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THE ALGEBRAIC STRUCTURE RELATED TO *L-A-B* TRIAD REPRESENTATIONS OF INTEGRABLE SYSTEMS*

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Evolution equations integrable by the inverse scattering transform are the compatibility condition of two linear eigenvalue problems

$$\begin{cases} L\psi=\lambda\psi, \ \lambda \text{ spectral parameter}, \\ \psi_t=A\psi. \end{cases}$$

Among these are the well-known KdV equation, AKNS equation in 1+1 dimensions and KP equation, Davey-Stewartson equation in 1+2 dimensions[1,2]. Manakov[3] has presented the *L-A-B* triad representation of the compatibility condition

$$\frac{\partial L}{\partial t}=[A, L]+BL, \tag{1}$$

which is more appropriate for the two-dimensional case. It has been demonstrated that many integrable systems possess this kind of *L-A-B* triad representations[3—8]. An illustrative example is the following integrable system[7]

$$\begin{cases} r_t=\Delta r+\alpha(r^2)_\xi-\beta((\partial_\xi^{-1}r_\eta)^2)_\xi+2\alpha\partial_\eta^{-1}s_{\xi\xi}-2\beta s_\eta, \\ s_t=-\Delta s+2\alpha(rs)_\xi-2\beta(s\partial_\xi^{-1}r_\eta)_\eta, \end{cases} \tag{2}$$

where $\partial_\eta=\partial_x+\partial_y$, $\partial_\xi=\partial_x-\partial_y$, $\Delta=-\alpha\partial_\xi^2+\beta\partial_\eta^2$, $f_\xi=\dfrac{\partial f}{\partial \xi}$, $f_\eta=\dfrac{\partial f}{\partial \eta}$, α and β are arbitrary constants. Three corresponding operators of its *L-A-B* triad representation are as follows:

$$\begin{cases} L=\partial_\eta\partial_\xi+r\partial_\eta+s, \\ A=-\alpha\partial_\xi^2-\beta\partial_\eta^2-2\beta(\partial_\xi^{-1}r_\eta)\partial_\eta-2\alpha\partial_\eta^{-1}s_\xi, \\ B=2\alpha r_\xi-2\beta\partial_\xi^{-1}r_{\eta\eta}=-2\partial_\xi^{-1}\Delta r. \end{cases} \tag{3}$$

When $\alpha=1$ and $\beta=0$, the integrable system (2) is reduced to the two-dimensional dispersive longwave equations[8]

$$r_t=-r_{\xi\xi}+(r^2)_\xi+2\partial_\eta^{-1}s_{\xi\xi}, \quad s_t=s_{\xi\xi}+2(rs)_\xi. \tag{4}$$

In this report, we shall construct the algebraic structure related to general

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((p+1)-dimensional) L-A-B triad representstions displayed in (1). A similar algebraic structure corresponding to the Lax representations has been discussed in Ref. [9].

Let $x\in\mathbb{R}^p$, $u=u(x,t)\in S^q(\mathbb{R}^p,\mathbb{R})$[9]. By $\mathscr{B}$ we denote all complex (or real) functions $P[u]=P(x,t,u)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u=u(x)$[9], and set $\mathscr{B}^r=\{(P_1,\cdots,P_r)^{\mathrm T}|P_i\in\mathscr{B}\}$. By $\mathscr{V}^r$ we denote all linear operators $\Phi=\Phi(x,t,u):\mathscr{B}^r\to\mathscr{B}^r$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u=u(x)$, and by $\mathscr{V}_0^r$ all matrix differential operators $L=L_{(x,t,u)}:\mathscr{B}^r\to\mathscr{B}^r$ with the following form[9]

$$L=(L_{ij})_{r\times r},\ L_{ij}=\sum_{|\alpha|\leqslant\alpha(i,j)}P_\alpha^{ij}[u]D^\alpha,\ P_\alpha^{ij}[u]\in\mathscr{B}. \tag{5}$$

For $\Phi\in\mathscr{V}^r$, we use Φ' to stand for the Gateaux derivative operator of Φ, namely

$$\Phi'[X]=\frac{\partial}{\partial\varepsilon}\Phi(u+\varepsilon X)\Big|_{\varepsilon=0},\ X\in\mathscr{B}^q. \tag{6}$$

In this report, we always assume that $L=L(x,u)\in\mathscr{V}_0^r$ and that $L':\mathscr{B}^q\to\mathscr{V}_0^r$ is an injective mapping. Evidently, if for $X\in\mathscr{B}^q$ there exists a pair of operators A, $B\in\mathscr{V}^r$ such that

$$[A,L]=L'[X]-BL, \tag{7}$$

then the evolution equation $u_t=X$ possesses an L-A-B triad representation (1).

Definition 1. Let A, $B\in\mathscr{V}^r$. If there exists $X\in\mathscr{B}^q$ such that (7) holds, then (A,B) is called a Manakov's pair of operators and X called an eigenvector field corresponding to the Manakov's pair (A,B). Moreover, we denote by $\mathscr{M}_0$ all Manakov's pairs, by $E(\mathscr{M}_0)$ all eigenvector fields, and by $\mathscr{G}_0$ all triples (A,B,X) satisfying (7).

It is easy to see that every Manakov's pair has only one eigenvector field. Therefore for the linear space $\mathscr{M}_0$, we can construct the following multiplication operation.

Definition 2. Let Manakov's pairs (A,B), $(\overline{A},\overline{B})\in\mathscr{M}_0$ have eigenvector fields X, $\overline{X}\in E(\mathscr{M}_0)$, respectively. We define the product of (A,B) and $(\overline{A},\overline{B})$ as

$$[\![(A,B),(\overline{A},\overline{B})]\!]=([\![A,\overline{A}]\!],[\![B,\overline{B}]\!]), \tag{8}$$

where

$$[\![A,\overline{A}]\!]=A'[\overline{X}]-\overline{A}'[X]+[A,\overline{A}], \tag{9}$$

$$[\![B,\overline{B}]\!]=B'[\overline{X}]-\overline{B}'[X]+[B,\overline{B}]+[B,\overline{A}]-[\overline{B},A]. \tag{10}$$

Note that the multiplication operation (8) is obviously a bilinear binary operation.

Theorem 1 *Let* (A,B,X), $(\overline{A},\overline{B},\overline{X})\in\mathscr{G}_0$. *Then* $([\![A,\overline{A}]\!],[\![B,\overline{B}]\!],[X,\overline{X}])\in\mathscr{G}_0$, *where*

$$[X,\overline{X}]=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}(X(u+\varepsilon\overline{X})-\overline{X}(u+\varepsilon X)).$$

Thus $\mathscr{M}_0$ *constitutes an algebra under the multiplication operation* (8).

Proof. Noticing that

$$[[A,\overline{A}],L]=-[[\overline{A},L],A]-[[L,A],\overline{A}]$$
$$=-[L'[\overline{X}],A]+[L'[X],\overline{A}]+[\overline{B}L,A]-[BL,\overline{A}]$$

and that (see Ref. [9])

$$(L'[X])'[\overline{X}]-(L'[\overline{X}])'[X]=L'[Y],\ Y=[X,\overline{X}],$$

we can directly calculate

$$\begin{aligned}[[\![A,\overline{A}]\!],L]&=[A'[\overline{X}]-\overline{A}'[X]+[A,\overline{A}],L]\\
&=[A,L]'[\overline{X}]-[\overline{A},L]'[X]+[\overline{B}L,A]-[BL,\overline{A}]\\
&=(L'[X])'[\overline{X}]-(BL)'[\overline{X}]-(L'[\overline{X}])'[X]+(\overline{B}L)'[X]\\
&\qquad+[\overline{B}L,A]-[BL,\overline{A}]\\
&=L'[[X,\overline{X}]]-B'[\overline{X}]L-BL'[\overline{X}]+\overline{B}'[X]L+\overline{B}L'[X]\\
&\qquad+[\overline{B}L,A]-[BL,\overline{A}]\\
&=L'[[X,\overline{X}]]-B'[\overline{X}]L+\overline{B}'[X]L-B([\overline{A},L]+\overline{B}L)\\
&\qquad+\overline{B}([A,L]+BL)+[\overline{B}L,A]-[BL,\overline{A}]\\
&=L'[[X,\overline{X}]]+(-B'[\overline{X}]+\overline{B}'[X]-[B,\overline{B}])L\\
&\qquad-B[\overline{A},L]+\overline{B}[A,L]+[\overline{B}L,A]-[BL,\overline{A}]\\
&=L'[[X,\overline{X}]]+(-B'[\overline{X}]+\overline{B}'[X]-[B,\overline{B}])L\\
&\qquad+[\overline{A},B]L-[A,\overline{B}]L\\
&=L'[[X,\overline{X}]]-[\![B,\overline{B}]\!]L,\end{aligned}$$

which just shows that $([\![A,\overline{A}]\!],[\![B,\overline{B}]\!],[X,\overline{X}])$ belongs to the space $\mathscr{K}_v$. It follows that the linear space $\mathscr{M}_v$ constitutes an algebra with the multiplication operation (8). The proof is complete.

Corollary 1. *$\langle E(\mathscr{M}_v),\ [\cdot,\cdot]\rangle$ forms a Lie algebra.*

Corollary 1 shows that if two evolution equations $u_t=X$, $u_t=\overline{X}$ ($X,\overline{X}\in\mathscr{B}^q$) all possess L-A-B triad representations, then the evolution equation $u_t=[X,\overline{X}]$ also possesses a L-A-B triad representation.

In the Cartesian product space $\mathscr{V}^r\times\mathscr{V}^r$, we define the following equivalent relation $\sim$:

$(A,B)\sim(\overline{A},\overline{B})\Longleftrightarrow[A,L]+BL=[\overline{A},L]+\overline{B}L$, (A,B), $(\overline{A},\overline{B})\in\mathscr{V}^r\times\mathscr{V}^r$. Set $K_v(L)=\{(A,B)\in\mathscr{V}^r\times\mathscr{V}^r\,|\,[A,L]+BL=0\}$. Obviously $K_v(L)$ is a subalgebra of $\langle\mathscr{M}_v,[\![\cdot,\cdot]\!]\rangle$.

Theorem 2. *The subalgebra $\langle K_v(L),\ [\![\cdot,\cdot]\!]\rangle$ is an ideal subalgebra of $\langle\mathscr{M}_v,[\![\cdot,\cdot]\!]\rangle$.*

Proof. For any $(A,B,X)\in\mathscr{K}_v$, $(\overline{A},\overline{B})\in K_v(L)$, it follows from Theorem 1 that

$$[[\![A,\overline{A}]\!],L]+[\![B,\overline{B}]\!]L=L'[[X,0]]=0,$$
$$[[\![\overline{A},A]\!],L]+[\![\overline{B},B]\!]L=L'[[0,X]]=0.$$

These show that $[\![(A, B), (\overline{A}, \overline{B})]\!], [\![(\overline{A}, \overline{B}), (A, B)]\!]$ all belong to $K_v(L)$. Therefore the statement in the theorem is really true. The proof is complete.

Let $(A, B)\in \mathscr{V}^r\times\mathscr{V}^r$. We use $CL(A, B)$ to stand for the equivalent class to which (A, B) belongs. Set $CL(\mathscr{M}_v)=\{CL(A, B)|(A, B)\in \mathscr{M}_v\}$. By Theorem 2 we see that $CL(\mathscr{M}_v) = \mathscr{M}_v/K_v(L)$ is a quotient algebra, whose multiplication operation is as follows:

$$[\![CL(A, B),\ CL(\overline{A}, \overline{B})]\!]=CL([\![(A, B),\ (\overline{A}, \overline{B})]\!]),\quad (A, B),\ (\overline{A}, \overline{B})\in \mathscr{M}_v. \tag{11}$$

Theorem 3. *The quotient algebra* $\langle CL(\mathscr{M}_v), [\![\cdot\ ,\ \cdot]\!]\rangle$ *is a Lie algebra and isomorphic to Lie algebra* $\langle E(\mathscr{M}_v),\ [\cdot\ ,\ \cdot]\rangle$.

Proof. For any (A_1, B_1, X), (A_2, B_2, Y), $(A_3, B_3, Z)\in \mathscr{S}_v$, by Theorem 1 we have

$$\begin{aligned}&[[\![[\![A_1, A_2]\!], A_3]\!]+\text{cycle}(A_1, A_2, A_3),\ L]\\ &=L'[[[X, Y], Z]+\text{cycle}(X, Y, Z)]-([\![[\![B_1, B_2]\!], B_3]\!]+\text{cycle}(B_1, B_2, B_3))L\\ &=-([\![[\![B_1, B_2]\!], B_3]\!]+\text{cycle}(B_1, B_2, B_3))L.\end{aligned}$$

This shows by (11) that

$$\begin{aligned}&[\![[\![CL(A_1, B_1),\ CL(A_2, B_2)]\!], CL(A_3, B_3)]\!]+\text{cycle}(CL(A_1, B_1),\ CL(A_2, B_2),\\ &\qquad CL(A_3, B_3))=0.\end{aligned}$$

Thus $\langle CL(\mathscr{M}_v),\ [\![\cdot\ ,\ \cdot]\!]\rangle$ constitutes a Lie algebra.

Now let us prove that two Lie algebras are isomorphic to each other. Make $\rho: CL(\mathscr{M}_v) \to E(\mathscr{M}_v)$, $CL(A, B)\mapsto X((A, B, X)\in\mathscr{S}_v)$. Obviously ρ is a linear mapping. Note that for (A, B, X), $(\overline{A}, \overline{B}, \overline{X})\in \mathscr{S}_v$, we have

$$\begin{aligned}&\rho([\![CL(A, B),\ CL(\overline{A}, \overline{B})]\!])=\rho(CL([\![(A, B),\ (\overline{A}, \overline{B})]\!])\\ &=[X, \overline{X}]=[\rho(CL(A, B)),\ \rho(CL(\overline{A}, \overline{B}))].\end{aligned}$$

Hence ρ is an isomorphism of Lie algebras, which implies that Lie algrbras $\langle CL(\mathscr{M}_v), [\![\cdot\ ,\ \cdot]\!]\rangle$ and $\langle E(\mathscr{M}_v),\ [\cdot\ ,\ \cdot]\rangle$ are isomorphic. The proof is complete.

Through the above theorem we easily see that when $L=L(x, u)\in\mathscr{V}_0^r$ is fixed and L' is injective, every evolution equation $u_t=X\ (X\in E(\mathscr{M}_v))$ has just a set of Manakov's pairs $CL(A, B)$ in L-A-B triad representations and no more. However, there exists an open problem in this kind of representations: How do we construct a corresponding Manakov's pair of operators for a given evolution equation $u_t=X\ (X\in E(\mathscr{M}_v))$? In addition, we have not known yet what relations there exist between two sorts of Manakov's pairs of operators corresponding to different spectral operators L_1, L_2 $(L_i=L_i(x, u)\in\mathscr{V}_0^r,\ i=1, 2)$. These need a further investigation.

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