A Unified Structure of Zero Curvature Representations of Integrable Hierarchies

MA Wen-Xiu (马文秀)

(*Institute of Mathematics, Fudan University, Shanghai 200433, PRC*)

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Let $x, t \in R$, $u=(u_1, \cdots, u_q)^T$, $u_i=u_i(x, t)$, $1 \leqslant i \leqslant q$. By $\mathscr{B}$ denote all C^∞-differentiable functions $P(x, t, u)$ and set $\mathscr{B}^r=\{(P_1, \cdots, P_r)^T | P_i \in \mathscr{B}\}$. Moreover, by $\mathscr{V}^r$ denote all C^∞-differentiable linear operators $\Phi=\Phi(x, t, u)$: $\mathscr{B}^r \to \mathscr{B}^{r[1]}$; by $\mathscr{V}^r_{(0)}$, all matrix multiplication operators $W=(W_{ij})_{r \times r}$ with $W_{ij} \in \mathscr{B}$, $1 \leqslant i, j \leqslant r$, and set $\tilde{\mathscr{V}}^r_{(0)}=\mathscr{V}^r_{(0)} \otimes C[\lambda, \lambda^{-1}]$. For $K \in \mathscr{B}^r$, $W \in \tilde{\mathscr{V}}^r_{(0)}$, we define their Gateaux derivatives in a direction $S \in \mathscr{B}^q$ as

$$K'[S]=\frac{\partial}{\partial \varepsilon} K(u+\varepsilon S)|_{\varepsilon=0}, \quad W'[S]=\frac{\partial}{\partial \varepsilon} W(u+\varepsilon S)|_{\varepsilon=0}.$$

The space $\mathscr{B}^q$ of vector fields constitutes an infinite-dimensional Lie algebra with the bilinear operation $[K, S]=K'[S]-S'[K]$ $(K, S \in \mathscr{B}^q)$[2].

Let us consider the spectral problem

$$\begin{cases} \varphi_x=U\varphi=U(u, \lambda)\varphi, \\ \varphi_t=V\varphi=V(u, \lambda)\varphi, \end{cases} \quad U, V \in \tilde{\mathscr{V}}^r_{(0)}. \tag{1}$$

Its integrability condition is a zero curvature equation

$$U_t-V_x+[U, V]=0. \tag{2}$$

Here zero curvature means that the curvature matrix $d\Theta-\Theta \wedge \Theta$ with $\Theta=Udx+Wdt$ takes zero.

Definition 1. If a system of evolution equations

$$u_t=K=K(x, t, u), \quad K \in \mathscr{B}^q \tag{3}$$

is equivalent to a zero curvature Eq. (2), then (2) is called a zero curvature representation of the system (3) and V is called a Lax operator of the system (3).

Zero curvature representations play an important role in the theory of integrable systems. In general, integrable systems may be derived from the zero curvature equation (2), and by using the spectral problem (1), infinitely many conservation laws of corresponding integrable systems can be engendered[3-5].

In the present note, we elaborate a unified structure of zero curvature representations of integrable hierarchies including isospectral ($\lambda_t=0$) and nonisospectral ($\lambda_t=\lambda^n$, $n\geqslant k$) cases. The key point is to convert a recursive property of integrable hierarchies into a characteristic operator equation. By our general structure, we shall give a hierarchy of nonisospectral ($\lambda_t=\lambda^n$, $n\geqslant k$) coupled KdV integrable systems associated with a known hierarchy of isospectral ($\lambda_t=0$) coupled KdV integrable systems, and further for those two hierarchies, we want to display the concrete construction of their zero curvature representations.

In what follows, we always assume that the spectral operator $U=U(u, x, \lambda)\in\mathscr{V}^r_{(0)}$ possesses an injective Gateaux derivative operator U': $\mathscr{B}^q\to\tilde{\mathscr{V}}^r_{(0)}$. Noticing that $U_t=U'[u_t]+\lambda_t U_\lambda$, we can obtain

Proposition 1. *The system* (3) *of evolution equations possesses a zero curvature representation* (2) *if and only if*

$$U'[K]+\lambda_t U_\lambda-V_x+[U, V]=0 \tag{4}$$

holds, which is a connection between the system (3) *and the spectral problem* (1).

First suppose that we have a hierarchy of isospectral integrable systems

$$u_t=K_m=\Phi^m f_0,\ \Phi\in\mathscr{V}^q,\ f_0\in\mathscr{B}^q,\ m\geqslant 0. \tag{5}$$

The operator Φ here is usually a hereditary symmetry determined by the spectral problem $\varphi_x=U\varphi$[6]. For a given vector field $X\in\mathscr{B}^q$, we introduce an operator equation of $\Omega\in\mathscr{V}^r_{(0)}$:

$$U'[\Phi X]-\lambda U'[X]=[\Omega, U]+\Omega_x, \tag{6}$$

and call it the characteristic operator equation of U at X. We assume that (6) has solutions and $\Omega=\Omega(X)$ is a particular solution at X.

Theorem 1. *Let the multiplication operator* $A_0\in\mathscr{V}^r_{(0)}$ *satisfy* $U'[f_0]-A_{0x}+[U, A_0]=0$ *and set*

$$V_m=\sum_{i=0}^{m}\lambda^{m-i}A_i=\lambda^m A_0+\sum_{i=1}^{m}\lambda^{m-i}\Omega(K_{i-1}),\ m\geqslant 0. \tag{7}$$

Then we have

$$U'[K_m]-V_{mx}+[U, V_m]=0,\ m\geqslant 0. \tag{8}$$

Therefore for any $m\geqslant 0$, *the integrable system* $u_t=K_m$ *possesses the isospectral* ($\lambda_t=0$) *zero curvature representation* $U_t-V_{mx}+[U, V_m]=0$.

Proof. Obviously by the hypothesis, the equality (7) holds for $m=0$. Below let $m\geqslant 1$. We may make the following calculation:

$$\begin{aligned}[U, V_m]&=[U, \sum_{i=0}^{m}\lambda^{m-i}A_i]=\sum_{i=0}^{m}\lambda^{m-i}[U, A_i]\\&=\lambda^m[U, A_0]+\sum_{i=1}^{m}\lambda^{m-i}[U, \Omega(K_{i-1})]\\&=\lambda^m(A_{0x}-U'[K_0])+\sum_{i=1}^{m}\lambda^{m-i}(\Omega(K_{i-1})_x-U'[K_i]-\lambda U'[K_{i-1}])\end{aligned}$$

$$= \lambda^m A_{0x} + \sum_{i=1}^{m} \lambda^{m-i} \Omega(K_{i-1})_x - U'[K_m]$$

$$= V_{mx} - U'[K_m],$$

which is just the equality (8). The rest is evident by Proposition 1. The proof is completed.

Next suppose that we have another hierarchy of nonisospectral integrable systems

$$u_t = \sigma_n = \Phi^n g_0, \ g_0 \in \mathscr{B}^q, \ n \geqslant 0, \tag{9}$$

where the operator $\Phi \in \mathscr{L}^q$ is the same as that of the isospectral hierarchy (5).

Theorem 2. *Let the multiplication operator $B_0 \in \widetilde{\mathscr{V}}^r_{(0)}$ satisfy $U'[g_0] + \lambda^k U_\lambda - B_{0x} + [U, B_0] = 0$ and set*

$$W_n = \sum_{j=0}^{n} \lambda^{n-j} B_j = \lambda^n B_0 + \sum_{j=1}^{n} \lambda^{n-j} \Omega(\sigma_{j-1}), \ n \geqslant 0. \tag{10}$$

Then we have

$$U'[\sigma_n] + \lambda^{n+k} U_\lambda - W_{nx} + [U, W_n] = 0, \ n \geqslant 0. \tag{11}$$

Therefore for any $n \geqslant 0$, the integrable system $u_t = \sigma_n$ possesses the nonisospectral ($\lambda_t = \lambda^{n+k}$) zero curvature representation $U_t - W_{nx} + [U, W_n] = 0$.

Proof. The equality (11) for $n=0$ is easily found by the hypothesis. Now let $n \geqslant 1$. Noting the characteristic operator equation (6), we have

$$[U, W_n] = [U, \sum_{j=0}^{n} \lambda^{n-j} B_j] = \sum_{j=0}^{n} \lambda^{n-j} [U, B_j]$$

$$= \lambda^n [U, B_0] + \sum_{j=1}^{n} \lambda^{n-j} [U, \Omega(\sigma_{j-1})]$$

$$= \lambda^n (B_{0x} - U'[\sigma_0] - \lambda^k U_\lambda) + \sum_{j=1}^{n} \lambda^{n-j} (\Omega(\sigma_{j-1})_x - U'[\sigma_j] - \lambda U'[\sigma_{j-1}])$$

$$= \lambda^n B_{0x} - \lambda^{n+k} U_\lambda + \sum_{j=1}^{n} \lambda^{n-j} \Omega(\sigma_{j-1})_x - U'[\sigma_n]$$

$$= W_{nx} - \lambda^{n+k} U_\lambda - U'[\sigma_n],$$

which is just the equality (11). The remaining result is true by Proposition 1. We complete the proof.

Now let us begin to consider an application of the above unified structure to a hierarchy of coupled KdV integrable systems:

$$u_t = K_m = \Phi^m f_0 = \Phi^m u_x, \ m \geqslant 0, \tag{12.1}$$

where

$$\Phi=\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{q-1} \end{bmatrix}, \quad \Phi=\begin{bmatrix} 0 & 0 & \cdots & 0 & R_0^* \\ 1 & 0 & \cdots & 0 & R_1^* \\ 0 & 1 & \cdots & 0 & R_2^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & R_{q-1}^* \end{bmatrix} \tag{12.2}$$

with

$$R_i^*=(v_i-\frac{1}{2}Iv_{ix}+\frac{1}{4}\delta_{il}\partial^2)^*=v_i+\frac{1}{2}v_{ix}I+\frac{1}{4}\delta_{il}\partial^2,\ 0\leqslant i\leqslant q-1,\ I=\partial^{-1}.$$

This hierarchy corresponds to the following spectral problem[7, 8]

$$\psi_{xx}+Q\psi=0,\ Q=Q(u,\ \lambda)=\lambda^{-l}\sum_{i=0}^{q}v_i\lambda^i,\ v_q=-1,\ q\geqslant 1,\ 0\leqslant l\leqslant q-1. \tag{13}$$

By setting $\varphi_1=\psi$, $\varphi_2=\psi_x$, (13) may be represented as

$$\varphi_x=U\varphi,\ \varphi=\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix},\ U=U(u,\ \lambda)=\begin{bmatrix}0 & 1\\ -Q & 0\end{bmatrix}. \tag{14}$$

We easily obtain

$$U'[X]=\begin{bmatrix}0 & 0\\ -\lambda^{-l}\sum_{i=1}^{q}X_i\lambda^{i-1} & 0\end{bmatrix},\quad X=\begin{bmatrix}X_1\\ \vdots \\ X_q\end{bmatrix}\in\mathscr{B}^q, \tag{15}$$

and it follows that U' is injective. The most important thing is to solve the corresponding characteristic operator equation (6). Let

$$X=\begin{bmatrix}X_1\\ \vdots\\ X_q\end{bmatrix}\in\mathscr{B}^q,\ \Phi X=\begin{bmatrix}(\Phi Z)_1\\ \vdots\\ (\Phi X)_q\end{bmatrix},\Omega=\begin{bmatrix}\Omega^{(1)} & \Omega^{(2)}\\ \Omega^{(3)} & -\Omega^{(1)}\end{bmatrix}\in\widetilde{\mathscr{E}}^2_{(0)}.$$

Then we can calculate

$$\begin{aligned}&U'[\Phi X]-\lambda U'[X]\\ &=-\lambda^{-l}\begin{bmatrix}0 & 0\\ (\Phi X)_1+\sum_{i=1}^{q-1}[(\Phi X)_{i+1}-X_i]\lambda^i-X_q\lambda^q & 0\end{bmatrix}\\ &=-\lambda^{-l}\begin{bmatrix}0 & 0\\ \sum_{i=0}^{q-1}(R_i^*X_q)\lambda^i-X_q\lambda^q & 0\end{bmatrix},\end{aligned}$$

$$[\Omega,\ U]+\Omega_x=\begin{bmatrix}-Q\Omega^{(2)}-\Omega^{(3)}+\Omega_x^{(1)} & 2\Omega^{(1)}+\Omega_x^{(2)}\\ 2Q\Omega^{(1)}+\Omega_x^{(3)} & \Omega^{(3)}+Q\Omega^{(2)}-\Omega_x^{(1)}\end{bmatrix}$$

Hence we have

$$\begin{cases} -Q\Omega^{(2)}-\Omega^{(3)}+\Omega_x^{(1)}=0,\ 2\Omega^{(1)}+\Omega_x^{(2)}=0. \\ 2Q\Omega^{(1)}+\Omega_x^{(3)}=-\lambda^{-l}\left[\sum_{i=0}^{q-1}(R_i^* X_q)\lambda^i - X_q\lambda^q\right]. \end{cases}$$

Further we find that

$$2Q\Omega^{(1)}+\Omega_x^{(3)}=-2\left(\frac{1}{2}Q_x+Q\partial+\frac{1}{4}\partial^3\right)\Omega^{(2)}$$

$$=-2\lambda^{-l}\left(\sum_{i=0}^{q-1}\lambda_i R_i^*\partial-\lambda_q\partial\right)\Omega^{(2)}. \tag{16}$$

By (16), we may choose $\Omega^{(2)}=\frac{1}{2}IX_q$, which gives rise to a special solution of the corresponding characteristic operator equation (6):

$$\Omega=\Omega(X)=\begin{bmatrix} -\frac{1}{4}X_q & \frac{1}{2}IX_q \\ -\frac{1}{2}QIX_q-\frac{1}{4}X_{qx} & \frac{1}{4}X_q \end{bmatrix},\quad X=\begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix}\in\mathscr{B}^q. \tag{17}$$

Notice that the simplest isospectral initial Lax operator can be chosen as $A_0=U$. According to Theorem 1, we obtain a hierarchy of isospectral Lax operators $V_m=\lambda^m U+\sum_{i=1}^{m}\lambda^{m-i}\Omega(K_{i-1})$, $m\geqslant 0$, of the coupled KdV integrable hierarchy (12). Particularly, the integrable system

$$u_t=K_1=\Phi f_0=\begin{bmatrix} v_0 v_{q-1,x}+\frac{1}{2}v_{0x}v_{q-1}+\frac{1}{4}\delta_{0l}v_{q-1,xxx} \\ v_{0x}+v_1 v_{q-1,x}+\frac{1}{2}v_{1x}v_{q-1}+\frac{1}{4}\delta_{1l}v_{q-1,xxx} \\ \vdots \\ v_{q-3,x}+v_{q-2}v_{q-1,x}+\frac{1}{2}v_{q-2,x}v_{q-1}+\frac{1}{4}\delta_{q-2,l}v_{q-1,xxx} \\ v_{q-2,x}+\frac{3}{2}v_{q-1,}v_{q-1x}+\frac{1}{4}\delta_{q-1,l}v_{q-1,xxx} \end{bmatrix}$$

has an isospectral Lax operator

$$V_1=\lambda A_0+A_1=\lambda A_0+\Omega(K_0)=\begin{bmatrix} -\frac{1}{4}v_{q-1,x} & \lambda+\frac{1}{2}v_{q-1} \\ -\left(\lambda+\frac{1}{2}v_{q-1}\right)Q-\frac{1}{4}v_{q-1,x} & \frac{1}{4}v_{q-1,x} \end{bmatrix}$$

Below we consider the nonisospectral case. We need to solve the following operator equation with respect to $g_0 \in \mathscr{B}^q$ and $B_0 \in \widetilde{\mathscr{V}}^2_{(0)}$:

$$U'[g_0] + \lambda^k U_\lambda - B_{0x} + [U, B_0] = 0. \tag{18}$$

Let us choose $k=1$ and

$$B_0 = \begin{bmatrix} B_0^{(1)} & B_0^{(2)} \\ B_0^{(3)} & -B_0^{(1)} \end{bmatrix} \in \widetilde{\mathscr{V}}^2_{(0)},\ B_0^{(2)} \in \mathscr{B}$$

Then similar to the deduction of (16), it is found that

$$g_0 = \left(qv_0 + \frac{1}{2}(q-l)xv_{0x}, \cdots, V_{q-1} + \frac{1}{2}(q-l)Xv_{q-1,x}\right)^T, \tag{19}$$

$$B_0 = \begin{bmatrix} -\frac{1}{4}(q-l) & \frac{1}{2}(q-l)x \\ -\frac{1}{2}(q-l)xQ & \frac{1}{4}(q-l) \end{bmatrix} \tag{20}$$

Now the nonisospectral ($\lambda_t = \lambda^{n+1}$) hierarchy of integrable systems reads

$$u_t = \sigma_n = \Phi^n g_0,\ n \geqslant 0, \tag{21}$$

where Φ is given by (12.2) and g_0, by (19). Moreover, for any $n \geqslant 0$, the integrable system $u_t = \sigma_n$ possesses the nonisospectral ($\lambda_t = \lambda^{n+1}$) Lax operator $W_n = \lambda^n B_0 + \sum_{j=1}^{n} \lambda^{n-j} \Omega(\sigma_{j-1})$. When $l=0$, we can also make $k=0$, $B_0=0$. This moment, the operator equation (18) has a solution $g_0 = (-v_1, 2v_2, \cdots, -(q-1)v_{q-1}, q)^T$. For the isospectral coupled KdV hierarchy (12), many integrable properties such as multi-Hamiltonian structures, infinitely many conservation laws, and compatible Poisson brackets have been presented in Refs. [7—10].

Note that the above deduction process also provides an approach for finding nonisospectral ($\lambda_t = \lambda^n$, $n \geqslant k$) integrable hierarchies. For the case of Lax representation, results similar to those of the present note may be found in Refs. [11, 12]. In addition, we would like to point out that the conditions in Theorems 1 and 2 are all natural and do not raise any new requirements. This is because those conditions just require that two initial integrable systems $u_t = f_0$ and $u_t = g_0$ possess isospectral ($\lambda_t = 0$) and nonisospectral ($\lambda_t = \lambda^k$) zero curvature representations, respectively. Two hierarchies of Lax operators determined by (7) and (10) may also constitute an infinite dimensional Lie algebra, which is left to a forthcoming paper. However, the geometrical meaning of the characteristic operator equation (6) and the interrelations between the hereditary symmetry Φ associated with the spectral problem (1) and the operator solution Ω of the characteristic operator equation (6) deserve a further investigation.

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