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Binary Darboux transformation for general matrix mKdV equations and reduced counterparts

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ABSTRACT

Based on Lax pairs and adjoint Lax pairs, a Darboux transformation is constructed for a class of general matrix mKdV equations and a kind of symmetric integrable reductions of the general case is analyzed. A fundamental step is to formulate a new type of Darboux matrices, in which eigenvalues could be equal to adjoint eigenvalues. From the zero seed solution, the resulting binary Darboux transformation is used to generate soliton solutions for the general matrix mKdV equations and the corresponding reduced counterparts.

1. Introduction

There are quite a few powerful analytical methods to construct exact solutions to nonlinear integrable equations in soliton theory [1–3]. Particularly, the Darboux transformation (DT) is an efficient approach to soliton solutions [4,5]. A kind of matrix spectral problems plays an essential role in producing DTs (see, e.g., [6]), which has a close connection with Riemann-Hilbert problems and the inverse scattering transform in soltion theory [1–3]. A binary DT is generated from a pair of matrix spectral problems being equivalent to a given equation, called a Lax pair, and another pair of adjoint matrix spectral problems being equivalent to the given equation as well, called an adjoint Lax pair. We would like to construct a binary DT for a class of general matrix mKdV equations and their reduced integrable counterparts.

Let x and t be two independent variables, and $u = u(x,t)$, a dependent variable or a vector of dependent variables. Assume that a Lax pair of spatial and temporal matrix spectral problems:

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \ \ -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{1.1}$$

where i stands for the unit imaginary number, λ is a spectral parameter and ϕ is a column eigenfunction, generates an integrable equation, through their compatibility condition, i.e., the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.2}$$

where $[\cdot,\cdot]$ denotes the matrix commutator. Associated with such integrable equations, there are nice algebraic structures behind their Lax pairs [7]. On the other hand, the adjoint matrix spectral problems of (1.1) are defined as follows:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u,\lambda), \ \ i\tilde{\phi}_t = \tilde{\phi}V = \tilde{\phi}V(u,\lambda). \tag{1.3}$$

Their compatibility condition presents the same zero curvature equation, and thus, it doesn't generate any new equations. For matrix spectral problems, we can often make appropriate reductions to obtain reduced integrable equations from the corresponding reduced zero curvature equations (see, e.g., [8]).

A binary DT of an integrable equation is given by

$$\phi' = T^+\phi = T^+(u,\lambda)\phi, \ \ \tilde{\phi}' = \tilde{\phi}T^- = \tilde{\phi}T^-(u,\lambda), \ \ u' = f(u), \tag{1.4}$$

where $(T^+)^{-1} = T^-$, provided that ϕ' and $\tilde{\phi}'$ satisfy the new matrix spectral problems:

$$-i\phi'_x = U'\phi', \ \ -i\phi'_t = V'\phi', \tag{1.5}$$

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and the new adjoint matrix spectral problems:

$$i\tilde{\phi}'_x = \tilde{\phi}' U', \;\; i\tilde{\phi}'_t = \tilde{\phi}' V', \tag{1.6}$$

where the new spectral matrices are given by

$$U' = U(u', \lambda) = U(f(u), \lambda), \;\; V' = V(u', \lambda) = V(f(u), \lambda). \tag{1.7}$$

The above condition for producing a binary DT is to just require that the Darboux matrices T^+ and T^- satisfy

$$U' = -iT^+_x T^- + T^+ U T^-, \;\; V' = -iT^+_t T^- + T^+ V T^-. \tag{1.8}$$

It is clear that either (1.5) or (1.6) ensures that the new spectral matrices, U' and V', generate the same zero curvature Eq. (1.2) with u replaced with u', and so u' gives us a new solution to the corresponding integrable equation. There exist various examples of binary DTs for scalar or coupled integrable equations (see, e.g., [4,9–12]), but very few examples for matrix integrable equations in the relevant literature (see, e.g., [13,14]).

In this paper, we would like to construct a binary DT for general matrix mKdV equations and reduced integrable counterparts, based on a Lax pair of arbitrary-order matrix spectral problems. An N-fold Darboux characteristics will be exhibited for the resulting binary DT in the standard case where eigenvalues are different from adjoint eigenvalues. Upon taking the zero seed solution, the resulting binary DT produces soliton solutions for the matrix mKdV equations and their reduced integrable counterparts. A few concluding remarks will be given in the last section.

2. Matrix mKdV equations

2.1. General equations

Let $m, n \geq 0$ be two arbitrarily given integers, and by I_s, we denote the identity matrix of size s ($s \in \mathbb{N}$). We consider a Lax pair of matrix spectral problems:

$$-i\phi_x = U\phi = U(p, q; \lambda)\phi, \;\; -i\phi_t = V\phi = V(p, q; \lambda)\phi, \tag{2.1}$$

where the two matrices of dependent variables are given by

$$p = (p_{jl})_{m\times n}, \;\; q = (q_{lj})_{n\times m}, \tag{2.2}$$

and the pair of spectral matrices, by

$$U = \lambda\Lambda + P, \;\; V = \lambda^3\Omega + Q. \tag{2.3}$$

The involved square matrices, Λ, Ω, P and Q, are defined by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n), \;\; \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.4}$$

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.5}$$

$$\begin{aligned} Q = Q(u,\lambda) &= \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) \\ &\quad - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3) \\ &= \frac{\beta}{\alpha}\lambda^2 \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\lambda \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} \\ &\quad - \frac{\beta}{\alpha^3}\begin{bmatrix} i(pq_x - p_x q) & p_{xx} + 2pqp \\ q_{xx} + 2qpq & i(qp_x - q_x p) \end{bmatrix}, \end{aligned} \tag{2.6}$$

where α_1, α_2 and β_1, β_2 are two pairs of different constants, $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$.

With one non-zero pair (p_{jl}, q_{lj}) $(1 \leq j, l \leq n)$, the spatial spectral problem in (2.1) reduces to the standard AKNS spectral problem [15]. Because of the existence of a multiple eigenvalue of Λ, the spatial matrix spectral problem in (2.1) with matrix potentials, p and q, is degenerate. However, this will not affect our analysis seriously.

The compatibility condition of the matrix spectral problems in (2.1) yields the following matrix mKdV equations:

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp), \\ q_t = -\dfrac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x). \end{cases} \tag{2.7}$$

When $m = 1$ and $n = 1$, we can obtain

$$p_{11,t} = p_{11,xxx} + 6p_{11}q_{11}p_{11,x}, \;\; q_{11,t} = q_{11,xxx} + 6p_{11}q_{11}q_{11,x}. \tag{2.8}$$

When $m = 1$ and $n = 2$, we can have

$$\begin{cases} p_{1l,t} = p_{1l,xxx} + 3(p_{11}q_{11} + p_{12}q_{21})p_{1l,x} + 3(p_{11,x}q_{11} + p_{12,x}q_{21})p_{1l}, \\ q_{l1,t} = q_{l1,xxx} + 3(p_{11}q_{11} + p_{12}q_{21})q_{l1,x} + 3(p_{11}q_{11,x} + p_{12}q_{21,x})q_{l1}, \end{cases} \tag{2.9}$$

where $1 \leq l \leq 2$. When $m = 2$ and $n = 2$, we can get

$$\begin{cases} p_{jl,t} = p_{jl,xxx} + 3\displaystyle\sum_{r,s=1}^{2} p_{jr}q_{rs}p_{sl,x} + 3\sum_{r,s=1}^{2} p_{jr,x}q_{rs}p_{sl}, \\ q_{lj,t} = q_{lj,xxx} + 3\displaystyle\sum_{r,s=1}^{2} q_{lr,x}p_{rs}q_{sj} + 3\sum_{r,s=1}^{2} q_{lr}p_{rs}q_{sj,x}, \end{cases} \tag{2.10}$$

where $1 \leq j, l \leq 2$.

2.2. Reduced counterparts

Let us now make integrable reductions (see also [8] for the basic idea). We take two constant invertible Hermitian matrices Σ_1, Σ_2 and introduce a particular reduction for the spectral matrix U:

$$U^\dagger(x,t,\lambda^*) = (U(x,t,\lambda^*))^\dagger = CU(x,t,\lambda)C^{-1}, \quad \text{where } C = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \;\; \Sigma_j^\dagger = \Sigma_j, \;\; j = 1, 2. \tag{2.11}$$

Henceforth, $\dagger$ denotes the Hermitian transpose and $*$, the complex conjugate. This reduction exactly requires

$$P^\dagger(x,t) = CP(x,t)C^{-1}. \tag{2.12}$$

The reduction in (2.12) engenders the reduction for the potential matrices:

$$q(x,t) = \Sigma_2^{-1}p^\dagger(x,t)\Sigma_1. \tag{2.13}$$

Such a reduction guarantees that

$$V^\dagger(x,t,\lambda^*) = CV(x,t,\lambda)C^{-1}, \;\; Q^\dagger(x,t,\lambda^*) = CQ(x,t,\lambda)C^{-1}, \tag{2.14}$$

where V and Q are determined by (2.3), (2.4) and (2.6).

We can then directly see that the reduction (2.12) (or (2.13)) agrees with the zero curvature equation of the reduced spatial and temporal matrix spectral problems of (2.1). Thus, under the reduction (2.12), the matrix mKdV Eq. (2.7) becomes the following reduced matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Sigma_2^{-1}p^\dagger\Sigma_1 p_x + 3p_x\Sigma_2^{-1}p^\dagger\Sigma_1 p), \tag{2.15}$$

where $p = (p_{jl})_{m\times n}$, and Σ_1, Σ_2 are two arbitrary invertible Hermitian matrices of sizes m and n, respectively.

When $n = 1$, taking $\alpha = -\beta = 1$ and $\Sigma_1 = 1, \Sigma_2 = \frac{1}{\sigma}$, we get the two scalar mKdV equations:

$$p_{11,t} = p_{11,xxx} + 6\sigma|p_{11}|^2 p_{11,x}, \;\; \sigma = \pm 1. \tag{2.16}$$

When $m=1$ and $n=2$, we can have a new system of integrable two-component mKdV equations:

$$\begin{cases} p_{11,t} = p_{11,xxx} + \left(c_1|p_{11}|^2 + c_2|p_{12}|^2\right)p_{11,x} \\ \qquad + \left(c_1 p_{11,x}p_{11}^* + c_2 p_{12,x}p_{12}*\right)p_{11}, \\ p_{12,t} = p_{12,xxx} + \left(c_1|p_{11}|^2 + c_2|p_{12}|^2\right)p_{12,x} \\ \qquad + \left(c_1 p_{11,x}p_{11}^* + c_2 p_{12,x}p_{12}^*\right)p_{12}, \end{cases} \tag{2.17}$$

where c_1 and c_2 are arbitrary nonzero real constants. One system of such mixed type mKdV equations has been solved by the inverse scattering transform [16]. When $m=2$ and $n=2$, we can get a more general system of mKdV equations

$$p_{jl,t} = p_{jl,xxx} + \sum_{r,s=1}^{2} c_r d_s p_{jr} p_{sr}^* p_{sl,x} + \sum_{r,s=1}^{2} c_r d_s p_{jr,x} p_{sr}^* p_{sl}, \tag{2.18}$$

where $1 \le j,l \le 2$, and $c_j, d_j,\ 1 \le j \le 2$, are arbitrary nonzero real constants.

3. Binary Darboux transformation

3.1. New type of Darboux matrices

Let us now formulate Darboux matrices in a general case, where eigenvalues could be equal to adjoint eigenvalues.

Let $N \ge 1$ be another arbitrarily given integer. We take two sets of eigenfunctions and adjoint eigenfunctions:

$$-i v_{k,x} = U(p,q;\lambda_k)v_k,\ \ -i v_{k,t} = V(p,q;\lambda_k)v_k,\ \ 1 \le k \le N, \tag{3.1}$$

and

$$i\hat{v}_{k,x} = \hat{v}_k U(p,q;\hat{\lambda}_k),\ \ i\hat{v}_{k,t} = \hat{v}_k V(p,q;\hat{\lambda}_k),\ \ 1 \le k \le N, \tag{3.2}$$

where λ_k and $\hat{\lambda}_k,\ 1 \le k \le N$, are arbitrary eigenvalues and adjoint eigenvalues, respectively, but some eigenvalues can be equal to some adjoint eigenvalues.

To make compact expressions, we define

$$v = (v_1, \cdots, v_N),\ \ \hat{v} = (\hat{v}_1^T, \cdots, \hat{v}_N^T)^T. \tag{3.3}$$

We can then state the equations for the eigenfunctions as follows:

$$\begin{cases} -i v_x = \Lambda v A + P v, \\ i\hat{v}_x = \hat{A}\hat{v}\Lambda + \hat{v}P, \end{cases} \tag{3.4}$$

and

$$\begin{cases} -i v_t = \Omega v A^3 + (Q(\lambda_1)v_1, \cdots, Q(\lambda_N)v_N), \\ i\hat{v}_t = \hat{A}^3\hat{v}\Omega + (\hat{v}_1 Q(\hat{\lambda}_1), \cdots, \hat{v}_N Q(\hat{\lambda}_N)), \end{cases} \tag{3.5}$$

where the four matrices Λ, Ω, P and Q are defined by (2.4), (2.5) and (2.6), and A and $\hat{A}$, by

$$\begin{cases} A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N), \\ \hat{A} = \mathrm{diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_N). \end{cases} \tag{3.6}$$

We consider the general situation, where eigenvalues could equal adjoint eigenvalues. To the end, we introduce a square M-matrix:

$$M = (m_{kl})_{N\times N},\ \ m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \ne \hat{\lambda}_k, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad \text{where } 1 \le k,l \le N. \tag{3.7}$$

This M-matrix involves zero entries, when $\lambda_l = \hat{\lambda}_k$ for a pair $1 \le k,l \le N$. Therefore, it generalizes the traditional case without zero entries (see, e.g., [3,17]) and can also yield soliton solutions to nonlocal integrable equations (see, e.g., [18]).

When M is invertible, we can formulate two Darboux matrices of new type:

$$\begin{cases} T^+ = T^+(\lambda) = I_{n+1} - \displaystyle\sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl}\hat{v}_l}{\lambda - \hat{\lambda}_l}, \\ T^- = T^-(\lambda) = I_{n+1} + \displaystyle\sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl}\hat{v}_l}{\lambda - \lambda_k}. \end{cases} \tag{3.8}$$

The two Darboux matrices can be expresssed in a compact form of partial fractional decomposition:

$$\begin{cases} T^+ = I_{n+1} - \displaystyle\sum_{l=1}^{N} \frac{v_l^M \hat{v}_l}{\lambda - \hat{\lambda}_l}, \\ T^- = I_{n+1} + \displaystyle\sum_{k=1}^{N} \frac{v_k \hat{v}_k^M}{\lambda - \lambda_k}, \end{cases} \tag{3.9}$$

where we define

$$\begin{cases} (v_1^M, \cdots, v_N^M) = (v_1, \cdots, v_N)M^{-1}, \\ ((\hat{v}_1^M)^T, \cdots, (\hat{v}_N^M)^T)^T = M^{-1}(\hat{v}_1^T, \cdots, \hat{v}_N^T)^T. \end{cases} \tag{3.10}$$

Further, we can more compactly rewrite the Darboux matrices as follows:

$$\begin{cases} T^+ = I_{n+1} - vM^{-1}\hat{R}\hat{v}, \\ T^- = I_{n+1} + vRM^{-1}\hat{v}, \end{cases} \tag{3.11}$$

upon introducing

$$\begin{cases} R = \mathrm{diag}\left(\dfrac{1}{\lambda - \lambda_1}, \cdots, \dfrac{1}{\lambda - \lambda_N}\right), \\ \hat{R} = \mathrm{diag}\left(\dfrac{1}{\lambda - \hat{\lambda}_1}, \cdots, \dfrac{1}{\lambda - \hat{\lambda}_N}\right). \end{cases} \tag{3.12}$$

Let us also define

$$T_1^\pm = \lim_{\lambda\to\infty} [\lambda(T^\pm(\lambda) - I_{n+1})]. \tag{3.13}$$

It is clear that

$$T_1^+ = -vM^{-1}\hat{v},\ \ T_1^- = vM^{-1}\hat{v}, \tag{3.14}$$

which also implies that

$$T_1^+ = -T_1^-.$$

Furthermore, it is direct to prove the following basic properties for the two Darboux matrices T^+ and T^-.

Theorem 3.1. *(i) A spectral property holds:*

$$\left(\prod_{l=1}^{N}(\lambda - \hat{\lambda}_l)T^+\right)(\lambda_k)v_k = 0,\ 1 \le k \le N,$$
$$\hat{v}_k\left(\prod_{l=1}^{N}(\lambda - \lambda_l)T^-\right)(\hat{\lambda}_k) = 0,\ 1 \le k \le N. \tag{3.15}$$

(ii) If an orthogonal condition holds:

$$\hat{v}_k v_l = 0 \text{ when } \lambda_l = \hat{\lambda}_k, \tag{3.16}$$

where $1 \le k,l \le N$, then we have

$$\hat{R}\hat{v}vR = MR - \hat{R}M, \tag{3.17}$$

which implies that T^+ and T^- are inverse to each other:

$$T^+(\lambda)T^-(\lambda) = I_{n+1}. \tag{3.18}$$

3.2. Binary DT for the general equations

In order to present a binary DT, we need to compute the derivatives of the M-matrix with respect to the independent variables x and t. A direct computation shows that

$$\hat{v}_k \Lambda v_l = 0 \text{ when } \lambda_l = \hat{\lambda}_k, \tag{3.19}$$

where $1 \le k, l \le N$, guarantees that

$$M_x = i\hat{v}\Lambda v; \tag{3.20}$$

and

$$\hat{v}_k \Omega_{[k,l]} v_l = 0 \text{ when } \lambda_l = \hat{\lambda}_k, \tag{3.21}$$

where $1 \le k, l \le N$, and

$$\Omega_{[k,l]} = \left(\hat{\lambda}_k^2 + \hat{\lambda}_k\lambda_l + \lambda_l^2\right)\Omega + \frac{\beta}{\alpha}(\hat{\lambda}_k + \lambda_l)P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + iP_x), \quad 1 \le k, l \le N, \tag{3.22}$$

guarantees that

$$M_t = i\left[\hat{v}\hat{A}^2\Omega v + \hat{v}\hat{A}\Omega A v + \hat{v}\Omega A^2 v + \frac{\beta}{\alpha}(\hat{v}\hat{A}Pv + \hat{v}PAv) - \frac{\beta}{\alpha^2}\hat{v}I_{m,n}(P^2 + iP_x)v\right]. \tag{3.23}$$

Based on those two properties, a general binary DT can be presented as follows.

Theorem 3.2. *Let $\Omega_{[k,l]}$ be defined by (3.22). If the conditions hold:*

$$\hat{v}_k v_l = \hat{v}_k \Lambda v_l = \hat{v}_k \Omega_{[k,l]} v_l = 0 \text{ when } \lambda_l = \hat{\lambda}_k, \tag{3.24}$$

where $1 \le k, l \le N$, then we have a binary DT:

$$\phi' = T^+\phi, \ \tilde{\phi}' = \tilde{\phi}T^-, \ P' = P + [T_1^+, \Lambda], \tag{3.25}$$

for the matrix mKdV Eq. (2.7).

Moreover, if $\{\lambda_k | 1 \le k \le N\} \cap \{\hat{\lambda}_k | 1 \le k \le N\} = \emptyset$, which is the standard case, we can decompose the above general binary DT into an N-fold binary DT.

To this end, let us define two new sets of single binary Darboux matrices $T^+[\![k]\!]$ and $T^-[\![k]\!]$, $1 \le k \le N$, recursively as follows:

$$\begin{cases} T^+[\![k]\!] = T^+[\![k]\!](\lambda) = I_{n+1} - \dfrac{\lambda_k - \hat{\lambda}_k}{\lambda - \hat{\lambda}_k}\dfrac{v_k'\hat{v}_k'}{\hat{v}_k'v_k'}, & 1 \le k \le N, \\ T^-[\![k]\!] = T^-[\![k]\!](\lambda) = I_{n+1} + \dfrac{\lambda_k - \hat{\lambda}_k}{\lambda - \lambda_k}\dfrac{v_k'\hat{v}_k'}{\hat{v}_k'v_k'}, & 1 \le k \le N, \end{cases} \tag{3.26}$$

with new pairs of eigenfunctions and adjoint eigenfunctions:

$$v_k' = T^+\{k-1\}(\lambda_k)v_k, \ \hat{v}_k' = \hat{v}_k T^-\{k-1\}(\hat{\lambda}_k), \ 1 \le k \le N, \tag{3.27}$$

where

$$\begin{cases} T^+\{0\} = T^-\{0\} = I_{n+1}, \\ T^+\{k\} = T^+[\![k]\!]\cdots T^+[\![2]\!]T^+[\![1]\!], \ 1 \le k \le N, \\ T^-\{k\} = T^-[\![1]\!]T^-[\![2]\!]\cdots T^-[\![k]\!], \ 1 \le k \le N. \end{cases} \tag{3.28}$$

Now we can have the following N-fold decompositions for the two Darboux matrices T^+ and T^-.

Theorem 3.3. *Let $\{\lambda_k | 1 \le k \le N\} \cap \{\hat{\lambda}_k | 1 \le k \le N\} = \emptyset$. Assume that the two Darboux matrices T^+ and T^- are defined by (3.8). Then we have the N-fold decomposition:*

$$\begin{aligned} T^+ &= T^+[\![N]\!]T^+[\![N-1]\!]\cdots T^+[\![1]\!], \\ T^- &= T^-[\![1]\!]\cdots T^-[\![N-1]\!]T^-[\![N]\!], \end{aligned} \tag{3.29}$$

where $T^+[\![k]\!]$ and $T^-[\![k]\!]$, $1 \le k \le N$, are determined by (3.26).

3.3. Binary DT for the reduced equations

Let us check how to satisfy the reduction property (2.11) for the new spectral matrix U' defined by (1.8) with (3.8). First, the crucial step is to take the adjoint eigenvalues:

$$\hat{\lambda}_k = \lambda_k^*, \ 1 \le k \le N. \tag{3.30}$$

Then, we can find that it will be sufficient for T_1^+ to satisfy an involution property:

$$(T_1^+(x,t))^\dagger = -CT_1^+(x,t)C^{-1}, \tag{3.31}$$

where C is defined as in (2.11). To satisfy this condition, we only need to take the adjoint eigenfunctions:

$$\hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(x,t,\lambda_k)C, \ 1 \le k \le N, \tag{3.32}$$

under which the three conditions in (3.24) become

$$v_k^\dagger C v_l = v_k^\dagger C\Lambda v_l = v_k^\dagger C\Omega_{[k,l]} v_l = 0 \text{ when } \lambda_l = \hat{\lambda}_k, \tag{3.33}$$

where $1 \le k, l \le N$.

Now, the general binary DT (3.25) presents a binary DT for the reduced matrix mKdV Eq. (2.15). We summarize such a binary DT for the reduced case as follows.

Theorem 3.4. *Let the adjoint eigenvalues $\{\hat{\lambda}_k | 1 \le k \le N\}$ be taken as in (3.30) and the associated adjoint eigenfunctions $\{\hat{v}_k | 1 \le k \le N\}$ be determined by (3.32). Then if the three orthogonal properties for $\{v_k | 1 \le k \le N\}$ in (3.33) hold, the binary Darboux transformation (3.25) is then reduced to a binary Darboux transformation for the reduced matrix mKdV Eq. (2.15).*

4. Applications to soliton solutions

4.1. Soliton solutions to the general equations

Let us first consider the general case. We take two arbitrary sets of eigenvalues and adjoint eigenvalues:

$$\{\lambda_k \in \mathbb{C} | 1 \le k \le N\}, \ \{\hat{\lambda}_k \in \mathbb{C} | 1 \le k \le N\}.$$

Beginning with the zero seed solution $P = 0$, we can obtain the corresponding eigenfunctions and adjoint eigenfunctions

$$v_k(x,t) = e^{i\lambda_k\Lambda x + i\lambda_k^3\Omega t} w_k, \ 1 \le k \le N, \tag{4.1}$$

$$\hat{v}_k(x,t) = \hat{w}_k e^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^3\Omega t}, \ 1 \le k \le N, \tag{4.2}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively, but need to satisfy the following three orthogonal conditions:

$$\hat{w}_k w_l = \hat{w}_k \Lambda w_l = (\hat{\lambda}_k^2 + \hat{\lambda}_k\lambda_l + \lambda_l^2)\hat{w}_k\Omega w_l = 0 \text{ when } \lambda_l = \hat{\lambda}_k, \tag{4.3}$$

where $1 \le k, l \le N$, and Λ and Ω are given by (2.4).

Now following the binary DT (3.25), a new potential matrix can be computed by

$$P' = [T_1^+, \Lambda], \ T_1^+ = -vM^{-1}\hat{v} = -\sum_{k,l=1}^N v_k (M^{-1})_{kl}\hat{v}_l. \tag{4.4}$$

Consequently, this yields a kind of soliton solutions to the matrix mKdV equations (2.7):

$$p = \alpha\sum_{k,l=1}^N v_k^1 (M^{-1})_{kl}\hat{v}_l^2, \ q = -\alpha\sum_{k,l=1}^N v_k^2 (M^{-1})_{kl}\hat{v}_l^1, \tag{4.5}$$

where we split $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, of which v_k^1 and v_k^2 are m- and n-dimensional column vectors, respectively, and $\hat{v}_k^1$ and $\hat{v}_k^2$ are m- and n-dimensional row vectors, respectively.

4.2. Soliton solutions to the reduced equations

Let us second consider the reduced case. The task is to satisfy the involution condition (3.31) in order to construct soliton solutions to the reduced matrix mKdV equations (2.15). Therefore, what we need to check is whether the new potential matrix P' generated via the binary DT satisfies the reduction property (2.12). If this is true, the soliton solutions in (4.5) to the matrix mKdV equations (2.7) are then reduced to the soliton solutions:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{4.6}$$

for the reduced matrix mKdV equations (2.15), where $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, $1 \le k \le N$, as before.

To satisfy the involution property (3.31), let us take N eigenvalues $\lambda_k \in \mathbb{C}$, $1 \le k \le N$, and define N adjoint eigenvalues $\{\hat{\lambda}_k | 1 \le k \le N\}$ via (3.30). Then, taking the zero seed solution $P = 0$, we can obtain the corresponding eigenfunctions v_k, $1 \le k \le N$:

$$v_k(x,t) = v_k(x,t,\lambda_k) = e^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_k, \ \ 1 \le k \le N, \tag{4.7}$$

where w_k, $1 \le k \le N$, are arbitrary column vectors. Further, according to our previous analysis on integrable reductions, the corresponding adjoint eigenfunctions $\hat{v}_k$, $1 \le k \le N$, will be

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(x,t,\lambda_k) C = w_k^\dagger e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^3 \Omega t} C, \quad 1 \le k \le N. \tag{4.8}$$

The three orthogonal properties in (3.33) become the following three new conditions:

$$w_k^\dagger C w_l = w_k^\dagger C \Lambda w_l = \left(\hat{\lambda}_k^2 + \hat{\lambda}_k \lambda_l + \lambda_l^2\right) w_k^\dagger C \Omega w_l = 0 \quad \text{when } \lambda_l = \hat{\lambda}_k, \tag{4.9}$$

where $1 \le k, l \le N$, on the constant vectors $\{w_k | 1 \le k \le N\}$. We point out that the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when taking $\lambda_k \in \mathbb{R}$. Because of $\alpha_1 \ne \alpha_2$ and $\beta_1 \ne \beta_2$, we directly see that the above three conditions exactly require that

$$\left(w_k^1\right)^\dagger \Sigma_1 w_l^1 = 0, \ \left(w_k^2\right)^\dagger \Sigma_2 w_l^2 = 0, \ \text{when } \lambda_l = \hat{\lambda}_k, \tag{4.10}$$

where $1 \le k, l \le N$, and we split $w_k = \left(\left(w_k^1\right)^T, \left(w_k^2\right)^T\right)^T$ as we did for v_k before.

Finally, the formula (4.6), together with (3.7), (4.7) and (4.8), presents soliton solutions to the reduced matrix mKdV equations (2.15).

5. Concluding remarks

The aim of the paper is to construct a binary Darboux transformation (DT) for a class of matrix mKdV equations and their reduced matrix integrable counterparts, associated with matrix spectral problems of arbitrary order. The crucial step is to apply two pairs of eigenfunctions and adjoint eigenfunctions, corresponding to two arbitrary sets of eigenvalues and adjoint eigenvalues. The resulting binary DTs have been used to derive soliton solutions to the matrix mKdV equations and their reduced integrable equations.

In our formulation of binary DTs, we introduce a generalized M-matrix innovatively, where eigenvalues could be equal to adjoint eigenvalues. The motivation for doing that is derived from recent studies on Riemann-Hilbert problems for nonlocal integrable equations (see, for example, [18]). The resulting general formulation of binary DTs can be applied to both local and nonlocal integrable equations (see, for example, [18–22] for nonlocal theories). Taking repeated eigenvalues or repeated adjoint eigenvalues engenders Darboux matrices with higher-order poles, and taking derivatives with respect to eigenvalues or adjoint eigenvalues leads to generalized Darboux transformations.

There are many other interesting problems in the theory of DTs, which include applications of DTs to other kinds of exact solutions, particularly lump solutions; systematical theories of binary DTs for integrable equations associated with non-semisimple Lie algebras; and connections with N-soliton solutions by the Hirota bilinear method and other approaches.

Data Availability Statements

All data generated or analyzed during this study are included in this published article.

Declaration of Competing Interest

None.

CRediT authorship contribution statement

Wen-Xiu Ma: Conceptualization, Methodology, Writing - original draft, Visualization, Investigation, Validation.

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