$$\begin{cases} u_{1,t_2} = u_{1,xx} - 2u_1^2u_2 + 8u_1u_3u_4 + 4u_2u_3^2, \\ u_{2,t_2} = -u_{2,xx} + 2u_1u_2^2 - 4u_1u_4^2 - 8u_2u_3u_4, \\ u_{3,t_2} = u_{3,xx} - 2u_1^2u_4 - 4u_1u_2u_3 + 4u_3^2u_4, \\ u_{4,t_2} = -u_{4,xx} + 4u_1u_2u_4 + 2u_2^2u_3 - 4u_3u_4^2, \end{cases} \tag{33}$$

and a coupled system of integrable mKdV models:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} - 6(u_1u_2 - 2u_3u_4)u_{1,x} + 12(u_1u_4 + u_2u_3)u_{3,x}, \\ u_{2,t_3} = u_{2,xxx} - 6(u_1u_2 - 2u_3u_4)u_{2,x} + 12(u_1u_4 + u_2u_3)u_{4,x}, \\ u_{3,t_3} = u_{3,xxx} - 6(u_1u_4 + u_2u_3)u_{1,x} - 6(u_1u_2 - 2u_3u_4)u_{3,x}, \\ u_{4,t_3} = u_{4,xxx} - 6(u_1u_4 + u_2u_3)u_{2,x} - 6(u_1u_2 - 2u_3u_4)u_{4,x}. \end{cases} \tag{34}$$

Upon taking

$$\epsilon = -3,\ \alpha = 1,\ \beta = 0,\ \gamma = 1,\ \delta = 1, \tag{35}$$

we immediately arrive at a coupled system of combined integrable NLS models:

$$\begin{cases} u_{1,t_2} = u_{1,xx} - 3u_{3,xx} - 2(u_2 - 3u_4)u_1^2 + 12(u_2 + u_4)u_1u_3 + 6(u_2 - 3u_4)u_3^2, \\ u_{2,t_2} = -u_{2,xx} + 3u_{4,xx} + 2(u_1 - 3u_3)u_2^2 - 12(u_1 + u_3)u_2u_4 - 6(u_1 - 3u_3)u_4^2, \\ u_{3,t_2} = u_{1,xx} + u_{3,xx} - 2(u_2 + u_4)u_1^2 - 4(u_2 - 3u_4)u_1u_3 + 6(u_2 + u_4)u_3^2, \\ u_{4,t_2} = -u_{2,xx} - u_{4,xx} + 2(u_1 + u_3)u_2^2 + 4(u_1 - 3u_3)u_2u_4 - 6(u_1 + u_3)u_4^2, \end{cases} \tag{36}$$

and a coupled system of combined integrable mKdV models:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} - 3u_{3,xxx} - 6[u_1(u_2 - 3u_4) - 3u_3(u_2 + u_4)]u_{1,x} \\ \qquad + 6[u_1(u_2 + u_4) + u_3(u_2 - u_4)]u_{3,x}, \\ u_{2,t_3} = u_{2,xxx} - 3u_{4,xxx} - 6[u_1(u_2 - 3u_4) - 3u_3(u_2 + u_4)]u_{2,x} \\ \qquad + 18[u_1(u_2 + u_4) + u_3(u_2 - 3u_4)]u_{4,x}, \\ u_{3,t_3} = u_{1,xxx} + u_{3,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 - 3u_4)]u_{1,x} \\ \qquad - 6[u_1(u_2 - 3u_4) - 3u_3(u_2 + u_4)]u_{3,x}, \\ u_{4,t_3} = u_{2,xxx} + u_{4,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 - 3u_4)]u_{2,x} \\ \qquad - 6[u_1(u_2 - 3u_4) - 3u_3(u_2 + u_4)]u_{4,x}. \end{cases} \tag{37}$$

These four specific systems provide representative coupled systems of integrable models, which broaden the family of coupled integrable NLS equations and mKdV equations (see, e,g., [22–25]).

3. Recursion operator

A recursion operator being hereditary for the soliton hierarchy (31) can be computed from the recursion relation $S^{\{m+1\}} = \Phi S^{\{m\}}$ [26]. Direct computation allows us to obtain this recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$.

Utilizing the recursion relations in (20), (21) and (22), we can conduct the following computation:

$$\begin{aligned} S_1^{\{m+1\}} &= \alpha b^{\{m+2\}} + \epsilon\beta f^{\{m+2\}} = \tfrac{1}{\alpha^2-\epsilon\beta^2}[\alpha S_{1,x}^{\{m\}} - \epsilon\beta S_{3,x}^{\{m\}} \\ &\quad -2\epsilon(\beta u_1 - \alpha u_3)(\beta a^{\{m+1\}} + \alpha e^{\{m+1\}}) \\ &\quad + 2(\alpha u_1 - \epsilon\beta u_3)(\epsilon\beta e^{\{m+1\}} + \alpha a^{\{m+1\}})], \\ S_2^{\{m+1\}} &= -\alpha c^{\{m+2\}} - \epsilon\beta g^{\{m+2\}} = \tfrac{1}{\alpha^2-\epsilon\beta^2}[-\alpha S_{2,x}^{\{m\}} + \epsilon\beta S_{4,x}^{\{m\}} \\ &\quad +2\epsilon(\beta u_2 - \alpha u_4)(\beta a^{\{m+1\}} + \alpha e^{\{m+1\}}) \\ &\quad - 2(\alpha u_2 - \epsilon\beta u_4)(\epsilon\beta e^{\{m+1\}} + \alpha a^{\{m+1\}})], \\ S_3^{\{m+1\}} &= \beta b^{\{m+2\}} + \alpha f^{\{m+2\}} = \tfrac{1}{\alpha^2-\epsilon\beta^2}[-\beta S_{1,x}^{\{m\}} + \alpha S_{3,x}^{\{m\}} \\ &\quad +2(\alpha u_1 - \epsilon\beta u_3)(\beta a^{\{m+1\}} + \alpha e^{\{m+1\}}) \\ &\quad - 2(\beta u_1 - \alpha u_3)(\epsilon\beta e^{\{m+1\}} + \alpha a^{\{m+1\}})], \\ S_4^{\{m+1\}} &= -\beta c^{\{m+2\}} - \alpha g^{\{m+2\}} = \tfrac{1}{\alpha^2-\epsilon\beta^2}[\beta S_{2,x}^{\{m\}} - \alpha S_{4,x}^{\{m\}} \\ &\quad -2(\alpha u_2 - \epsilon\beta u_4)(\beta a^{\{m+1\}} + \alpha e^{\{m+1\}}) \\ &\quad + 2(\beta u_2 - \alpha u_4)(\epsilon\beta e^{\{m+1\}} + \alpha a^{\{m+1\}})]. \end{aligned}$$

On the other hand, we can perform the computation

$$\begin{aligned} \beta a^{\{m+1\}} + \alpha e^{\{m+1\}} &= -\partial^{-1}(u_1S_4^{\{m\}} + u_2S_3^{\{m\}} + u_3S_2^{\{m\}} + u_4S_1^{\{m\}}), \\ \epsilon\beta e^{\{m+1\}} + \alpha a^{\{m+1\}} &= -\partial^{-1}(u_1S_2^{\{m\}} + u_2S_1^{\{m\}} + \epsilon u_3S_4^{\{m\}} + \epsilon u_4S_3^{\{m\}}). \end{aligned}$$

It then follows that the recursion operator appears as

$$\begin{aligned} \Phi = \frac{1}{\alpha^2 - \epsilon\beta^2}\Bigg\{ & \begin{bmatrix} \alpha & 0 & -\epsilon\beta & 0 \\ 0 & -\alpha & 0 & \epsilon\beta \\ -\beta & 0 & \alpha & 0 \\ 0 & \beta & 0 & -\alpha \end{bmatrix}\partial \\ & + 2\begin{bmatrix} \epsilon(\beta u_1 - \alpha u_3) \\ -\epsilon(\beta u_2 - \alpha u_4) \\ -(\alpha u_1 - \epsilon\beta u_3) \\ \alpha u_2 - \epsilon\beta u_4 \end{bmatrix}\begin{bmatrix} \partial^{-1}u_4 \\ \partial^{-1}u_3 \\ \partial^{-1}u_2 \\ \partial^{-1}u_1 \end{bmatrix}^T \\ & +2\begin{bmatrix} -(\alpha u_1 - \epsilon\beta u_3) \\ \alpha u_2 - \epsilon\beta u_4 \\ \beta u_1 - \alpha u_3 \\ -(\beta u_2 - \alpha u_4) \end{bmatrix}\begin{bmatrix} \partial^{-1}u_2 \\ \partial^{-1}u_1 \\ \epsilon\partial^{-1}u_4 \\ \epsilon\partial^{-1}u_3 \end{bmatrix}^T\Bigg\}. \end{aligned} \tag{38}$$

In this expression for the recursion operator, the second and third terms are two matrix products.

This hereditary recursion operator involves five constant parameters, exhibiting the diversity of the recursion structure in the integrable hierarchy. Despite the nonlocality of the recursion operator, the locality of the isospectral ($\lambda_{t_m} = 0$) flows is maintained. This implies that each flow in the hierarchy preserves the integrable structure, ensuring that the derived soliton equations remain solvable by inverse scattering techniques and other methods applicable to local equations.

4. Bi-Hamiltonian structures

In order to equip Hamiltonian structures with the soliton hierarchy (31), one can apply the classical trace identity (8) to the spatial matrix eigenvalue problem (10). Let us consider $\epsilon \neq 0$ below. The case of $\epsilon = 0$ needs an application of the variational identity (see [21]).

Taking advantage of the solution W defined by (14) and applying the trace identity to the current spatial matrix eigenvalue problem (10), we can explicitly derive the Hamiltonian densities and the associated flows and then integrate the Hamiltonian structures into the resulting hierarchy of soliton equations. Specifically, we have

$$\mathrm{tr}\Big(W\frac{\partial \mathcal{P}}{\partial \lambda}\Big) = 2\alpha a + 2\epsilon\beta e,\ \mathrm{tr}\Big(W\frac{\partial \mathcal{P}}{\partial u}\Big) = (2c, 2b, 2\epsilon g, 2\epsilon f)^T, \tag{39}$$

and consequently, the indicated trace identity leads to

$$\frac{\delta}{\delta u}\int \lambda^{-(n+1)}(\alpha a^{\{n+1\}}+\epsilon\beta e^{\{n+1\}})\,dx$$
$$=\lambda^{-\kappa}\frac{\partial}{\partial\lambda}\lambda^{\kappa-n}(c^{\{n\}},b^{\{n\}},\epsilon g^{\{n\}},\epsilon f^{\{n\}})^T,\ n\geq 0. \tag{40}$$

A verification with $n=2$ shows $\kappa=0$, and as a consequence, one obtains

$$\frac{\delta}{\delta u}\mathcal{H}^{\{n\}}=(c^{\{n+1\}},b^{\{n+1\}},\epsilon g^{\{n+1\}},\epsilon f^{\{n+1\}})^T,\ n\geq 0, \tag{41}$$

where the indicated Hamiltonian quantities appear as:

$$\mathcal{H}^{\{n\}}=-\int\frac{\alpha a^{\{n+2\}}+\epsilon\beta e^{\{n+2\}}}{n+1}\,dx,\ n\geq 0. \tag{42}$$

This allows us to furnish the Hamiltonian structures for the soliton hierarchy (31):

$$u_{t_m}=\mathcal{S}^{\{m\}}=J_1\frac{\delta\mathcal{H}^{\{m\}}}{\delta u},\ J_1=\begin{bmatrix}0\alpha & 0 & \beta & \\ -\alpha & 0 & -\beta & 0\\ 0 & \beta & 0 & \frac{\alpha}{\epsilon}\\ -\beta & 0 & -\frac{\alpha}{\epsilon} & 0\end{bmatrix},\ m\geq 0, \tag{43}$$

where J_1 is, obviously, Hamiltonian, and the Hamiltonian quantities $\mathcal{H}^{\{m\}}$ are directly provided by (42). We point out that an important property exhibited by the Hamiltonian structures is an interrelation between conserved quantities and symmetries of the same model.

The standard soliton theory tells that those vector fields $\mathcal{S}^{\{n\}}$ commute:

$$[\![\mathcal{S}^{\{n_1\}},\mathcal{S}^{\{n_2\}}]\!]=\mathcal{S}^{\{n_1\}\prime}(u)[\mathcal{S}^{\{n_2\}}]-\mathcal{S}^{\{n_2\}\prime}(u)[\mathcal{S}^{\{n_1\}}]=0,\ n_1,n_2\geq 0, \tag{44}$$

which can bee seen from a Lax algebra of operators:

$$[\![\mathcal{Q}^{\{n_1\}},\mathcal{Q}^{\{n_2\}}]\!]=\mathcal{Q}^{\{n_1\}\prime}(u)[\mathcal{S}^{\{n_2\}}]-\mathcal{Q}^{\{n_2\}\prime}(u)[\mathcal{S}^{\{n_1\}}]$$
$$+[\mathcal{Q}^{\{n_1\}},V\mathcal{Q}^{\{n_2\}}]=0,\ n_1,n_2\geq 0. \tag{45}$$

One can also verify this property directly by examining the relation between the isospectral ($\lambda_{t_m}=0$) zero curvature equations (see [27] for details).

Further, with some detailed analysis, we can see that J_1 and $J_2=\Phi J_1$ form a Hamiltonian pair. Thus, the soliton hierarchy (31) possesses the following bi-Hamiltonian structures [28]:

$$u_{t_m}=\mathcal{S}^{\{m\}}=J_1\frac{\delta\mathcal{H}^{\{m\}}}{\delta u}=J_2\frac{\delta\mathcal{H}^{\{m-1\}}}{\delta u},\quad m\geq 1. \tag{46}$$

It follows that under the resulting two Poisson brackets, the associated Hamiltonian functionals commute:

$$\{\mathcal{H}^{\{n_1\}},\mathcal{H}^{\{n_2\}}\}_{J_i}=0,\ n_1,n_2\geq 0,\ i=1,2, \tag{47}$$

where

$$\{\mathcal{F},\mathcal{G}\}_{J_i}=\int\Big(\frac{\delta\mathcal{F}}{\delta u}\Big)^TJ_i\frac{\delta\mathcal{G}}{\delta u}\,dx,\ i=1,2. \tag{48}$$

The above two equalities, (44) and (47), imply that all derived isospectral flows have infinitely many conserved quantities and symmetries inherent to the integrable system. Moreover, by virtue of the recursion and bi-Hamiltonian structures, the resulting conserved quantities and symmetries can be effectively computed to explore Liouville integrability. This property is essential for building practical applications and conducting analysis of these integrable models. It guarantees that exact solutions can reveal important physical behavior and can be precisely explored in mathematical ways.

To summarize, the soliton hierarchy (31) possesses specific bi-Hamiltonian structures, exhibiting Liouville integrability, Every model has two infinite sequences of symmetries $\{\mathcal{S}^{\{n\}}\}_{n=0}^{\infty}$ and conserved quantities $\{\mathcal{H}^{\{n\}}\}_{n=0}^{\infty}$, and all in the same sequence commute with each other. The concrete examples, presented in (33) and (34), yield unique nonlinear coupled Liouville integrable models with bi-Hamiltonian structures, enhancing and refining the existing studies in the literature (see, e.g., [19,20,29]).

5. Concluding remarks

This paper has explored a special 4th-order matrix eigenvalue problem and constructed its affiliated integrable Hamiltonian hierarchies involving five free parameters through the Lax pair formulation over the real field. The analysis we have conducted is helpful in generating integrable models with bi-Hamiltonian structures and understanding the underlying dynamics of these systems. The use of Laurent series matrix solutions to solve the affiliated stationary zero curvature equation is crucial, as it allows us to explore the integrability properties of the presented models. Utilizing the classical trace identity to the specific matrix isospectral problem elucidates the bi-Hamiltonian structures inherent in these systems. Several illustrative examples of lower orders exhibit the characteristics of the presented integrable models.

Further investigation is warranted to explore the soliton configurations for the obtained integrable models using effective and efficient methods in the field of integrable models. These methods include the Riemann–Hilbert technique [30], the Zakharov–Shabat dressing method [31], the determinant approach [32], the Darboux transformation [33–35] and the deep neural networks [36]. Alternative significant solutions, such as kink, anti-kink, breather, complexiton, lump, and rogue wave solutions, including various interaction solutions (see, e.g., [37–44]), can be derived from solitons under specific wave number reductions. Performing nonlocal group reductions on matrix eigenvalue problems can also result in reduced nonlocal integrable models (see, e.g., [45]). Nonlocality exposes a variety of unique phenomena and solutions (see, e.g., [46,47]).

Clearly, incorporating additional dependent variables into the spatial eigenvalue matrix can lead to larger integrable models [48–51]. Such generalizations can result in systems with richer dynamics and more complex interactions between the variables, including focusing, defocusing, and mixed-type interactions [52–55]. However, as more dependent variables are introduced, the resulting equations become increasingly complex. This increased complexity can make the analysis and understanding of the system more challenging, especially in the real field. Investigating larger integrable models will provide a more profound understanding of the core principles underlying nonlinear dynamics and the integrable nature of nonlinear models in mathematical physics.

CRediT authorship contribution statement

Wen-Xiu Ma: Writing – review & editing, Writing – original draft, Visualization, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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