THE ALGEBRAIC STRUCTURE RELATED TO 
L-A-B TRIAD REPRESENTATIONS OF 
INTEGRABLE SYSTEMS*

MA WENXIU (马文秀)
(Institute of Mathematics, Fudan University, Shanghai 200433, PRC)

Received September 5, 1991

Keywords: integrable system, L-A-B triad representation, quotient algebra.

Evolution equations integrable by the inverse scattering transform are the compatibility condition of two linear eigenvalue problems

\[
\begin{align*}
I \psi &= \lambda \psi, \quad \lambda \text{ spectral parameter}, \\
\psi_t &= A \psi.
\end{align*}
\]

Among these are the well-known KdV equation, AKNS equation in 1+1 dimensions and KP equation, Davey-Stewartson equation in 1+2 dimensions. Manakov\cite{1} has presented the L-A-B triad representation of the compatibility condition

\[
\frac{\partial L}{\partial t} = [A, L] + BL,
\]

which is more appropriate for the two-dimensional case. It has been demonstrated that many integrable systems possess this kind of L-A-B triad representations. An illustrative example is the following integrable system\cite{11}

\[
\begin{align*}
\begin{cases}
\xi_t &= \Delta \xi + \alpha (\chi^2)_{xx} - \beta ((\partial_{xx}^{-1} \eta)^2)_{xx} + 2 \alpha \partial_{xx}^{-1} \eta \chi - 2 \beta \partial_{xx}^{-1} \eta \\
\eta_t &= -\Delta \eta + 2 \alpha (m)_{xx} - 2 \beta ((\partial_{xx}^{-1} \eta)^2)_{xx}
\end{cases}
\end{align*}
\]

where \( \partial_x = \partial_x + \partial_y, \partial_{xx} = \partial_x - \partial_y, \Delta = -\alpha \partial_x^2 + \beta \partial_x^2, f_x = \frac{\partial f}{\partial x}, f_\eta = \frac{\partial f}{\partial \eta} \), \( \alpha \) and \( \beta \) are arbitrary constants. Three corresponding operators of its L-A-B triad representation are as follows:

\[
\begin{align*}
L &= \partial_x \partial_{\xi_t} + \partial_\eta \partial_{\eta_t} + s, \\
A &= -\alpha \partial_x^2 - \beta \partial_x^2 + 2 \beta ((\partial_{xx}^{-1} \eta)^2)_{xx} - 2 \alpha \partial_x^{-1} \chi, \\
B &= 2 \alpha \partial_x - 2 \beta \partial_x^2 \partial_{\eta_t} = -2 \partial_x^{-1} \Delta \eta.
\end{align*}
\]

When \( \alpha = 1 \) and \( \beta = 0 \), the integrable system (2) is reduced to the two-dimensional dispersive longwave equations\cite{12}

\[
\begin{align*}
\xi_t &= -\eta_x + (\chi^2)_{xx} + 2 \partial_{\eta}^{-1} \chi \xi_x, \\
\eta_t &= \eta_{xx} + 2 (\eta \chi)_{xx}.
\end{align*}
\]

In this report, we shall construct the algebraic structure related to general

\* Project supported by the National Science Foundation of Postdoctor of China.
Let $x \in \mathbb{R}^n$, $u = u(x, t) \in \mathcal{S}^n([0, t])$. By $\mathcal{A}$ we denote all complex (or real) functions $P[u] = P(x, t, u)$ which are $C^\infty$-differentiable with respect to $x$, $t$ and $C^\infty$-Gâteaux differentiable with respect to $u = u(x)$. By $\mathcal{Y}'$, we denote all linear operators $\Phi = \Phi(x, t, u) : \mathcal{A} \to \mathcal{Y}'$ which are $C^\infty$-differentiable with respect to $x$, $t$ and $C^\infty$-Gâteaux differentiable with respect to $u = u(x)$, and by $\mathcal{Y}_0$ all matrix differential operators $L = L(x, \Theta) : \mathcal{Y}' \to \mathcal{Y}'$ with the following form:

$$L = (L_0)_{x, u}, \quad L_0 = \sum_{\mu \in A(u)} P^{[\alpha]}_{\mu} [u] D^\alpha, \quad P^{[\alpha]}_{\mu}[u] \in \mathcal{A}. \quad (5)$$

For $\Phi \in \mathcal{Y}'$, we use $\Phi'$ to stand for the Gâteaux derivative operator of $\Phi$, namely

$$\Phi'[X] = \frac{\partial}{\partial \varepsilon} \Phi(u + \varepsilon X) \bigg|_{\varepsilon=0}, \quad X \in \mathcal{A}. \quad (6)$$

In this report, we always assume that $L = L(x, u) \in \mathcal{Y}_0$ and that $L' : \mathcal{Y} \to \mathcal{Y}_0$ is an injective mapping. Evidently, if for $X \in \mathcal{A}$ there exists a pair of operators $A, B \in \mathcal{Y}'$ such that

$$[A, L] = L'[X] - BL, \quad (7)$$

then the evolution equation $u = X$ possesses an $L \cdot A \cdot B$ triad representation (1).

**Definition 1.** Let $A, B \in \mathcal{Y}'$. If there exists $X \in \mathcal{A}$ such that (7) holds, then $(A, B)$ is called a Manakov's pair of operators and $X$ called an eigenvector field corresponding to the Manakov's pair $(A, B)$. Moreover, we denote by $\mathcal{M}$ all Manakov's pairs, by $E(\mathcal{M})$ all eigenvector fields, and by $\mathcal{N}$ all triples $(A, B, X)$ satisfying (7).

It is easy to see that every Manakov's pair has only one eigenvector field. Therefore for the linear space $\mathcal{M}$, we can construct the following multiplication operation.

**Definition 2.** Let Manakov's pairs $(A, B), (\tilde{A}, \tilde{B}) \in \mathcal{M}$ have eigenvector fields $X, \tilde{X} \in E(\mathcal{M})$, respectively. We define the product of $(A, B)$ and $(\tilde{A}, \tilde{B})$ as

$$[(A, B), (\tilde{A}, \tilde{B})] = ([A, \tilde{A}], [B, \tilde{B}]), \quad (8)$$

where

$$[A, \tilde{A}] = A'[\tilde{X}] - \tilde{A}'[X] + [A, \tilde{A}], \quad (9)$$

$$[B, \tilde{B}] = B'[\tilde{X}] - \tilde{B}'[X] + [B, \tilde{B}] + [B, \tilde{A}] - [\tilde{B}, A]. \quad (10)$$

Note that the multiplication operation (8) is obviously a bilinear binary operation.

**Theorem 1** Let $(A, B, X), (\tilde{A}, \tilde{B}, \tilde{X}) \in \mathcal{M}$. Then $([A, \tilde{A}], [B, \tilde{B}], [X, \tilde{X}]) \in \mathcal{M}$, where

$$[X, \tilde{X}] = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} (X(u + \varepsilon \tilde{X}) - \tilde{X}(u + \varepsilon X)).$$

Thus $\mathcal{M}$ constitutes an algebra under the multiplication operation (8).
Proof. Noticing that
\[[[A, \overline{A}], L] = -([[A, L], A] - [[L, A], \overline{A}])
= -[[L'(X), A] + [L'(X), \overline{A}] + [BL, A] - [BL, \overline{A}]\]
and that (see Ref. [9])
\[(L'(X)Y(X) - (L'(X)Y(X)) = L'(Y), Y = [X, \overline{X}],\]
we can directly calculate
\[[[A, \overline{A}], L] = [A'([X] - \overline{X}) + A, \overline{A}], L]
= [A, L]'X - [A, L]'X + [BL, A] - [BL, \overline{A}]
= (L'(X)'X - (BL)'X) - (L'(X)'X) + (BL)'X + [BL, A] - [BL, \overline{A}]
\]
\[= L'([X, \overline{X}]) - B'([X] + \overline{B}[X]) + \overline{B'}[X] + \overline{B}[X] + [BL, A] - [BL, \overline{A}]
\]
which just shows that \([[A, \overline{A}], [B, \overline{B}]], [X, \overline{X}]\) belongs to the space \(\mathcal{S}\). It follows that the linear space \(\mathcal{A}\) constitutes an algebra with the multiplication operation (8). The proof is complete.

**Corollary 1.** \(<E(\mathcal{A}), [\cdot, \cdot]\) forms a Lie algebra.

Corollary 1 shows that if two evolution equations \(u = X, \overline{u} = \overline{X}\) of all possess \(A-B\) triad representations, then the evolution equation \(u = [X, \overline{X}]\) also possesses a \(A-B\) triad representation.

In the Cartesian product space \(\mathcal{S} \times \mathcal{S} \times \mathcal{S}\), we define the following equivalent relation \(\sim:\)
\[(A, B) \sim (\overline{A}, \overline{B}) \iff [A, L] + BL = [\overline{A}, L] + \overline{B[L, A]]} + [BL, A] - [BL, \overline{A}].\]
Set \(K_s(L) = \{(A, B) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S} | [A, L] + BL = 0\}\). Obviously \(K_s(L)\) is a subalgebra of \(<\mathcal{S}, [\cdot, \cdot]\>\).

**Theorem 2.** The subalgebra \(<K_s(L), [\cdot, \cdot]\>\) is an ideal subalgebra of \(<\mathcal{S}, [\cdot, \cdot]\>\).

**Proof.** For any \((A, B, X) \in \mathcal{S}, (\overline{A}, \overline{B}) \in K_s(L)\), it follows from Theorem 1 that
\[[[A, \overline{A}], L] + [B, \overline{B}] L = L'[[X, 0]] = 0;
[[A, \overline{A}], L] + [\overline{B}, B] L = L'[[0, X]] = 0.\]
These show that \([ (A, B), (\overline{A}, \overline{B}) ] \), \([ (\overline{A}, \overline{B}), (A, B) ] \) all belong to \(K_e(L) \). Therefore the statement in the theorem is really true. The proof is complete.

Let \((A, B) \in \mathcal{X} \times \mathcal{X} \). We use \(CL(A, B) \) to stand for the equivalent class to which \((A, B) \) belongs. Set \(CL(\mathcal{X}) = \{ (CL(A, B))(A, B) \in \mathcal{X} \} \). By Theorem 2 we see that \(CL(\mathcal{X}) = \mathcal{X} / K_e(L) \) is a quotient algebra, whose multiplication operation is as follows:

\[
[CL(A, B), CL(\overline{A}, \overline{B})] = CL([[(A, B), (\overline{A}, \overline{B})]], (A, B), (\overline{A}, \overline{B}) \in \mathcal{X}).
\] (11)

**Theorem 3.** The quotient algebra \(\langle CL(\mathcal{X}), [\cdot, \cdot] \rangle \) is a Lie algebra and isomorphic to Lie algebra \(\langle E(\mathcal{X}), [\cdot, \cdot] \rangle \).

**Proof.** For any \((A_1, B_1, X), (A_2, B_2, Y), (A_3, B_3, Z) \in \mathcal{X} \), by Theorem 1 we have

\[
[[(A_1, A_2), A_3]] + \text{cycle}(A_1, A_2, A_3), L)
\]

\[
= L' [[[(X, Y), Z]] + \text{cycle}(X, Y, Z)] - (\text{cycle}(B_1, B_2, B_3) + \text{cycle}(B_1, B_2, B_3)) L
\]

\[
= - (\text{cycle}(B_1, B_2, B_3) + \text{cycle}(B_1, B_2, B_3)) L.
\]

This shows by (11) that

\[
[[CL(A_1, B_1), CL(A_2, B_2)], CL(A_3, B_3)] + \text{cycle}(CL(A_1, B_1), CL(A_2, B_2), CL(A_3, B_3)) = 0
\]

Thus \(\langle CL(\mathcal{X}), [\cdot, \cdot] \rangle \) constitutes a Lie algebra.

Now let us prove that two Lie algebras are isomorphic to each other. Make \(\rho: CL(\mathcal{X}) \rightarrow E(\mathcal{X}), CL(A, B) \mapsto X((A, B, X) \in \mathcal{X}) \). Obviously \(\rho \) is a linear mapping. Note that for \((A, B, X), (\overline{A}, \overline{B}, \overline{X}) \in \mathcal{X} \), we have

\[
\rho([CL(A, B), CL(\overline{A}, \overline{B})]) = \rho(CL([[(A, B), (\overline{A}, \overline{B})]]))
\]

\[
= [X, \overline{X}] = \rho(CL(\overline{A}, B))\rho(CL(\overline{A}, \overline{B})).
\]

Hence \(\rho \) is an isomorphism of Lie algebras, which implies that Lie algebras \(\langle CL(\mathcal{X}), [\cdot, \cdot] \rangle \) and \(\langle E(\mathcal{X}), [\cdot, \cdot] \rangle \) are isomorphic. The proof is complete.

Through the above theorem we easily see that when \(L = L(x, u) \in \mathcal{X} \) is fixed and \(L' \) is injective, every evolution equation \(u_t = X(XE(\mathcal{X})) \) has just a set of Manakov's pairs \(CL(A, B) \) in \(L' \)-A-B triad representations and no more. However, there exists an open problem in this kind of representations: How do we construct a corresponding Manakov's pair of operators for a given evolution equation \(u_t = X(X \in E(\mathcal{X})) \)? In addition, we have not known yet what relations there exist between two sorts of Manakov's pairs of operators corresponding to different spectral operators \(L_1, L_2 \) \((L_i = L_i(x, u) \in \mathcal{X}, i = 1, 2) \). These need a further investigation.

The author would like to thank Profs. Gu Chaohao and Hu Hesheng for their enthusiastic guidance.

**References**


[2] Calogero, F.
1982; 1986


[4] Zakharov,

[5] Veselov A. I


[7] Konopelchen

[8] Bolti, M.,

[9] Ma, W. X.
(L): Therefore

as to which (A) that \( CL(\mathcal{M}_e) \)

ows:

\[ \mathcal{Y} \in \mathcal{M}_e \quad (11) \]

nd isomorphic to

em 1 we have

\[ B_3 \} L \]

\[ A_2 , B_2 \} \]

ike \( \rho: CL(\mathcal{M}_e) \)

3: Note that for

ms \( \langle CL(\mathcal{M}_e) \). \)

fixed and \( L' \) is

anakov's pairs

an open prob-

anakov's pair

we have not

of operators

=1, 2). These

their enthusiastic

hia 1981.