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Wronskians, generalized Wronskians and solutions to the Korteweg–de Vries equation

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Abstract

A bridge going from Wronskian solutions to generalized Wronskian solutions of the Korteweg–de Vries (KdV) equation is built. It is then shown that generalized Wronskian solutions can be viewed as Wronskian solutions. The idea is used to generate positons, negatons and their interaction solutions to the KdV equation. Moreover, general positons and negatons are constructed through the Wronskian formulation. A few new exact solutions to the KdV equation are explicitly presented as examples of Wronskian solutions.

1. Introduction

The Korteweg–de Vries (KdV) equation is one of the most important models exhibiting the soliton phenomenon [1]. Its multi-soliton solutions [2] can be expressed by using a Wronskian determinant [3,4]. Matveev found that there also exists another class of explicit solutions, called positons, to the KdV equation, which can be presented by a generalized Wronskian determinant [5]; and afterwards, negatons were also constructed through taking advantage of the generalized Wronskian determinant [6].

Solutions determined by the technique of Wronskian determinant and the technique of generalized Wronskian determinant are called Wronskian solutions and generalized Wronskian solutions, respectively. Solitons are examples of Wronskian solutions, and positons and negatons are examples of generalized Wronskian solutions. A natural question to ask is whether there is any interrelation between Wronskian solutions and generalized Wronskian solutions. What kind of relation one can have if it exists?

In this paper, we would like to build a bridge between Wronskian solutions and generalized Wronskian solutions. This gives us a way to obtain generalized Wronskian solutions through Wronskian solutions to the KdV equation. The basic idea is used to generate positons, negatons and their interaction solutions to the KdV equation. Moreover, general positons and negatons are constructed through the Wronskian formulation. A few new exact solutions to the KdV equation are explicitly presented as examples of Wronskian solutions.

2. Bridge between Wronskians and generalized Wronskians

Let us specify the KdV equation as follows

$$u_t - 6uu_x + u_{xxx} = 0, \tag{2.1}$$

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where (also in the rest of the paper) $g_{y_1\cdots y_i}$ is the conventional notation denoting the ith order partial derivative $\partial^i g/\partial y_1\cdots\partial y_i$. Hirota introduced the transformation [2]:

$$u = -2\partial_x^2 \ln f = -\frac{2(ff_{xx} - f_x^2)}{f^2}, \tag{2.2}$$

between the KdV equation (2.1) and the following bilinear equation

$$(D_xD_t + D_x^4)f\cdot f = f_{xt}f - f_tf_x + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0, \tag{2.3}$$

where D_x and D_t, called the Hirota operators, are defined by

$$f(x+k, t+h)g(x-k, t-h) = \sum_{i,j=0}^{\infty} \frac{1}{i!j!}(D_x^iD_t^jf\cdot g)k^ih^j. \tag{2.4}$$

Solutions to the above bilinear KdV equation (2.3) can be given by the Wronskian determinant [3,4]:

$$W(\phi_1,\phi_2,\ldots,\phi_N) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \quad N \geqslant 1, \tag{2.5}$$

where

$$\phi_i^{(0)} = \phi_i, \quad \phi_i^{(j)} = \frac{\partial^j}{\partial x^j}\phi_i, \quad j\geqslant 1, \quad 1\leqslant i\leqslant N. \tag{2.6}$$

Sirianunpiboon et al. [7] furnished the following conditions

$$-\phi_{i,xx} = \sum_{j=1}^{i}\lambda_{ij}\phi_j, \quad \phi_{i,t} = -4\phi_{i,xxx}, \quad 1\leqslant i\leqslant N, \tag{2.7}$$

where λ_{ij} are arbitrary real constants, to make the Wronskian determinant a solution to the bilinear KdV equation (2.3), and showed that rational function solutions and their interaction solutions with multi-soliton ones to the KdV equation can be obtained this way.

Matveev found [5] that there exists so-called generalized Wronskian solutions to the bilinear KdV equation (2.3), which read as

$$W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi), \quad \partial_k^i = \frac{\partial^i}{\partial k^i}, \quad 1\leqslant i\leqslant n, \tag{2.8}$$

where the function ϕ satisfies

$$-\phi_{xx} = k^2\phi, \quad \phi_t = -4\phi_{xxx}, \quad k\in\mathbb{R}.$$

The resulting solutions to the KdV equation (2.1) are called positons, since the corresponding eigenfunctions are associated with positive eigenvalues of the Schrödinger spectral problem.

Generally, the generalized Wronskian determinant (2.8) gives rise to a solution to the bilinear KdV equation (2.3), provided that the function ϕ satisfies

$$-\phi_{xx} = \alpha(k)\phi, \quad \phi_t = -4\phi_{xxx}, \tag{2.9}$$

with α being an arbitrary function of $k\in\mathbb{R}$. Observe that if we have (2.9), then the function ϕ also satisfies

$$-(\partial_k^m\phi)_{xx} = \sum_{i=0}^{m}\binom{m}{i}(\partial_k^i\alpha)(\partial_k^{m-i}\phi), \ (\partial_k^m\phi)_t = -4(\partial_k^m\phi)_{xxx}, \ m\geqslant 0. \tag{2.10}$$

Upon setting

$$\psi_{i+1} = \frac{1}{i!}\frac{\partial^i\phi}{\partial k^i}, \quad \alpha_{i+1} = \frac{1}{i!}\frac{\partial^i\alpha}{\partial k^i}, \quad i\geqslant 0, \tag{2.11}$$

it follows that the functions ψ_i, $1\leqslant i\leqslant n+1$, satisfy a lower triangular system of second-order differential equations:

$$\begin{cases} -\psi_{1,xx} = \alpha_1\psi_1, \\ -\psi_{2,xx} = \alpha_2\psi_1 + \alpha_1\psi_2, \\ \cdots \\ -\psi_{n+1,xx} = \alpha_{n+1}\psi_1 + \alpha_n\psi_2 + \cdots + \alpha_1\psi_{n+1}. \end{cases} \tag{2.12}$$

This is to say,

$$-\Psi_{xx} = \Lambda\Psi, \quad \Lambda = \begin{bmatrix} \alpha_1 & & & 0 \\ \alpha_2 & \alpha_1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_{n+1} & \cdots & \alpha_2 & \alpha_1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n+1} \end{bmatrix}, \tag{2.13}$$

which is a special case of the conditions (2.7). Obviously, under the transformation (2.11), the above system of differential equations, together with

$$\psi_{i,t} = -4\psi_{i,xxx}, \quad 1 \leqslant i \leqslant n+1,$$

is equivalent to the conditions (2.9). Therefore, summing up, there exists a bridge going from the Wronskian to the generalized Wronskian:

$$W(\psi_1, \psi_2, \ldots, \psi_{n+1}) = \left(\prod_{i=1}^{n} \frac{1}{i!}\right) W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi). \tag{2.14}$$

The constant factor in (2.14) does not affect the final solution determined by (2.2), i.e., we have

$$u = -2\,\partial_x^2 \ln W(\psi_1, \psi_2, \ldots, \psi_{n+1}) = -2\,\partial_x^2 \ln W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi). \tag{2.15}$$

This implies that

$$u = -2\,\partial_x^2 \ln W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi) \tag{2.16}$$

gives a solution to the KdV equation (2.1) if (2.9) holds, and that all such generalized Wronskian solutions to the KdV equation can be obtained through the Wronskian formulation. However, (2.12) can have other solutions besides $(\phi, \partial_k\phi, \ldots, \partial_k^n\phi)$ with ϕ solving (2.9), and thus not all Wronskian solutions are of generalized Wronskian type.

3. Positons, negatons and their interaction solutions

Two particular classes of generalized Wronskian solutions to the KdV equation are positons and negatons. It is known that positons of order n are represented by using the generalized Wronskian determinant [5]:

$$f = W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi), \quad \phi = \cos(kx + 4k^3t + \gamma(k)), \tag{3.1}$$

or

$$f = W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi), \quad \phi = \sin(kx + 4k^3t + \gamma(k)), \tag{3.2}$$

where γ is an arbitrary function of k; and that negatons of order n, by using the generalized Wronskian determinant [6]:

$$f = W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi), \quad \phi = \cosh(kx - 4k^3t + \gamma(k)), \tag{3.3}$$

or

$$f = W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi), \quad \phi = \sinh(kx - 4k^3t + \gamma(k)), \tag{3.4}$$

where γ is an arbitrary function of k as well. Note that two kinds of positons are equivalent to each other, due to the existence of an arbitrary function $\gamma(k)$. But two kinds of negatons are functionally independent.

In the case of positons, we have

$$\alpha(k) = k^2, \quad k \in \mathbb{R} \tag{3.5}$$

in the conditions (2.9), which implies that the Schrödinger operator $-\partial^2/\partial x^2 + u$ with zero potential has a positive eigenvalue. Further, we have

$$\alpha_1 = k^2, \quad \alpha_2 = 2k, \quad \alpha_3 = 1, \quad \alpha_i = 0, \quad i \geqslant 4, \tag{3.6}$$

and the corresponding coefficient matrix becomes

$$\Lambda = \begin{bmatrix} k^2 & & & & 0 \\ 2k & k^2 & & & \\ 1 & 2k & k^2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 2k & k^2 \end{bmatrix}. \tag{3.7}$$

In the case of negatons, we have

$$\alpha(k) = -k^2, \quad k \in \mathbb{R} \tag{3.8}$$

in the conditions (2.9), which implies that the Schrödinger operator $-\partial^2/\partial x^2 + u$ with zero potential has a negative eigenvalue. Thus, we have

$$\alpha_1 = -k^2, \quad \alpha_2 = -2k, \quad \alpha_3 = -1, \quad \alpha_i = 0, \quad i \geqslant 4, \tag{3.9}$$

and the corresponding coefficient matrix reads as

$$\Lambda = \begin{bmatrix} -k^2 & & & & 0 \\ -2k & -k^2 & & & \\ -1 & -2k & -k^2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & -2k & -k^2 \end{bmatrix}. \tag{3.10}$$

Therefore, all positons and negatons can be presented through the Wronskian formulation.

The manipulation in the previous section also allows us to generate interaction solutions among positons and negatons. Let us choose the functions ϕ_i, $1 \leqslant i \leqslant m$, among the functions

$$\cos(k_i x + 4k_i^3 t + \gamma_i(k_i)), \quad \sin(k_i x + 4k_i^3 t + \gamma_i(k_i)), \tag{3.11}$$

and the functions

$$\cosh(k_i x - 4k_i^3 t + \gamma_i(k_i)), \quad \sinh(k_i x - 4k_i^3 t + \gamma_i(k_i)), \tag{3.12}$$

where the k_i's are arbitrary constants and the γ_i's are arbitrary functions. Then we have a new class of exact solutions to the KdV equation (2.1):

$$u_m = -2\partial_x^2 \ln f = -\frac{2(ff_{xx} - f_x^2)}{f^2}, \tag{3.13}$$

where the function f is given by

$$f = W(\phi_1, \partial_k \phi_1, \ldots, \partial_k^{n_1} \phi_1; \ldots; \phi_m, \partial_k \phi_m, \ldots, \partial_k^{n_m} \phi_m) \tag{3.14}$$

with arbitrary non-negative integers $n_1, n_2, \ldots, n_m$. This is an abroad class of interaction solutions among positons and negatons to the KdV equation (2.1).

In particular, let us fix

$$\phi_1 = \sin(kx + 4k^3 t + \gamma), \quad \phi_2 = \sinh(kx - 4k^3 t + \gamma), \tag{3.15}$$

where k and γ are two arbitrary constants. Then the Wronskian determinant $f = W(\phi_1, \phi_2)$ becomes

$$f = W(\phi_1, \phi_2) = k(\sin \xi_+ \cosh \xi_- - \sinh \xi_- \cos \xi_+), \tag{3.16}$$

and the corresponding interaction solution between a simple positon and a simple negaton reads as

$$u = -2\partial_x^2 \ln f = -2\partial_x^2 \ln W(\phi_1, \phi_2) = \frac{4k^2(\sinh^2 \xi_- - \sin^2 \xi_+)}{(\sin \xi_+ \cosh \xi_- - \sinh \xi_- \cos \xi_+)^2}, \tag{3.17}$$

where

$$\xi_+ = kx + 4k^3 t + \gamma, \quad \xi_- = kx - 4k^3 t + \gamma. \tag{3.18}$$

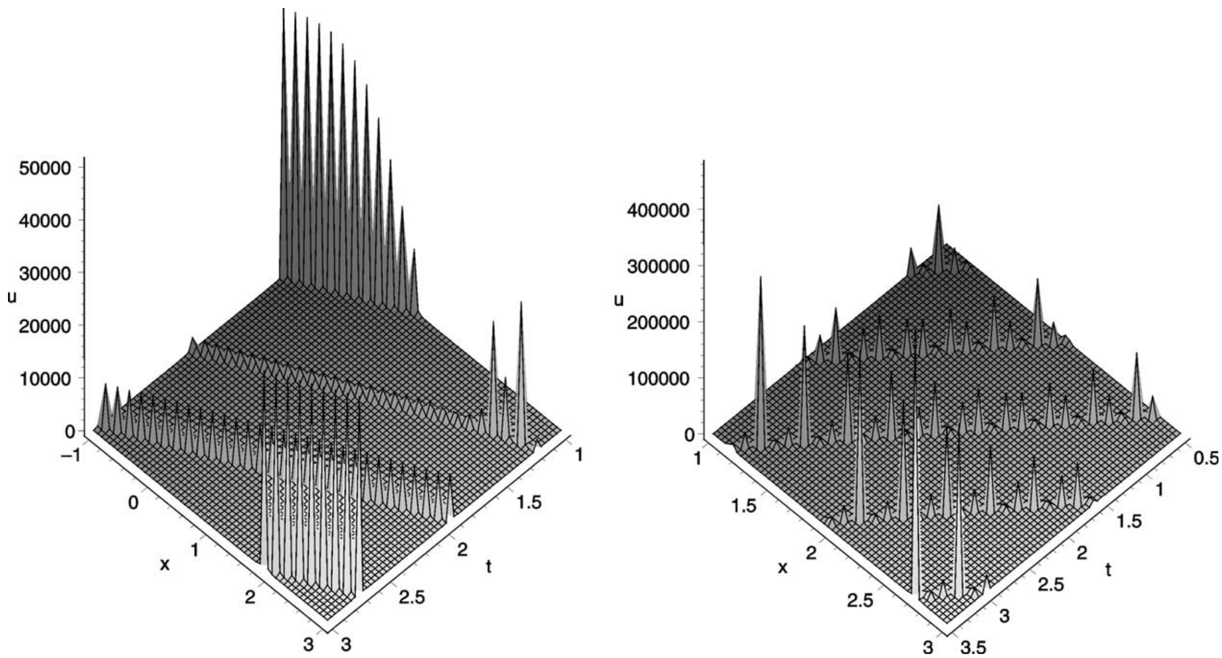

Fig. 1. Interaction solution: $k = 1$, $\gamma = 1$ (left) and $k = 2$, $\gamma = -2$ (right).

The graphs of the solution in two cases are depicted with grid = [60, 60] in Fig. 1, which show the distribution of some singularities.

4. General positons and negatons

In this section, we are going to present two classes of general positons and negatons to the KdV equation (2.1), which provide us with new exact solutions to the KdV equation (2.1).

Let us first start from

$$-\phi_{+,xx} = k^2\phi_+, \quad \phi_{+,t} = -4\phi_{+,xxx}, \quad k \in \mathbb{R}. \tag{4.1}$$

A general solution to the system (4.1) is given by

$$\phi_+(k) = c(k)\cos(kx + 4k^3t) + d(k)\sin(kx + 4k^3t), \tag{4.2}$$

where c and d are two arbitrary functions of k. Then based on our construction in Section 2, we obtain a class of exact solutions to the KdV equation (2.1):

$$u = -2\,\partial_x^2 \ln W(\phi_+(k), \partial_k\phi_+(k), \ldots, \partial_k^n\phi_+(k)), \tag{4.3}$$

where $\phi_+(k)$ is given by (4.2). Such solutions correspond to the positive eigenvalue of the Schrödinger spectral problem, and simple positons determined by (3.1) and (3.2) are just their two examples in the cases of

$$c = \cos(\gamma(k)), \quad d = -\sin(\gamma(k)) \quad \text{and} \quad c = \sin(\gamma(k)), \quad d = \cos(\gamma(k)).$$

If we choose c and d to be constants, then the Wronskian determinant involving the first-order derivative $\partial_k\phi_+$ becomes

$$W(\phi_+(k), \partial_k\phi_+(k)) = -\frac{1}{2}(c^2 - d^2)\sin(2\xi_+) + cd\cos(2\xi_+) - (c^2 + d^2)kx - 12(c^2 + d^2)k^3t, \tag{4.4}$$

and the corresponding general positon of order 1 reads as

$$
\begin{aligned}
u &= -2\partial_x^2 \ln W(\phi_+(k), \partial_k \phi_+(k)) \\
&= \frac{4k^2(c^2+d^2)[(c^2-d^2)xk + 12(c^2-d^2)k^3t + 2cd]\sin(2\xi_+)}{\left[-(1/2)(c^2-d^2)\sin(2\xi_+) + cd\cos(2\xi_+) - (c^2+d^2)kx - 12(c^2+d^2)k^3t\right]^2} \\
&\quad - \frac{4k^2(c^2+d^2)[(2cdkx + 24cdk^3t - c^2 + d^2)\cos(2\xi_+) - (c^2+d^2)]}{\left[-(1/2)(c^2-d^2)\sin(2\xi_+) + cd\cos(2\xi_+) - (c^2+d^2)kx - 12(c^2+d^2)k^3t\right]^2},
\end{aligned}
\tag{4.5}
$$

where c, d and k are arbitrary constants and ξ_+ is given by

$$\xi_+ = kx + 4k^3t. \tag{4.6}$$

The graphs of the solution in two cases are depicted with grid $= [60, 60]$ in Fig. 2, which exhibit some singularities of the solution.

Let us second start from

$$-\phi_{-,xx} = -k^2\phi_-, \quad \phi_{-,t} = -4\phi_{-,xxx}, \quad k \in \mathbb{R}. \tag{4.7}$$

A general solution to the system (4.7) is given by

$$\phi_-(k) = c(k)\exp(kx - 4k^3t) + d(k)\exp(-kx + 4k^3t), \tag{4.8}$$

where c and d are two arbitrary functions of k. Then similarly, based on our construction in Section 2, we obtain another class of exact solutions to the KdV equation (2.1):

$$u = -2\partial_x^2 \ln W(\phi_-(k), \partial_k\phi_-(k), \ldots, \partial_k^n\phi_-(k)), \tag{4.9}$$

where $\phi_-(k)$ is given by (4.8). Such solutions correspond to the negative eigenvalue of the Schrödinger spectral problem, and simple negatons determined by (3.3) and (3.4) are just their two special examples in the cases of

$$c = \frac{1}{2}e^{\gamma(k)}, \quad d = \frac{1}{2}e^{-\gamma(k)} \quad \text{and} \quad c = \frac{1}{2}e^{\gamma(k)}, \quad d = -\frac{1}{2}e^{-\gamma(k)}.$$

If we choose c and d to be constants, then the Wronskian determinant involving the first-order derivative $\partial_k\phi_-$ becomes

$$W(\phi_-(k), \partial_k\phi_-(k)) = c^2e^{2\xi_-} - d^2e^{-2\xi_-} + 4cdk(x - 12k^2t), \tag{4.10}$$

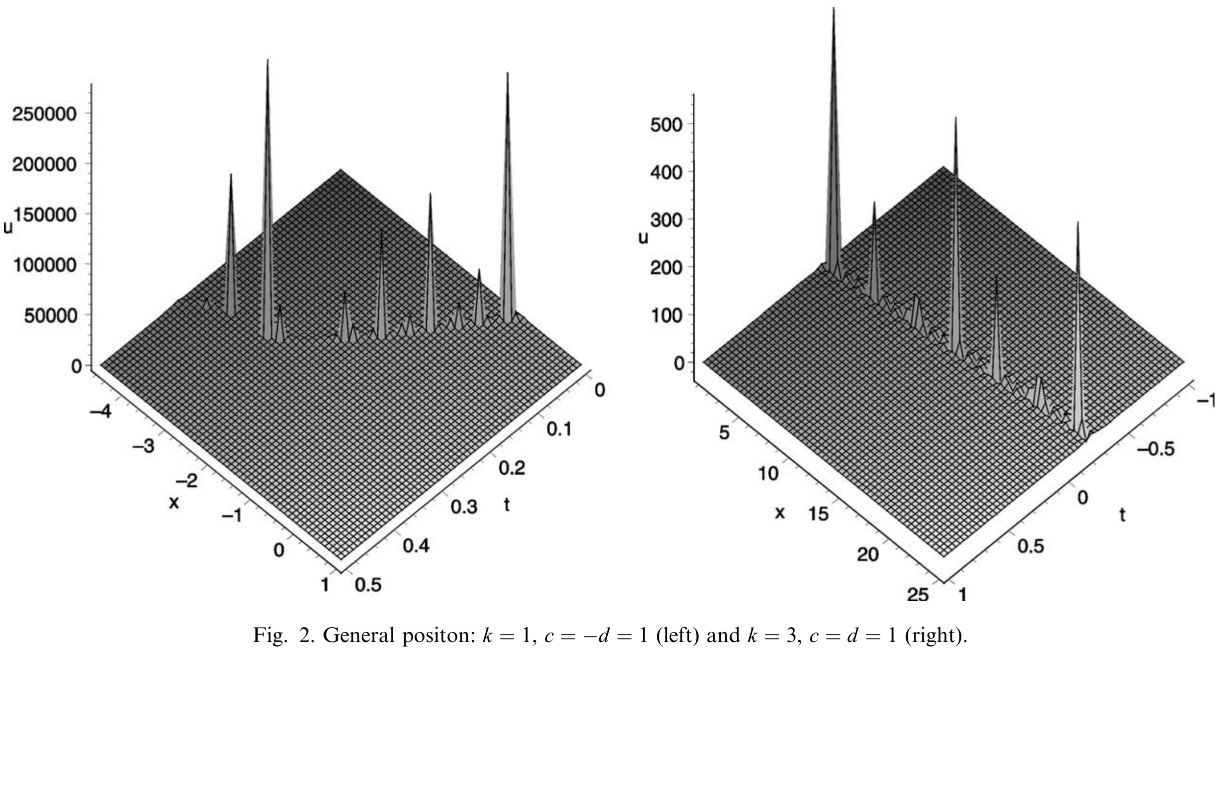

Fig. 2. General positon: $k = 1$, $c = -d = 1$ (left) and $k = 3$, $c = d = 1$ (right).

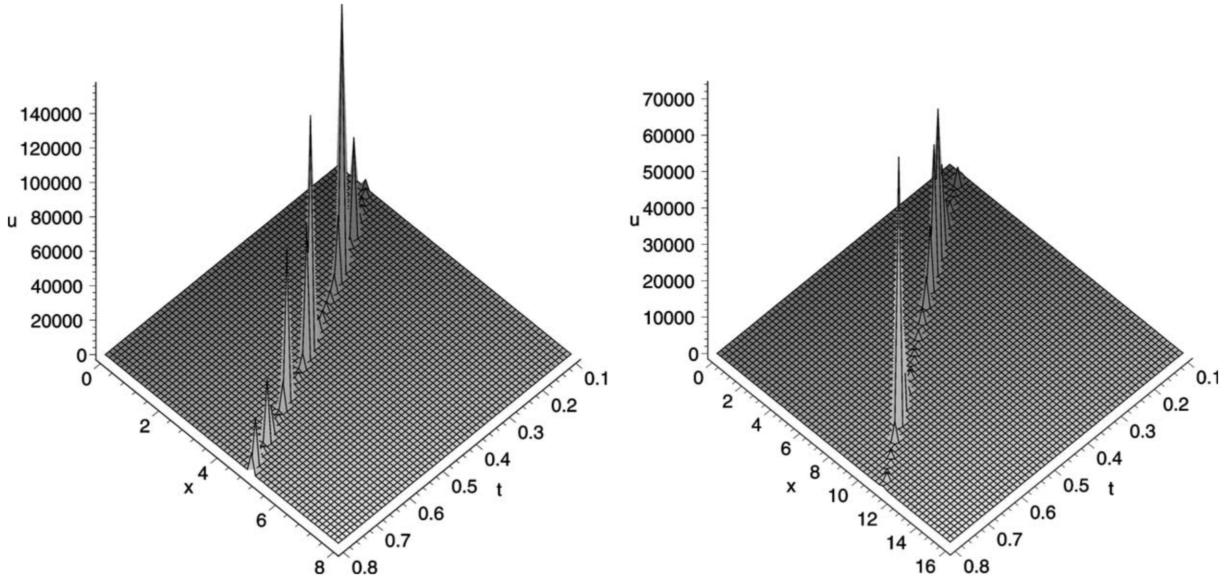

Fig. 3. General negaton: $k = 1$, $2c = d = 2$ (left) and $k = 2$, $c = -d = 1$ (right).

and the corresponding general negaton of order 1 reads as

$$u = -2\partial_x^2 \ln W(\phi_-(k), \partial_k\phi_-(k)) = \frac{-32cdk^2\left[c^2(kx - 12k^3t - 1)\mathrm{e}^{2\xi_-} - d^2(kx - 12k^3t + 1)\mathrm{e}^{-2\xi_-} - 2cd\right]}{\left[c^2\mathrm{e}^{2\xi_-} - d^2\mathrm{e}^{-2\xi_-} + 4cdk(x - 12k^2t)\right]^2}, \tag{4.11}$$

where c, d and k are arbitrary constants and ξ_- is given by

$$\xi_- = kx - 4k^3t. \tag{4.12}$$

The graphs of the solution in two cases are depicted with grid = [60, 60] in Fig. 3, which show the distribution of some singularities of the solution.

5. Conclusion and remarks

On one hand, a bridge between Wronskian solutions and generalized Wronskian solutions to the KdV equation was built. It gives us a way to obtain generalized Wronskian solutions simply from Wronskian determinants. The basic idea was used to generate positons, negatons and their interaction solutions to the KdV equation through the Wronskian formulation. A specific interaction solution between two simple positon and negaton to the KdV equation (2.1) was given by (3.17). On the other hand, general positons and negatons were also presented through the Wronskian formulation. They provide new examples of Wronskian solutions. Two new specific solutions of general positons and negatons to the KdV equation (2.1) were given by (4.5) and (4.11).

There are also interaction solutions between positons and solitons [8]. Such solutions can also be constructed through the Wronskian determinant

$$f = W(\phi, \partial_k\phi, \ldots, \partial_k^n; \phi_1, \ldots, \phi_N), \tag{5.1}$$

where the function ϕ is chosen from the functions in (3.11) and the functions ϕ_i are chosen as

$$\phi_i = \cosh(k_ix - 4k_i^3t + \gamma_i), \quad \gamma_i = \text{constant}, \quad \text{if } i \text{ odd},$$
$$\phi_i = \sinh(k_ix - 4k_i^3t + \gamma_i), \quad \gamma_i = \text{constant}, \quad \text{if } i \text{ even}.$$

Moreover, if we choose the function ϕ from the functions in (3.12), then the Wronskian determinant given by (5.1) generates interaction solutions between negatons and solitons to the KdV equation (2.1). Combining the constructions in (3.14) and (5.1) will give rise to more general interaction solutions among positons, negatons and solitons. We believe that such an idea of constructing interaction solutions should work for other soliton equations, especially for the perturbation KdV equations [9].

Finally, we point out that there is also another class of explicit exact solutions to the KdV equation (2.1), called complexitons [10]. One-complexiton is given by

$$u = \frac{-4\beta^2\left[1+\cos(2\delta(x-\bar{\beta}t)+\kappa)\cosh(2\Delta(x+\bar{\alpha}t)+\gamma)\right]}{\left[\Delta\sin(2\delta(x-\bar{\beta}t)+\kappa)+\delta\sinh(2\Delta(x+\bar{\alpha}t)+\gamma)\right]^2} + \frac{4\alpha\beta\sin(2\delta(x-\bar{\beta}t)+\kappa)\sinh(2\Delta(x+\bar{\alpha}t)+\gamma)}{\left[\Delta\sin(2\delta(x-\bar{\beta}t)+\kappa)+\delta\sinh(2\Delta(x+\bar{\alpha}t)+\gamma)\right]^2}, \tag{5.2}$$

where $\alpha, \beta > 0$, κ and γ are arbitrary real constants, and Δ, δ, $\bar{\alpha}$ and $\bar{\beta}$ are given by

$$\Delta = \sqrt{\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}}, \quad \delta = \sqrt{\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}},$$

$$\bar{\alpha} = 4\sqrt{\alpha^2+\beta^2}+8\alpha, \quad \bar{\beta} = 4\sqrt{\alpha^2+\beta^2}-8\alpha.$$

It requires a generalization of the conditions (2.7) to construct interaction solutions among rational solutions, solitons, positons, negatons and complexitons, which will be discussed elsewhere. Positons, most of negatons (except solitons) and complexitons exhibit different singularities. A general theory on singularity of the KdV equation needs to be explored.

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