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Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations

Wen-Xiu Ma *

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

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ABSTRACT

The aim of this paper is to study vector nonlocal reverse-time NLS (nonlinear Schrödinger) equations and present a binary Darboux transformation by utilizing two sets of eigenfunctions and adjoint eigenfunctions. A product of N single binary Darboux transformations is explored for the resultant binary Darboux transformation. A class of soliton solutions is generated by an application starting from the zero seed potential.

1. Introduction

In the field of integrable equations, the Darboux transformation is among powerful approaches to soliton solutions, and it also has applications to matrix factorization and Riemann–Hilbert problems. A Lax pair consisting of spatial and temporal matrix eigenvalue problems is the basic tool to create Darboux transformations. Besides such a Lax pair of matrix eigenvalue problems, a binary Darboux transformation requires to use an equivalent pair of adjoint matrix eigenvalue problems. We would like to study a general formulation for binary Darboux transformations associated with a vector reduced Ablowitz-Kaup-Newell-Segur (AKNS) spatial eigenvalue problem.

Let ν be an eigenvalue parameter, and $u = u(x,t)$, a potential vector, where x denotes the spatial variable and t, the temporal variable. To begin understanding a binary Darboux transformation, we take a Lax pair of matrix eigenvalue problems:

$$-i\eta_x = U\eta = U(u,\nu)\eta, \ \ -i\eta_t = V\eta = V(u,\nu)\eta, \tag{1.1}$$

where i be the unit imaginary number and η is an $m \times m$ matrix eigenfunction. The involved pair of square matrices, U and V, is defined by

$$U = A(\nu) + P(u,\nu), \tag{1.2}$$

and

$$V = B(\nu) + Q(u, u_x, \dots, u^{(n_0)}; \nu). \tag{1.3}$$

Here the two $m\times m$ matrices A and B are constant and commuting with each other, but the other two $m \times m$ matrices P and Q are trace-less and depend on the potential vector u. Represented as the compatibility condition of the two matrix eigenvalue problems, the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.4}$$

where $[\cdot,\cdot]$ stands for the matrix commutator, leads to an integrable equation, whose Cauchy problem is solvable through utilizing the inverse scattering transform [1–3]. We formulate reduced Lax pairs and zero curvature equations so that we can generate nonlocal integrable counterparts (see, for example, [4,5]).

In soliton theory, we also use the adjoint pair of matrix eigenvalue problems:

$$i\tilde{\eta}_x = \tilde{\eta}U, \ \ i\tilde{\eta}_t = \tilde{\eta}V. \tag{1.5}$$

Obviously, the compatibility condition of this adjoint pair of eigenvalue problems does not generate any new condition, except the original zero curvature equation. Moreover, the inverse η^{-1} of a matrix eigenfunction η presents an adjoint matrix eigenfunction defined by the adjoint pair (1.5). Such a connection has also been used in the study of Riemann–Hilbert problems in the field of local integrable equations (see, for example, [6]).

A binary Darboux transformation is constituted of

$$\eta' = S^{+}\eta, \ \ \tilde{\eta}' = \tilde{\eta}S^{-}, \ \ u' = f(u), \tag{1.6}$$

* Correspondence to: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.
E-mail address: mawx@cas.usf.edu.

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as long as

$$U' = -iS^+_x(S^+)^{-1} + S^+U(S^+)^{-1},\ V' = -iS^+_t(S^+)^{-1} + S^+V(S^+)^{-1}, \quad (1.7)$$

where two Darboux matrices S^+ and S^- are inverse to each other, and

$$U' = U'(u', v) = U(u', v),\ V' = V'(u', v) = V(u', v). \quad (1.8)$$

Therefore, η' and $\tilde{\eta}'$ present a new pair of eigenfunction and adjoint eigenfunction, i.e., they satisfy

$$-i\eta'_x = U'\eta',\ -i\eta'_t = V'\eta', \quad (1.9)$$

and

$$i\tilde{\eta}'_x = \tilde{\eta}'U',\ i\tilde{\eta}'_t = \tilde{\eta}'V', \quad (1.10)$$

respectively. Either of these two pairs guarantees that the new Lax pair, consisting of U' and V', just yields the original zero curvature equation, with u being replaced with u'. Accordingly, the novel argument u' engenders another solution to the considered integrable equation. There are diverse examples of applications of Darboux transformations and binary Darboux transformations to integrable nonlinear Schrödinger (NLS) equations and integrable modified Korteweg–de Vries (mKdV) equations in the literature (see, for example, [7–16]).

In this paper, we would like to present a binary Darboux transformation for vector nonlocal reverse-time nonlinear Schrödinger (NLS) equations and explore its link with a kind of N-fold binary Darboux transformation. By taking the zero seed potential, an application of the obtained binary Darboux transformation yields a class of soliton solutions, and illustrative examples are nonlocal vector integrable NLS equations. The conclusion section gives a few comments and remarks.

2. Vector nonlocal NLS equations

Let $n \in \mathbb{N}$ be an arbitrary natural number, I_n stand for the nth-order identity matrix, and $\{\delta_1, \delta_2\}$ and $\{\gamma_1, \gamma_2\}$ be two pairs of arbitrary but different constants. Many references exhibit that the vector NLS equations are generated from the matrix eigenvalue problems (see, for example, [17]):

$$-i\eta_x = U\eta = U(u, v)\eta,\ -i\eta_t = V\eta = V(u, v)\eta, \quad (2.1)$$

whose Lax pair is introduced to be

$$U = v\Lambda + P,\ V = v^2\Theta + Q. \quad (2.2)$$

In the above, the involved four matrices are defined as follows:

$$\Lambda = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 I_n \end{bmatrix},\ P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad (2.3)$$

$$\Theta = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 I_n \end{bmatrix},\ Q = Q(u, v) = \frac{\gamma}{\delta} v \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\gamma}{\delta^2}\begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}, \quad (2.4)$$

where the potential column vector is taken as $u = (p, q^T)^T$, for which $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n)^T$, and two constants are defined by $\delta = \delta_1 - \delta_2$ and $\gamma = \gamma_1 - \gamma_2$. We can easily observe that through utilizing the square potential matrix P, the above involved matrix Q can be defined in the following way:

$$\begin{aligned} Q &= \frac{\gamma}{\delta} vP - \frac{\gamma}{\delta^2}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix} P^2 + \begin{bmatrix} i & 0 \\ 0 & -iI_n \end{bmatrix} P_x \right\} \\ &= \frac{\gamma}{\delta} vP - \frac{\gamma}{\delta^2} I_{1,n}(P^2 + iP_x), \end{aligned} \quad (2.5)$$

where $I_{1,n} = \mathrm{diag}(1, I_n)$.

When only one pair of p_j and q_j, $1 \le j \le n$, is nonzero, the above spatial matrix eigenvalue problem in (2.1) becomes the scalar AKNS eigenvalue problem [18]. Obviously, the vector AKNS spatial matrix eigenvalue problem in (2.1) is a degenerate problem, since the matrix Λ has a multiple eigenvalue δ_2. Nevertheless, this will not be a problem for our discussion.

The compatibility condition of the above matrix eigenvalue problems engenders the classical local vector NLS equations:

$$p_{j,t} = -\frac{\gamma}{\delta^2} i[p_{j,xx} + 2(\sum_{r=1}^n p_r q_r)p_j],\ q_{j,t} = \frac{\gamma}{\delta^2} i[q_{j,xx} + 2(\sum_{r=1}^n p_r q_r)q_j],\ 1 \le j \le n. \quad (2.6)$$

When $n = 2$, under a special kind of reductions, the above vector integrable NLS equations (2.6) can give rise to the Manokov system [19]. This system has been shown to possess an analytic decomposition into N integrable Hamiltonian systems of ordinary differential equations [20].

Following an idea of conducting reductions by similarity transformations in [21], we can formulate a particular sort of nonlocal reductions:

$$U^T(x, -t, -v)C + CU(x, t, v) = 0,\ C = \mathrm{diag}(1, \Sigma),\ \Sigma^\dagger = \Sigma, \quad (2.7)$$

for the given spectral matrix U [22]. Equivalently, this requires a condition on P:

$$P^T(x, -t)C + CP(x, t) = 0. \quad (2.8)$$

From now on, we use T to denote the matrix transpose, and Σ stands for a constant invertible symmetric matrix.

Conducting the nonlocal reductions defined by (2.8), we can achieve the reductions between the two potential matrices:

$$\Sigma q(x, t) + p^T(x, -t) = 0, \quad (2.9)$$

where Σ is an $n \times n$ matrix as adopted earlier. Based on this, one can also directly verify that

$$V^T(x, -t, -v)C = CV(x, t, v),\ Q^T(x, -t, -v)C = CQ(x, t, v), \quad (2.10)$$

where the two $(n+1)\times(n+1)$ matrices V and Q are determined earlier by (2.2) and (2.4), respectively.

It is not difficult to see that the nonlocal reductions in (2.8) (or equivalently, (2.9)) keep the original zero curvature equation to be form-invariant. Consequently, under the reductions in (2.8), the spatial and temporal matrix eigenvalue problems in (2.1) yield the following vector nonlocal reverse-time integrable NLS equations:

$$ip_t(x, t) = \frac{\gamma}{\delta^2}[p_{xx}(x, t) - 2p(x, t)\Sigma^{-1}p^T(x, -t)p(x, t)], \quad (2.11)$$

where the symmetric matrix Σ is arbitrary but invertible.

When $n = 1$, we can arrive at a well-known example [23]:

$$ip_{1,t}(x, t) = p_{1,xx}(x, t) + 2\sigma p_1(x, -t)p_1^2(x, t),\ \sigma = \pm 1.$$

When $n = 2$, we can obtain a new system of nonlocal reverse-time integrable NLS equations:

$$\begin{cases} ip_{1,t}(x, t) = p_{1,x,x}(x, t) + (\zeta_1 p_1(x, -t)p_1(x, t) + \zeta_2 p_2(x, -t)p_2(x, t))p_1(x, t), \\ ip_{2,t}(x, t) = p_{2,x,x}(x, t) + (\zeta_1 p_1(x, -t)p_1(x, t) + \zeta_2 p_2(x, -t)p_2(x, t))p_2(x, t), . \end{cases}$$

where ζ_1 and ζ_2 are arbitrary but nonzero constants.

3. Binary Darboux transformation

3.1. A general skeleton

Let us first discuss a binary Darboux transformation, through using a single pair of eigenfunction and adjoint eigenfunction:

$$-iv_{1,x} = U(u, v_1)v_1,\ -iv_{1,t} = V(u, v_1)v_1, \quad (3.1)$$

and

$$i\hat{v}_{1,x} = \hat{v}_1 U(u, \hat{v}_1),\ i\hat{v}_{1,t} = \hat{v}_1 V(u, \hat{v}_1), \quad (3.2)$$

where ν_1 and $\hat{\nu}_1$ are a pair of arbitrary complex eigenvalue and adjoint eigenvalue. Such an idea of adopting both kinds of eigenfunctions was also used in conducting symmetry constraints [24].

A binary Darboux transformation using a single pair of eigenfunctions can be stated as follows.

Theorem 3.1. *Assume that*

$$\begin{cases} S^+[1] = S^+[1](\nu) = I_{n+1} - \dfrac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\dfrac{v_1\hat{v}_1}{\hat{v}_1 v_1}, \\ S^-[1] = S^-[1](\nu) = I_{n+1} + \dfrac{\nu_1-\hat{\nu}_1}{\nu-\nu_1}\dfrac{v_1\hat{v}_1}{\hat{v}_1 v_1}, \end{cases} \tag{3.3}$$

and

$$S_1^+[1] = \lim_{\nu\to\infty}[\nu(S^+[1]-I_{n+1})],\ \ S_1^-[1] = \lim_{\nu\to\infty}[\nu(S^-[1]-I_{n+1})]. \tag{3.4}$$

Then we have

$$S^+[1](\nu_1)v_1 = 0,\ \ \hat{v}_1 S^-[1](\hat{\nu}_1) = 0, \tag{3.5}$$

$$S^+[1](\nu)S^-[1](\nu) = I_{n+1},\quad S_1^+[1] = -S_1^-[1]; \tag{3.6}$$

and moreover,

$$\eta' = S^+[1]\eta,\ \ \tilde{\eta}' = \tilde{\eta}S^-[1],\ \ P' = P + S_1^+[1]\Lambda + \Lambda S_1^-[1] = P + [S_1^+[1],\Lambda], \tag{3.7}$$

constitute a binary Darboux transformation for the vector classical local integrable NLS equations (2.6).

Proof. Evidently, we have

$$S_1^+[1] = -(\nu_1-\hat{\nu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1},\ \ S_1^-[1] = (\nu_1-\hat{\nu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}. \tag{3.8}$$

The properties in (3.5) and (3.6) just need some straightforward computations.

What we want to verify now are the following two equations:

$$-i(S^+[1])_x = U'S^+[1] - S^+[1]U, \tag{3.9}$$

where $U' = \nu\Lambda + P'$ with P' given by (3.7), and

$$-i(S^+[1])_t = V'S^+[1] - S^+[1]V, \tag{3.10}$$

where $V' = \nu\Theta + Q'$ with $Q' = Q|_{P=P'}$. Those two equations ensure that (3.7) provides us with a binary Darboux transformation.

The case of $\nu_1 = \hat{\nu}_1$ is evident. Therefore, we assume that $\nu_1 \neq \hat{\nu}_1$ in our computation below. We first consider the x-part of the above equations. Observe that

$$\begin{aligned} &(\frac{\hat{v}_1 v_1}{\nu_1-\hat{\nu}_1})_x = \frac{1}{\nu_1-\hat{\nu}_1}(\hat{v}_{1,x}v_1 + \hat{v}_1 v_{1,x}) \\ &= \frac{1}{\nu_1-\hat{\nu}_1}[-i\hat{v}_1 U(\hat{\nu}_1)v_1 + i\hat{v}_1 U(\nu_1)v_1] \\ &= i\hat{v}_1\frac{U(\nu_1)-U(\hat{\nu}_1)}{\nu_1-\hat{\nu}_1}v_1 = i\hat{v}_1\Lambda v_1. \end{aligned} \tag{3.11}$$

On one hand, we compute that

$$\begin{aligned} &-i(\nu-\hat{\nu}_1)(S^+[1])_x = i(v_1\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}\hat{v}_1)_x \\ &= iv_{1,x}\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}\hat{v}_1 + iv_1(\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1})_x\hat{v}_1 + iv_1\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}\hat{v}_{1,x} \\ &= -U(\nu_1)v_1\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}\hat{v}_1 + v_1\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}(\hat{v}_1\Lambda v_1)\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}\hat{v}_1 + v_1\frac{\nu_1-\hat{\nu}_1}{\hat{v}_1 v_1}\hat{v}_1 U(\hat{\nu}_1) \\ &:= T_1 + T_2 + T_3, \end{aligned} \tag{3.12}$$

where we have used the result in (3.11) and the formula for the derivative of an inverse matrix:

$$(M^{-1})_x = -M^{-1}M_x M^{-1}. \tag{3.13}$$

On the other hand, we have

$$U'S^+[1] - S^+[1]U = [U, S^+[1]] + [S_1^+[1],\Lambda]S^+[1].$$

We can further compute the two terms in the above sum as follows:

$$\begin{aligned} &[U, S^+[1]] = [U, -\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}] \\ &= -(U(\nu_1)+(\nu-\nu_1)\Lambda)\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} + \frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}(U(\hat{\nu}_1)+(\nu-\hat{\nu}_1)\Lambda) \\ &= -U(\nu_1)\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} + \frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}U(\hat{\nu}_1) \\ &\quad -(\nu-\nu_1)\Lambda\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} + \frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}(\nu-\hat{\nu}_1)\Lambda \\ &= \frac{T_1}{\nu-\hat{\nu}_1} + \frac{T_3}{\nu-\hat{\nu}_1} - (\nu-\nu_1)\Lambda\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} - S_1^+[1]\Lambda, \end{aligned}$$

and

$$\begin{aligned} &[S_1^+[1],\Lambda]S^+[1] \\ &= -[S_1^+[1],\Lambda]\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} + [S_1^+[1],\Lambda] \\ &= (\nu_1-\hat{\nu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}\Lambda\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} - \Lambda(\nu_1-\hat{\nu}_1)\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} \\ &\quad + S_1^+[1]\Lambda + \Lambda(\nu_1-\hat{\nu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} \\ &= \frac{T_2}{\nu-\hat{\nu}_1} + S_1^+[1]\Lambda - \Lambda(\nu_1-\hat{\nu}_1)\frac{\nu_1-\hat{\nu}_1}{\nu-\hat{\nu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1 v_1} + \Lambda(\nu_1-\hat{\nu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}. \end{aligned}$$

At this moment, we see that $(\nu-\hat{\nu}_1)(U'S^+[1]-S^+[1]U)$ is independent of the spectral parameter ν, and it is equal to $T_1+T_2+T_3$, which is $-i(\nu-\hat{\nu}_1)(S^+[1])_x$. Accordingly, Eq. (3.9) is satisfied.

Note that we can present Q by (2.5) in terms of P. An analogous but lengthy argument can verify the t-part (3.10) of the equations. The proof is, thus, complete. □

Let us below discuss a general case of using N pairs of eigenfunctions and adjoint eigenfunctions. Associated with arbitrarily given eigenvalues and adjoint eigenvalues, ν_k and $\hat{\nu}_k$, $1 \le k \le N$, denote the two sets of corresponding eigenfunctions and adjoint eigenfunctions by

$$-iv_{k,x} = U(u,\nu_k)v_k,\ \ -iv_{k,t} = V(u,\nu_k)v_k,\ \ 1 \le k \le N, \tag{3.14}$$

and

$$i\hat{v}_{k,x} = \hat{v}_k U(u,\hat{\nu}_k),\ \ i\hat{v}_{k,t} = \hat{v}_k V(u,\hat{\nu}_k),\ \ 1 \le k \le N. \tag{3.15}$$

A key step to define Darboux matrices is the following. Formulate a square matrix:

$$M = (m_{kl})_{N\times N},\ \ m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\nu_l-\hat{\nu}_k}, & \text{if } \nu_l \neq \hat{\nu}_k, \\ m_{kl}^c(x,t), & \text{if } \nu_l = \hat{\nu}_k, \end{cases} \quad \text{where } 1 \le k,l \le N. \tag{3.16}$$

The spatial and temporal derivatives of m_{kl}^c will be given later, but no condition on m_{kl}^c is needed in the following discussion until Theorem 3.2.

Now, when the matrix M is invertible, let us define

$$\begin{cases} S^+ = S^+(\nu) = I_{n+1} - \displaystyle\sum_{k,l=1}^N \frac{v_k(M^{-1})_{kl}\hat{v}_l}{\nu-\hat{\nu}_l}, \\ S^- = S^-(\nu) = I_{n+1} + \displaystyle\sum_{k,l=1}^N \frac{v_k(M^{-1})_{kl}\hat{v}_l}{\nu-\nu_k}, \end{cases} \tag{3.17}$$

and

$$S_1^\pm(\nu) = \lim_{\nu\to\infty}[\nu(S^\pm(\nu)-I_{n+1})], \tag{3.18}$$

where I_{n+1} is the $(n+1)$th-order identity matrix. We remark that if we have the conditions:

$$\{\nu_k \mid 1\le k\le N\}\cap\{\hat\nu_k \mid 1\le k\le N\}=\emptyset;\ \nu_k\ne\nu_l,\ \hat\nu_k\ne\hat\nu_l,\ 1\le k,l\le N;$$

then we arrive at the standard Darboux transformations, analyzed in various studies (see, for example, [3,25,26]). However, the nonempty intersection is the situation that produces soliton solutions to interesting nonlocal integrable equations, and illustrative examples will also be given in the next section.

We can easily determine the following results.

Lemma 3.1. *The properties hold:*

$$S^+(\nu_k)v_k=0,\ \hat v_k S^-(\hat\nu_k)=0,\ 1\le k\le N, \tag{3.19}$$

if two matrices $S^+(\nu_k)$ and $S^-(\hat\nu_k)$ make sense, respectively.

Employing the formulas of the two Darboux matrices S^+ and S^- in (3.17), we can readily explore their partial fraction decompositions, which are exhibited in the following lemma.

Lemma 3.2. *Let the Darboux matrices S^+ and S^- be defined by (3.17). Then the following partial fraction decompositions hold:*

$$S^+=I_{n+1}-\sum_{k=1}^N\frac{v_k^M\hat v_k}{\nu-\hat\nu_k},\ S^-=I_{n+1}+\sum_{k=1}^N\frac{v_k\hat v_k^M}{\nu-\nu_k} \tag{3.20}$$

where

$$\begin{cases}(v_1^M,v_2^M,\dots,v_N^M)M=(v_1,v_2,\dots,v_N),\\ M((\hat v_1^M)^T,(\hat v_2^M)^T,\dots,(\hat v_N^M)^T)^T=(\hat v_1^T,\hat v_2^T,\dots,\hat v_N^T)^T.\end{cases} \tag{3.21}$$

Further, we can arrive at the following conclusion.

Lemma 3.3. *Under the orthogonal condition*

$$v_k^T\hat v_l^T=\hat v_l v_k=0,\ \text{if}\ \nu_k=\hat\nu_l,\ \text{where}\ 1\le k,l\le N, \tag{3.22}$$

we have

$$S^+(\nu)S^-(\nu)=I_{n+1},\ S_1^+=-S_1^-, \tag{3.23}$$

the first result of which means that $S^+(\nu)$ and $S^-(\nu)$ are inverse to each other.

Proof. For the sake of good order, we express

$$v=(v_1,v_2,\dots,v_N),\ \hat v=(\hat v_1^T,\hat v_2^T,\dots,\hat v_N^T)^T, \tag{3.24}$$

and

$$F=\begin{bmatrix}\frac{1}{\nu-\nu_1} & & & 0\\ & \frac{1}{\nu-\nu_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\nu-\nu_N}\end{bmatrix},\quad \hat F=\begin{bmatrix}\frac{1}{\nu-\hat\nu_1} & & & 0\\ & \frac{1}{\nu-\hat\nu_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\nu-\hat\nu_N}\end{bmatrix}. \tag{3.25}$$

One can now prove that

$$\hat F\hat v vF=MF-\hat FM. \tag{3.26}$$

Let us take a pair of integers $1\le k,l\le N$. If $\nu_l=\hat\nu_k$, then we first have $(\hat F\hat v vF)_{kl}=0$, due to (3.22). Second, we can compute

$$(MF-\hat FM)_{kl}=m_{kl}\frac{1}{\nu-\nu_l}-\frac{1}{\nu-\hat\nu_k}m_{kl}=0,$$

based on the definition of M in (3.16). Therefore, $(\hat F\hat v vF)_{kl}=(MF-\hat FM)_{kl}$. On the other hand, when $\nu_l\ne\hat\nu_k$, we can have

$$(\hat F\hat v vF)_{kl}=\frac{\hat v_k}{\nu-\hat\nu_k}\frac{v_l}{\nu-\nu_l}=\Big(\frac{1}{\nu-\nu_l}-\frac{1}{\nu-\hat\nu_k}\Big)\frac{\hat v_k v_l}{\nu_l-\hat\nu_k}=(MF-\hat FM)_{kl}.$$

Consequently, the property in (3.26) holds.

Further, taking advantage of (3.26), we can see that

$$S^+S^-=I_{n+1}-vM^{-1}\hat F\hat v+vFM^{-1}\hat v-vM^{-1}\hat F\hat v vFM^{-1}\hat v=I_{n+1},$$

because S^+ and S^- can be restated as

$$S^+=I_{n+1}-vM^{-1}\hat F\hat v,\ S^-=I_{n+1}+vFM^{-1}\hat v. \tag{3.27}$$

The second result of (3.23) comes from an observation

$$S_1^+=-\sum_{k,l=1}^N(M^{-1})_{kl}v_k\hat v_l=-vM^{-1}\hat v,\ S_1^-=\sum_{k,l=1}^N(M^{-1})_{kl}v_k\hat v_l=vM^{-1}\hat v. \tag{3.28}$$

Therefore, the proof is finished. □

At this moment, we are ready to express the required general binary Darboux transformation for the vector classical local integrable NLS equations (2.6) as follows.

Theorem 3.2. *If the condition in (3.22) and*

$$m^c_{kl,x}=i\hat v_k\Lambda v_l,\ \text{if}\ \nu_l=\hat\nu_k,\ \text{where}\ 1\le k,l\le N, \tag{3.29}$$

and

$$m^c_{kl,t}=i\hat v_k\Xi_{[k,l]}v_l,\ \text{if}\ \nu_l=\hat\nu_k,\ \text{where}\ 1\le k,l\le N, \tag{3.30}$$

where

$$\Xi_{[k,l]}=\frac{V(u,\nu_l)-V(u,\hat\nu_k)}{\nu_l-\hat\nu_k}=(\hat\nu_k^2+\hat\nu_k\nu_l+\nu_l^2)\Theta+\frac{\gamma}{\delta}(\hat\nu_k+\nu_l)P-\frac{\gamma}{\delta^2}I_{1,n}(P^2+iP_x), \tag{3.31}$$

are all satisfied, then the vector classical local integrable NLS equations (2.6) has the following binary Darboux transformation:

$$\eta'=S^+\eta,\ \tilde\eta'=\tilde\eta S^-,\ P'=P+S_1^+\Lambda+\Lambda S_1^-=P+[S_1^+,\Lambda]. \tag{3.32}$$

Proof. Our proof is to verify the conditions in (3.9) and (3.10), which imply the binary Darboux transformation (3.32).

In the same manner as in Theorem 3.1, let us below verify the x-part (3.9) of the equations. First note that we have

$$v_x=i(\Lambda vD+Pv),\ \hat v_x=-i(\hat D\hat v\Lambda+\hat vP), \tag{3.33}$$

where D and $\hat D$ are given by

$$D=\mathrm{diag}(\nu_1,\nu_2,\dots,\nu_N),\ \hat D=\mathrm{diag}(\hat\nu_1,\hat\nu_2,\dots,\hat\nu_N). \tag{3.34}$$

Moreover, like (3.11), we can see

$$M_x=i\hat v\Lambda v, \tag{3.35}$$

by virtue of (3.30).

Now taking the matrix expressions for S^+ and S_1^+ in (3.27) and (3.28) into consideration, and applying the derivative formulas in (3.13), (3.33) and (3.35), we can compute that

$$\begin{aligned}-iS_x^+&=(-\Lambda vDM^{-1}\hat F\hat v-PvM^{-1}\hat F\hat v)+vM^{-1}\hat v\Lambda vM^{-1}\hat F\hat v\\&\quad+(vM^{-1}\hat F\hat D\hat v\Lambda+vM^{-1}\hat F\hat vP),\\ U'S^+&=(\nu\Lambda+P-vM^{-1}\hat v\Lambda+\Lambda vM^{-1}\hat v)-(\nu\Lambda vM^{-1}\hat F\hat v+PvM^{-1}\hat F\hat v\\&\quad-vM^{-1}\hat v\Lambda vM^{-1}\hat F\hat v+\Lambda vM^{-1}\hat v vM^{-1}\hat F\hat v),\\ S^+U&=(\nu\Lambda+P)-(\nu vM^{-1}\hat F\hat v\Lambda+vM^{-1}\hat F\hat vP).\end{aligned}$$

Upon some further simplification, we can make the following computation:

$$\begin{aligned}
&-iS^+_x - U'S^+ + S^+U\\
&= -\Lambda vAM^{-1}\hat{F}\hat{v} + vM^{-1}\hat{F}\hat{D}\hat{v}\Lambda - vvM^{-1}\hat{F}\hat{v}\Lambda + vM^{-1}\hat{v}\Lambda\\
&\quad -\Lambda vM^{-1}\hat{v} + v\Lambda vM^{-1}\hat{F}\hat{v} + \Lambda vM^{-1}\hat{v}vM^{-1}\hat{F}\hat{v}\\
&= (vM^{-1}\hat{F}\hat{D}\hat{v}\Lambda - vvM^{-1}\hat{F}\hat{v}\Lambda + vM^{-1}\hat{v}\Lambda)\\
&\quad +(v\Lambda vM^{-1}\hat{F}\hat{v} - \Lambda vDM^{-1}\hat{F}\hat{v}) - (\Lambda vM^{-1}\hat{v} - \Lambda vM^{-1}\hat{v}vM^{-1}\hat{F}\hat{v})\\
&= (-vM^{-1}\hat{F}\hat{F}^{-1}\hat{v}\Lambda + vM^{-1}\hat{v}\Lambda) + \Lambda vF^{-1}M^{-1}\hat{F}\hat{v}\\
&\quad -(\Lambda vM^{-1}\hat{v} - \Lambda vM^{-1}\hat{v}vM^{-1}\hat{F}\hat{v})\\
&= \Lambda vF^{-1}M^{-1}\hat{F}\hat{v} - (\Lambda vM^{-1}\hat{v} - \Lambda vM^{-1}\hat{v}vM^{-1}\hat{F}\hat{v}) = 0,
\end{aligned}$$

the last step of which comes from (3.26). Therefore, the x-part (3.9) of the equations is satisfied.

Another similar argument can prove that the t-part (3.10) of the equations is satisfied, too. The proof is, thus, complete. □

3.2. N-Fold decomposition

Furthermore, by some detailed computation, we would like to prove that the above general binary Darboux transformation can be factored into a product of N single binary Darboux transformations.

We first define two sequences of basic matrices $S^+\{k\}$ and $S^-\{k\}$, $1 \le k \le N$, recursively as follows:

$$\begin{cases} S^+\{k\} = S^+\{k\}(\nu) = I_{n+1} - \dfrac{\nu_k - \hat{\nu}_k}{\nu - \hat{\nu}_k}\dfrac{v'_k\hat{v}'_k}{\hat{v}'_k v'_k},\ 1 \le k \le N,\\[2ex] S^-\{k\} = S^-\{k\}(\nu) = I_{n+1} + \dfrac{\nu_k - \hat{\nu}_k}{\nu - \nu_k}\dfrac{v'_k\hat{v}'_k}{\hat{v}'_k v'_k},\ 1 \le k \le N,\end{cases} \tag{3.36}$$

with

$$v'_k = S^+[\![k-1]\!](\nu_k)v_k,\ \hat{v}'_k = \hat{v}_k S^-[\![k-1]\!](\hat{\nu}_k),\ 1 \le k \le N, \tag{3.37}$$

where

$$\begin{cases} S^+[\![0]\!] = S^-[\![0]\!] = I_{n+1},\\ S^+[\![k]\!] = S^+\{k\}\cdots S^+\{2\}S^+\{1\},\ 1 \le k \le N-1,\\ S^-[\![k]\!] = S^-\{1\}S^-\{2\}\cdots S^-\{k\},\ 1 \le k \le N-1.\end{cases} \tag{3.38}$$

We point out that $S^+\{1\}$ and $S^-\{1\}$ above are the same as $S^+[1]$ and $S^-[1]$ determined by (3.3).

Theorem 3.3. *Assume that $\{\nu_k \,|\, 1 \le k \le N\} \cap \{\hat{\nu}_k \,|\, 1 \le k \le N\} = \emptyset$. Then S^+ and S^- can be factored into the N-fold products:*

$$S^+ = S^+\{N\}S^+\{N-1\}\cdots S^+\{1\},\ S^- = S^-\{1\}\cdots S^-\{N-1\}S^-\{N\}, \tag{3.39}$$

where $S^+\{k\}$ and $S^-\{k\}$, $1 \le k \le N$, are recursively defined by (3.36) .

Proof. Using the definition of $S^+[\![k]\!]$ and $S^-[\![k]\!]$ in (3.38), we can see

$$S^+\{k\}(\nu_k)v'_k = 0,\ \hat{v}'_k S^-\{k\}(\hat{\nu}_k) = 0,\ 1 \le k \le N. \tag{3.40}$$

It thus follows that

$$\begin{cases} S^+[\![N]\!](\nu_k)v_k = S^+\{N\}(\nu_k)\cdots S^+\{k+1\}(\nu_k)S^+\{k\}(\nu_k)v'_k = 0,\ 1 \le k \le N,\\ \hat{v}_k S^-[\![N]\!](\hat{\nu}_k) = \hat{v}'_k S^-\{k\}(\hat{\nu}_k)S^-\{k+1\}(\hat{\nu}_k)\cdots S^-\{N\}(\hat{\nu}_k) = 0,\ 1 \le k \le N,\end{cases}$$

where

$$S^+[\![N]\!] = S^+\{N\}\cdots S^+\{2\}S^+\{1\},\ S^-[\![N]\!] = S^-\{1\}S^-\{2\}\cdots S^-\{N\}. \tag{3.41}$$

Now, due to the same property for S^+ and S^- in Lemma 3.1, this means that

$$S^+ = S^+[\![N]\!],\ S^- = S^-[\![N]\!], \tag{3.42}$$

which are exactly the decompositions in (3.39). The proof is then complete. □

3.3. Nonlocal reduction

To satisfy the nonlocal reduction condition for U' in (2.7), we choose

$$\hat{\nu}_k = -\nu_k,\ 1 \le k \le N, \tag{3.43}$$

Then, one can directly verify that

$$(S_1^+(x,-t))^T C = CS_1^+(x,t) \tag{3.44}$$

guarantees the nonlocal reduction condition (2.7) for the new spectral matrix U'. To have the property in (3.44), let us further take

$$\hat{v}_k(x,t,\hat{\nu}_k) = v_k^T(x,-t,\nu_k)C,\ 1 \le k \le N, \tag{3.45}$$

and demand that

$$\begin{cases} v_k^T(x,-t,\nu_k)Cv_l(x,t,\nu_l) = 0,\\ m^c_{kl,x} = iv_k^T(x,-t,\nu_k)C\Lambda v_l(x,t,\nu_l), & \text{if } \nu_l = \hat{\nu}_k,\\ m^c_{kl,t} = iv_k^T(x,-t,\nu_k)C\Xi_{[k,l]}v_l(x,t,\nu_l),\end{cases} \tag{3.46}$$

where $1 \le k, l \le N$. Accordingly, the reduced binary Darboux transformation generates a binary Darboux transformation for the nonlocal integrable NLS equations in (2.11). We summarize this obtained result on the required Darboux transformation in the following theorem.

Theorem 3.4. *Let us take the set of adjoint eigenvalues, $\{\hat{\nu}_k \,|\, 1 \le k \le N\}$, as in (3.43) and the set of adjoint eigenfunctions, $\{\hat{v}_k \,|\, 1 \le k \le N\}$, by (3.45). If the three basic conditions in (3.46) are satisfied, then the reduced binary Darboux transformation (3.32) yields a binary Darboux transformation for the vector nonlocal reverse-time integrable NLS Eqs. (2.11).*

4. Soliton solutions

4.1. Classical unreduced case

Let us take two arbitrary sets of complex numbers $\{\nu_k \,|\, 1 \le k \le N\}$ and $\{\hat{\nu}_k \,|\, 1 \le k \le N\}$ as eigenvalues and adjoint eigenvalues. Starting from the zero seed potential $P = 0$, we can obtain the associated eigenfunctions and adjoint eigenfunctions

$$v_k(x,t) = e^{i\nu_k\Lambda x + i\nu_k^2\Theta t}w_k,\ 1 \le k \le N, \tag{4.1}$$

$$\hat{v}_k(x,t) = \hat{w}_k e^{-i\hat{\nu}_k\Lambda x - i\hat{\nu}_k^2\Theta t},\ 1 \le k \le N, \tag{4.2}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively. Then if we go with $m^c_{kl} = 0$, we need to impose the conditions:

$$\hat{w}_k w_l = \hat{w}_k\Lambda w_l = \hat{w}_k\Theta w_l = 0,\ \text{if } \nu_l = \hat{\nu}_k,\ \text{where } 1 \le k, l \le N. \tag{4.3}$$

At this moment, from the binary Darboux transformation in (3.32), we directly determine a novel potential matrix:

$$P' = [S_1^+, \Lambda],\ S_1^+ = -\sum_{k,l=1}^N (M^{-1})_{kl}v_k\hat{v}_l. \tag{4.4}$$

This generate a sort of N-soliton solutions to the local NLS equations in (2.6):

$$p_j = \delta\sum_{k,l=1}^N (M^{-1})_{kl}v_k^{(1)}\hat{v}_l^{(j+1)},\ q_j = -\delta\sum_{k,l=1}^N (M^{-1})_{kl}v_k^{(j+1)}\hat{v}_l^{(1)},\ 1 \le j \le n, \tag{4.5}$$

where $v_k = (v_k^{(1)}, v_k^{(2)}, \dots, v_k^{(n+1)})^T$ and $\hat{v}_k = (\hat{v}_k^{(1)}, \hat{v}_k^{(2)}, \dots, \hat{v}_k^{(n+1)})$, $1 \le k \le N$.

4.2. Nonlocal reduced case

Considering this nonlocal situation, we have to check if the involution condition

$$(S_1^+(-x,t))^\dagger C = C S_1^+(x,t) \tag{4.6}$$

is satisfied so that we can get soliton solutions to the nonlocal NLS equations (2.11). Equivalently, this essential property demands that the novel potential matrix P' determined by the resultant binary Darboux transformation needs to satisfy the reduction condition in (2.8). If the above condition (4.6) holds, then the presented soliton solution to the local NLS equations (2.6) yields a sort of soliton solutions:

$$p_j = \delta \sum_{k,l=1}^{N} (M^{-1})_{kl} v_k^{(1)} \hat{v}_l^{(j+1)},\ 1 \le j \le n, \tag{4.7}$$

for the nonlocal NLS equations (2.11).

To satisfy the key property (4.6), we introduce the required adjoint eigenvalues $\hat{\nu}_k$, $1 \le k \le N$, as in (3.43), upon choosing a set of arbitrary eigenvalues

$$\nu_k \in \mathbb{C},\ 1 \le k \le N. \tag{4.8}$$

Moreover, the associated eigenfunctions v_k, $1 \le k \le N$, can be taken as follows:

$$v_k(x,t) = v_k(x,t,\nu_k) = \mathrm{e}^{i\nu_k \Lambda x + i\nu_k^2 \Theta t} w_k,\ 1 \le k \le N, \tag{4.9}$$

respectively, where w_k, $1 \le k \le N$, are arbitrary column vectors. Furthermore, based on the previous analysis on the nonlocal reductions, the corresponding adjoint eigenfunctions $\hat{v}_k$, $1 \le k \le N$, can be chosen as

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\nu}_k) = v_k^T(x,-t,\nu_k) C = w_k^T \mathrm{e}^{-i\hat{\nu}_k \Lambda x - i\hat{\nu}_k^2 \Theta t} C,\ 1 \le k \le N, \tag{4.10}$$

respectively. In this way, when taking $P = 0$, the properties in (3.46) become the following orthogonal conditions

$$w_k^T C w_l = w_k^T C \Lambda w_l = w_k^T C \Theta w_l = 0,\ \text{if}\ \nu_l = \hat{\nu}_k,\ \text{where}\ 1 \le k,l \le N, \tag{4.11}$$

for $\{w_k | 1 \le k \le N\}$, if we select $m_{kl}^c = 0$. Obviously, those crucial requirements in (4.11) equivalently yields

$$\begin{cases} w_k^{(1)} w_l^{(1)} = 0, \\ (w_k^{(2)}, \dots, w_k^{(n+1)}) \Sigma (w_l^{(2)}, \dots, w_l^{(n+1)})^T = 0, \end{cases} \quad \text{if}\ \nu_l = \hat{\nu}_k,\ \text{where}\ 1 \le k,l \le N, \tag{4.12}$$

with $w_k^{(j)}$ denoting the jth component of w_k, where $1 \le j \le n+1$, for each $1 \le k \le N$.

To summarize, if we define an M-matrix by (3.16), which satisfies (3.46), then the formula (4.7) presents a kind of soliton solutions to the nonlocal NLS equations (2.11). When we take (4.9) and (4.10) and choose $m_{kl}^c = 0$, the conditions in (3.46) reduces to the conditions in (4.12).

5. Conclusion

We aim to propose a binary Darboux transformation for a class of vector nonlocal reverse-time integrable NLS (nonlinear Schrödinger) equations. The crucial idea is to take advantage of two sets of eigenfunctions and adjoint eigenfunctions. A general skeleton of binary Darboux transformations and N-fold decompositions was explored. Taking a reduction of the M-matrix generates the orthogonal conditions on eigenfunctions and adjoint eigenfunctions. The resultant binary Darboux transformation was applied to soliton solutions.

The primary result in our analysis is a general framework of Darboux matrices for both local and nonlocal cases. The contributions consist of a generalized M-matrix and the realization of nonlocal reductions. While eigenvalues equal to adjoint eigenvalues, the M-matrix takes zero entries in the existing studies (see, e.g., [14–16]), but the M-matrix takes non-zero entries in our framework above, which satisfy the most general conditions. Our particular case with repeated eigenvalues or repeated adjoint eigenvalues yields generalized Darboux transformations. The study was inspired by recent works on nonlocal reduced integrable equations, including nonlocal integrable NLS and mKdV equations (see, for example, [4,27–31]).

Applications of binary Darboux transformations to other kinds of exact solutions, e.g., lump solutions [32] and breather waves [33–35], could significantly improve our understanding of nonlinear waves. It is also expected that one could extend the resultant theory of binary Darboux transformations to nonlocal integrable models with self-consistent sources (see, e.g., [9,36]), integrable lattice equations (see, e.g., [37–39] for Darboux transformations), and other multi-component integrable equations (see, e.g., [40–42]).

CRediT authorship contribution statement

Wen-Xiu Ma: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The author declares that there is no financial/personal interest or belief that could affect the research objectivity.

Data availability

No data was used for the research described in the article.

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