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Multi-component bi-Hamiltonian Dirac integrable equations

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Abstract

A specific matrix iso-spectral problem of arbitrary order is introduced and an associated hierarchy of multi-component Dirac integrable equations is constructed within the framework of zero curvature equations. The bi-Hamiltonian structure of the obtained Dirac hierarchy is presented be means of the variational trace identity. Two examples in the cases of lower order are computed.

1. Introduction

The integrability of nonlinear differential equations [1] is an important problem in the field of differential equations. It is in the direction of solving the problem to look for integrable differential equations and to explore characteristics of integrability. Integrability of ordinary differential equations and 1+1 dimensional partial differential equations, particularly scalar equations, was extensively studied, and various criteria such as the inverse scattering transform, the Hamiltonian formulation and the Painlevé series were proposed for testing integrability [2,3].

However, we lack a thorough theory for determining integrability of multi-component differential equations and higher-dimensional differential equations. Both the multiplicity and the dimension bring a diversity of mathematical structures of integrable equations. The integrable couplings are one of the examples which show rich mathematical properties that multi-component integrable equations possess [4–7]. The way of using semi-direct sums of Lie algebras [8,9] provides a thought essential to analyze and classify multi-component integrable equations.

In this paper, we are going to construct a multi-component Dirac integrable hierarchy from an arbitrary order matrix spectral problem, which possesses a bi-Hamiltonian formulation. The paper is organized as follows. In Section 2, a new higher-order matrix iso-spectral problem is introduced and the associated Lax integrable equations of multi-components are computed. In Section 3, an integrable hierarchy among the resulting Lax integrable equations is worked out and proved to possess a bi-Hamiltonian structure using the variational trace identity. This implies that the the resulting Dirac hierarchy possesses infinitely many commuting symmetries and conserved densities. A few concluding remarks are given in Section 4.

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2. Lax pairs and associated integrable equations

We consider a matrix iso-spectral problem:

$$\phi_x = U\phi = U(u,\lambda)\phi, \qquad U = \begin{bmatrix} \lambda J_2 & q \\ -q^{\mathrm{T}} & r \end{bmatrix},\ J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.1}$$

where λ is a spectral parameter, r is a skew-symmetric (i.e., $r^{\mathrm{T}} = -r$) matrix of arbitrary order and $u = p(q, r)$ is a vector potential. Note that p is a kind of arrangement of all entries in q and r into a vector.

To derive associated integrable equations, we first solve the stationary zero curvature equation

$$V_x = [U, V] \tag{2.2}$$

of the spectral problem (2.1). We assume that a solution V can be given by

$$V = \begin{bmatrix} aJ_2 & b \\ -b^{\mathrm{T}} & c \end{bmatrix},$$

where a is scalar and $c^{\mathrm{T}} = -c$. Then we have

$$[U, V] = \begin{bmatrix} -qb^{\mathrm{T}} + bq^{\mathrm{T}} & \lambda J_2 b - aJ_2 q + qc - br \\ \lambda b^{\mathrm{T}} J_2 - aq^{\mathrm{T}} J_2 - rb^{\mathrm{T}} + cq^{\mathrm{T}} & -q^{\mathrm{T}} b + b^{\mathrm{T}} q + [r, c] \end{bmatrix}.$$

Therefore, the stationary zero curvature Eq. (2.2) becomes

$$a_x J_2 = bq^{\mathrm{T}} - qb^{\mathrm{T}}, \tag{2.3a}$$

$$b_x = \lambda J_2 b - aJ_2 q + qc - br, \tag{2.3b}$$

$$c_x = b^{\mathrm{T}} q - q^{\mathrm{T}} b + [r, c]. \tag{2.3c}$$

Let us seek a formal solution of the type

$$V = \begin{bmatrix} aJ_2 & b \\ -b^{\mathrm{T}} & c \end{bmatrix} = \sum_{k=0}^{\infty} V_k \lambda^{-k}, \quad V_k = \begin{bmatrix} a_k J_2 & b_k \\ -(b_k)^{\mathrm{T}} & c_k \end{bmatrix}, \tag{2.4}$$

where $(c_k)^{\mathrm{T}} = -c_k$, $k \geqslant 0$. Then Eq. (2.3) recursively define all a_k, b_k and c_k, $k \geqslant 0$, with the initial values satisfying

$$b_0 = 0, \quad a_{0,x} = 0, \quad c_{0,x} = [r, c_0].$$

Now for any integer $n \geqslant 1$, we introduce

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n = \sum_{j=0}^{n} V_j \lambda^{n-j} + \Delta_n, \quad \Delta_n = \begin{bmatrix} 0 & 0 \\ 0 & \delta_n \end{bmatrix}, \tag{2.5}$$

δ_n being skew-symmetric and of the same size as c, and define the time evolution law for the eigenfunction ϕ:

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u,\lambda)\phi. \tag{2.6}$$

Setting $U_1 = U|_{\lambda=0}$, we can compute that

$$((\lambda^n V)_+)_x - [U, (\lambda^n V)_+] = V_{n,x} - [U_1, V_n] = \begin{bmatrix} 0 & b_{n,x} + a_n J_2 q - qc_n + b_n r \\ * & 0 \end{bmatrix},$$

$$\Delta_{n,x} - [U, \Delta_n] = \Delta_{n,x} - [U_1, \Delta_n] = \begin{bmatrix} 0 & -q\delta_n \\ -\delta_n q^{\mathrm{T}} & \delta_{n,x} - [r, \delta_n] \end{bmatrix},$$

where $n \geqslant 1$. It follows then that the compatibility conditions of (2.1) and (2.6) for all $n \geqslant 1$, i.e., the zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geqslant 1, \tag{2.7}$$

associate with the Lax pairs of U and $V^{(n)}$, $n \geqslant 1$, give rise to a hierarchy of Lax integrable evolution equations

$$q_{t_n} = b_{n,x} + a_n J_2 q - qc_n + b_n r - q\delta_n, \quad r_{t_n} = \delta_{n,x} - [r, \delta_n],\ n \geqslant 1. \tag{2.8}$$

3. Multi-component Dirac integrable hierarchy

3.1. Integrable hierarchy

When $r=0$, we have to take $\delta_{n,x}=0$, $n\geqslant 1$. Then, the Lax integrable hierarchy (2.8) becomes

$$q_{t_n}=b_{n,x}+a_nJ_2q-qc_n-q\delta_n,\quad n\geqslant 1. \tag{3.1}$$

To obtain Hamiltonian integrable equations, we further take

$$\delta_n=0,\quad n\geqslant 1. \tag{3.2}$$

The corresponding integrable hierarchy (3.1) now reads

$$q_{t_n}=b_{n,x}+a_nJ_2q-qc_n,\quad n\geqslant 1. \tag{3.3}$$

Owing to $r=0$, (2.3) implies

$$a_{n,x}J_2=b_nq^{\mathrm{T}}-qb_n^{\mathrm{T}},\qquad J_2b_{n+1}=b_{n,x}+a_nJ_2q-qc_n,\quad c_{n,x}=b_n^{\mathrm{T}}q-q^{\mathrm{T}}b_n,\ n\geqslant 1. \tag{3.4}$$

This tells that the resulting hierarchy (3.3) can be rewritten as

$$q_{t_n}=X_n(q)=J_{\mathrm{c}}b_{n+1}=J_2b_{n+1}=M_{\mathrm{c}}b_n=b_{n,x}+[\partial^{-1}(b_nq^{\mathrm{T}}-qb_n^{\mathrm{T}})]q-q\partial^{-1}(b_n^{\mathrm{T}}q-q^{\mathrm{T}}b_n),\quad n\geqslant 1, \tag{3.5}$$

where J_{c} and M_{c} denote the compact form of two matrix operators J and M:

$$(p(q))_{t_n}=Jp(b_{n+1})=Mp(b_n),\quad n\geqslant 1. \tag{3.6}$$

On one hand, note that the inner product between two vectors $p(A)$ and $p(B)$ is

$$(p(A),p(B))=\int \mathrm{tr}(A^{\mathrm{T}}B)\mathrm{d}x,$$

where A and B denote matrices of the same size as q. Obviously, J and M are skew-symmetric. That is, for any two matrices A and B of the same size as q, we have

$$(p(A),p(O_{\mathrm{c}}B))=-(p(O_{\mathrm{c}}A),p(B)),$$

when $O_{\mathrm{c}}=J_{\mathrm{c}}$ or $O_{\mathrm{c}}=M_{\mathrm{c}}$. Since the operator J is independent of the dependent variables in q, J automatically satisfies the Jacobi identity

$$(p(A),p((J_{\mathrm{c}})'[J_{\mathrm{c}}B]C))+\mathrm{cycle}(A,B,C)=0,$$

where $(J_{\mathrm{c}})'$ denotes the Gateaux derivative, and A,B and C are arbitrary matrices of the same size as q. Therefore, J is a Hamiltonian operator. Moreover, it can be directly verified that J and M form a Hamiltonian pair, i.e., any linear combination of J and M is still Hamiltonian.

On the other hand, let us recall the variational trace identity

$$\frac{\delta}{\delta u}\int\left\langle V,\frac{\partial U}{\partial\lambda}\right\rangle\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V,\frac{\partial U}{\partial u}\right\rangle, \tag{3.7}$$

where the Killing form $\langle P,Q\rangle=\mathrm{tr}(PQ)$ and the constant $\gamma=-\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle V,V\rangle|$ (see [10–13]). In our case, we have

$$\left\langle V,\frac{\partial U}{\partial q}\right\rangle=-2b,\quad \left\langle V,\frac{\partial U}{\partial\lambda}\right\rangle=-2a. \tag{3.8}$$

Accordingly, using the variational trace identity, we obtain

$$\frac{\delta}{\delta q}\widetilde{H}_n=b_n,\quad \widetilde{H}_n=\int\left(-\frac{a_{n+1}}{n}\right)\mathrm{d}x,\ n\geqslant 1. \tag{3.9}$$

It naturally follows that the integrable hierarchy (3.5) has a bi-Hamiltonian structure:

$$q_{t_n}=J_{\mathrm{c}}\frac{\delta}{\delta q}\widetilde{H}_{n+1}=M_{\mathrm{c}}\frac{\delta}{\delta q}\widetilde{H}_n,\quad n\geqslant 1. \tag{3.10}$$

The bi-Hamiltonian structure guarantees [14] that

$$\{\widetilde{H}_k, \widetilde{H}_l\}_J = \{\widetilde{H}_k, \widetilde{H}_l\}_M = 0, \quad k, l \geqslant 1, \tag{3.11}$$

$$[p(X_k), p(X_l)] = J\frac{\delta}{\delta u}\{\widetilde{H}_{k+1}, \widetilde{H}_{l+1}\}_J = 0, \quad k, l \geqslant 1, \tag{3.12}$$

where the Poisson bracket associated with a Hamiltonian operator O is given by

$$\{\widetilde{H}, \widetilde{K}\}_O = \left(\frac{\delta \widetilde{H}}{\delta u}, O\frac{\delta \widetilde{K}}{\delta u}\right) = \int \left(\frac{\delta \widetilde{H}}{\delta u}\right)^{\mathrm{T}} O\frac{\delta \widetilde{K}}{\delta u}\,\mathrm{d}x,$$

and the commutator of vector functions is defined by

$$[X, Y] = X'(u)[Y] - Y'(u)[X] = \frac{\partial}{\partial \varepsilon}(X(u+\varepsilon Y) - Y(u+\varepsilon X))|_{\varepsilon=0}.$$

The equalities in (3.11) and (3.12) imply that each system in the hierarchy (3.10) has infinitely many commuting symmetries and conserved densities (see [14–16] for a general theory on the bi-Hamiltonian formulation).

3.2. The case of scalar r

When r is a 1×1 matrix, let us choose $c = 0$, and set

$$q = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad b_n = \begin{bmatrix} d_n \\ e_n \end{bmatrix}, \ n \geqslant 0, \tag{3.13}$$

where u_1 and u_2 are scalar variables, and d_n and e_n are scalar functions. Then, the recursion relation (3.4) becomes

$$\begin{cases} d_{n+1} = -e_{n,x} + u_1 a_n, \\ e_{n+1} = d_{n,x} + u_2 a_n, \\ a_{n+1,x} = u_2 d_{n+1} - u_1 e_{n+1}, \end{cases}$$

where $n \geqslant 0$, and thus, the Dirac integrable hierarchy (3.5) (i.e., (3.10)) reduces to the two-component Dirac hierarchy [17] associated with a Dirac spectral problem [18]:

$$u_{t_n} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{t_n} = \begin{bmatrix} e_{n+1} \\ -d_{n+1} \end{bmatrix} = J\begin{bmatrix} d_{n+1} \\ e_{n+1} \end{bmatrix} = J\frac{\delta}{\delta u}\widetilde{H}_{n+1} = \begin{bmatrix} d_{n,x} + u_2 a_n \\ e_{n,x} - u_1 a_n \end{bmatrix} = M\begin{bmatrix} d_n \\ e_n \end{bmatrix} = M\frac{\delta}{\delta u}\widetilde{H}_n, \quad n \geqslant 1, \tag{3.14}$$

where the Hamiltonian functionals $\widetilde{H}_n,\ n \geqslant 1$, are defined by (3.9) and the Hamiltonian pair reads

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \partial + u_2\partial^{-1}u_2 & -u_2\partial^{-1}u_1 \\ -u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 \end{bmatrix}.$$

The corresponding hereditary recursion operator [19] of the hierarchy (3.14) is

$$\Phi = MJ^{-1} = \begin{bmatrix} -u_2\partial^{-1}u_1 & -\partial - u_2\partial^{-1}u_2 \\ \partial + u_1\partial^{-1}u_1 & u_1\partial^{-1}u_2 \end{bmatrix}.$$

We call the proposed hierarchy (3.5) the multi-component Dirac hierarchy, because the reduction of the matrix q to a vector yields this two-component Dirac hierarchy.

Upon choosing $a_0 = 1$ and setting every integration constant to zero, the first nonlinear system of integrable equations in the hierarchy (3.14) is the Dirac nonlinear Schrödinger equations:

$$\begin{cases} u_{1,t_2} = -u_{2,xx} - \frac{1}{2}u_2(u_1^2 + u_2^2), \\ u_{2,t_2} = u_{1,xx} + \frac{1}{2}u_1(u_1^2 + u_2^2). \end{cases} \tag{3.15}$$

Its bi-Hamiltonian structure is the following

$$u_{t_2} = J\frac{\delta}{\delta u}\widetilde{H}_3 = M\frac{\delta}{\delta u}\widetilde{H}_2,$$

where two Hamiltonian functionals are

$$\widetilde{H}_2 = \int \frac{1}{2}(u_{1,x}u_2 - u_1u_{2,x})\mathrm{d}x, \quad \widetilde{H}_3 = \int \left[-\frac{1}{2}u_1u_{1,xx} - \frac{1}{2}u_2u_{2,xx} - \frac{1}{8}(u_1^2 + u_2^2)^2\right]\mathrm{d}x.$$

3.3. The case of 2×2 matrix r

When r is a 2×2 matrix, let us set

$$q=\begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix},\quad b_n=\begin{bmatrix} d_n & f_n \\ e_n & g_n \end{bmatrix},\quad c_n=\begin{bmatrix} 0 & h_n \\ -h_n & 0 \end{bmatrix},\ n\geqslant 0, \tag{3.16}$$

where u_i, $1\leqslant i\leqslant 5$, are scalar variables, and d_n, e_n, f_n, g_n and h_n are scalar functions. Then, the recursion relation (3.4) becomes

$$\begin{cases} d_{n+1}=-e_{n,x}+u_1a_n-u_4h_n, \\ e_{n+1}=d_{n,x}+u_2a_n+u_3h_n, \\ f_{n+1}=-g_{n,x}+u_3a_n+u_2h_n, \\ g_{n+1}=f_{n,x}+u_4a_n-u_1h_n, \\ a_{n+1,x}=u_2d_{n+1}+u_4f_{n+1}-u_1e_{n+1}-u_3g_{n+1}, \\ h_{n+1,x}=u_3d_{n+1}+u_4e_{n+1}-u_1f_{n+1}-u_2g_{n+1}, \end{cases}$$

where $n\geqslant 0$, and so, the Dirac integrable hierarchy (3.5) (i.e., (3.10)) gives rise to

$$u_{t_n}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_{t_n}=\begin{bmatrix} e_{n+1} \\ -d_{n+1} \\ g_{n+1} \\ -f_{n+1} \end{bmatrix}=J\begin{bmatrix} d_{n+1} \\ e_{n+1} \\ f_{n+1} \\ g_{n+1} \end{bmatrix}=J\frac{\delta}{\delta u}\widetilde{H}_{n+1}=\begin{bmatrix} d_{n,x}+u_2a_n+u_3h_n \\ e_{n,x}-u_1a_n+u_4h_n \\ f_{n,x}+u_4a_n-u_1h_n \\ g_{n,x}-u_3a_n-u_2h_n \end{bmatrix}=M\begin{bmatrix} d_n \\ e_n \\ f_n \\ g_n \end{bmatrix}=M\frac{\delta}{\delta u}\widetilde{H}_n,\quad n\geqslant 1, \tag{3.17}$$

where the Hamiltonian pair is given by

$$J=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and

$$M=\begin{bmatrix} \partial+p_{22}+p_{33} & -p_{21}+p_{34} & p_{24}-p_{31} & -p_{23}-p_{32} \\ -p_{12}+p_{43} & \partial+p_{11}+p_{44} & -p_{14}-p_{41} & p_{13}-p_{42} \\ p_{42}-p_{13} & -p_{41}-p_{14} & \partial+p_{44}+p_{11} & -p_{43}+p_{12} \\ -p_{32}-p_{23} & p_{31}-p_{24} & -p_{34}+p_{21} & \partial+p_{33}+p_{22} \end{bmatrix},$$

with $p_{ij}=u_i\partial^{-1}u_j$, $1\leqslant i,j\leqslant 4$. The corresponding hereditary recursion operator [19] for the hierarchy (3.17) is

$$\Phi=MJ^{-1}=\begin{bmatrix} -p_{21}+p_{34} & -\partial-p_{22}-p_{33} & -p_{23}-p_{32} & -p_{24}+p_{31} \\ \partial+p_{11}+p_{44} & p_{12}-p_{43} & p_{13}-p_{42} & p_{14}+p_{41} \\ -p_{41}-p_{14} & -p_{42}+p_{13} & -p_{43}+p_{12} & -\partial-p_{44}-p_{11} \\ p_{31}-p_{24} & p_{32}+p_{23} & \partial+p_{33}+p_{22} & p_{34}-p_{21} \end{bmatrix}.$$

Upon choosing $a_0=1$ and $h_0=0$, and setting every integration constant to zero, the first nonlinear system of integrable equations in the hierarchy (3.17) reads

$$\begin{cases} u_{1,t_2}=-u_{2,xx}-\frac{1}{2}u_2(u_1^2+u_2^2+u_3^2+u_4^2)+u_3(u_1u_4-u_2u_3), \\ u_{2,t_2}=u_{1,xx}+\frac{1}{2}u_1(u_1^2+u_2^2+u_3^2+u_4^2)+u_4(u_1u_4-u_2u_3), \\ u_{3,t_2}=-u_{4,xx}-\frac{1}{2}u_4(u_1^2+u_2^2+u_3^2+u_4^2)-u_1(u_1u_4-u_2u_3), \\ u_{4,t_2}=u_{3,xx}+\frac{1}{2}u_3(u_1^2+u_2^2+u_3^2+u_4^2)-u_2(u_1u_4-u_2u_3). \end{cases} \tag{3.18}$$

It has a bi-Hamiltonian structure:

$$u_{t_2}=J\frac{\delta}{\delta u}\widetilde{H}_3=M\frac{\delta}{\delta u}\widetilde{H}_2,$$

with two Hamiltonian functionals being given by

$$\widetilde{H}_2 = \int \frac{1}{2}(u_{1,x}u_2 - u_1u_{2,x} + u_{3,x}u_4 - u_3u_{4,x})\mathrm{d}x,$$
$$\widetilde{H}_3 = \int \left[-\frac{1}{2}u_1u_{1,xx} - \frac{1}{2}u_2u_{2,xx} - \frac{1}{2}u_3u_{3,xx} - \frac{1}{2}u_4u_{4,xx} - \frac{1}{8}(u_1^2+u_2^2+u_3^2+u_4^2)^2 - \frac{1}{2}(u_1u_4 - u_2u_3)^2\right]\mathrm{d}x.$$

If taking $u_3 = u_4 = 0$, this system reduces to the Dirac nonlinear Schrödinger Eq. (3.15).

4. Concluding remarks

A higher-order matrix iso-spectral problem was introduced and an associated multi-component integrable hierarchy was presented. The obtained Dirac integrable hierarchy was proved to possess a bi-Hamiltonian structure by using the variational trace identity. The recursion structure guarantees infinitely many commuting symmetries and conserved densities.

There are other ways to construct multi-component integrable equations such as perturbations [20,5,6], loop algebra methods [21,7] and enlarging spectral problems based on semi-direct sums of loop algebras [22,8,9]. The related results and other existing theories [23–25] provide hints for classifying multi-component integrable equations and setting criteria for integrability. It is also interesting to find conditions on matrix forms, even sufficient conditions, under which matrix spectral problems can engender multi-component integrable equations with Hamiltonian structures.

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