Matrix Integrable Fourth-Order Nonlinear Schrödinger Equations and Their Exact Soliton Solutions

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We construct matrix integrable fourth-order nonlinear Schrödinger equations through reducing the Ablowitz–Kaup–Newell–Segur matrix eigenvalue problems. Based on properties of eigenvalue and adjoint eigenvalue problems, we solve the corresponding reflectionless Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues, and formulate their soliton solutions via those reflectionless Riemann–Hilbert problems. Soliton solutions are computed for three illustrative examples of scalar and two-component integrable fourth-order nonlinear Schrödinger equations.

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Abundant nonlinear phenomena in physical sciences and engineering can be described by integrable models.[1] When lower-order perturbations are taken into account, two prototypical examples of integrable models, the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation, are introduced to learn and identify wave propagation in nonlinear media.[2] By presenting integrable models as the compatibility conditions (i.e., zero curvature equations), matrix eigenvalue problems are used to solve their Cauchy problems by the inverse scattering transform.[3,4]

By taking account of the invariance of zero curvature equations under group constraints, nonlocal integrable models can be generated from Lax pairs of matrix eigenvalue problems. Such integrable models keep the integrable structures that the original integrable models exhibit (see, e.g., Refs. [5–7] for nonlocal reduced integrable models). If one group constraint is taken into consideration, we can formulate three classes of local integrable models, i.e., one class of NLS type and two classes of mKdV type,[8,9] and five classes of nonlocal integrable models, i.e., three classes of NLS type and two classes of mKdV type.[10] On the other hand, Riemann–Hilbert problems have been widely applied for constructing soliton solutions.[11,12] Indeed, many local and nonlocal integrable models have been investigated by considering the associated Riemann–Hilbert problems (see, e.g., Refs. [13–15] and Refs. [7,16–19] for details in the local and nonlocal cases, respectively).

It should also be noted that the standard integrable NLS model of second-order plays a significant role in nonlinear optics, used as the governing equation for the propagation of wave field envelope in weakly nonlinear dispersive media (see, e.g., Ref. [20]). Its vector and matrix integrable generalizations were proposed, for which there exist higher-order symmetries;[21] and its continuous, semi-discrete and fully discrete matrix versions were analyzed and solved in terms of bidifferential graded algebras.[22] A fractional generalized NLS equation with a combined second- and third-order nonlinearity was recently studied, with its symmetric and antisymmetric soliton solutions being constructed and the influence of the Lévy index on diverse solitons being analyzed.[23] A (3+1)-dimensional normalized NLS equation with variable-coefficients was used to describe light bullets in a tapered graded-index waveguide with parity-time-symmetric potentials.[24] An energy-conservation deep-learning method was constructed to study a coupled system of NLS equations and to analyze its formation mechanism of vector solitons in birefringent fibers.[25] Moreover, higher-order analogous integrable models were formulated and studied to explore nonlinear dispersive waves in optical fibers and water waves (see, e.g., Refs. [20,26]).

In this Letter, we present a paradigmatic example of applications of Riemann–Hilbert problems. We formulate a class of reduced matrix hierarchies of integrable models of AKNS type, including reduced matrix integrable fourth-order models of NLS type, and apply the Riemann–Hilbert technique to construction of their soliton solutions. Specifically, we present a kind of group constraints of the AKNS matrix eigenvalue problems to formulate reduced matrix hierarchies of integrable models of AKNS type, including matrix integrable fourth-order models of NLS type. Based on properties of eigenvalue and adjoint eigenvalue problems, we solve the corresponding reflectionless Riemann–Hilbert problems, in which eigenvalues could be equal to adjoint eigenvalues, and compute soliton solutions to the

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resulting reduced matrix integrable models of AKNS type, particularly to one scalar integrable fourth-order model of NLS type and two two-component systems of integrable fourth-order models of NLS type. The scalar integrable fourth-order model reads

$$\begin{aligned}-ip_{1,t} = & p_{1,xxxx} + 6\sigma p_1^* p_{1,x}^2 + 4\sigma p_1 |p_{1,x}|^2 + 8\sigma |p_1|^2 p_{1,xx} \\ & + 2\sigma p_1^2 p_{1,xx}^* + 6\sigma^2 |p_1|^4 p_1, \end{aligned} \quad (1)$$

where $\sigma \in \mathbb{C}$ is an arbitrary nonzero constant, $*$ stands for the complex conjugation, and $|p_1|$ denotes the absolute value of p_1. The final section concludes our findings and gives several remarks for future studies.

Reduced Matrix Integrable mKdV Hierarchies—The Matrix AKNS Integrable Hierarchies Revisited. To express the subsequent analysis clearly, let us first recall the AKNS hierarchies of matrix integrable models. As usual, we assume that p and q denote two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n}, \quad q = q(x,t) = (q_{kj})_{n\times m}, \quad (2)$$

where m, n are two given natural numbers. The matrix AKNS eigenvalue problems are defined by

$$\begin{aligned} & -i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda \Lambda + P)\phi, \\ & -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi = (\lambda^r \Omega + Q^{[r]})\phi, \ r\ge 0, \end{aligned} \quad (3)$$

where λ stands for the spectral parameter. The constant square matrices Λ and Ω are given by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n), \quad \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \quad (4)$$

where I_s denotes the identity matrix of size s, and α_1, α_2 and β_1, β_2 are two arbitrary pairs of distinct real constants. Moreover, the other two involved square matrices of size $m+n$ are defined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad (5)$$

called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \quad (6)$$

where $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ are determined recursively by

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a^{[0]} = \beta_1 I_m, \quad d^{[0]} = \beta_2 I_n, \quad (7a)$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p), \quad s\ge 0, \quad (7b)$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q), \quad s\ge 0, \quad (7c)$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q), \quad d_x^{[s]} = i(qb^{[s]} - c^{[s]}p), \quad s\ge 1, \quad (7d)$$

with $\alpha = \alpha_1 - \alpha_2$ and zero constants of integration being chosen. Based on the relations in (7), we can see that

$$W = \sum_{s\ge 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \quad (8)$$

presents a solution to the stationary zero curvature equation

$$W_x = i[U, W], \quad (9)$$

which plays a crucial role in defining an integrable hierarchy.

By using computer algebra systems, we can directly work out all sets of $a^{[s]}$, $b^{[s]}$, $c^{[s]}$ and $d^{[s]}$, $s\ge 1$, which are differential polynomials of p and q with respect to x. For example, we can have

$$\begin{aligned} b^{[5]} = & \frac{\beta}{\alpha^5}[p_{xxxx} + 4pqp_{xx} + (6p_xq + 2pq_x)p_x \\ & + (4p_{xx}q + 2p_xq_x + 2pq_{xx} + 6pqpq)p], \\ c^{[5]} = & \frac{\beta}{\alpha^5}[q_{xxxx} + 4q_{xx}pq + q_x(6pq_x + 2p_xq) \\ & + q(4pq_{xx} + 2p_xq_x + 2p_{xx}q + 6pqpq)], \\ a^{[5]} = & \frac{\beta}{\alpha^5} i[6pq(pq_x - p_xq) + pq_{xxx} - p_{xxx}q \\ & + p_{xx}q_x - p_xq_{xx}], \\ d^{[5]} = & \frac{\beta}{\alpha^5} i[2q(p_xq - pq_x)p + 4qpqp_x - 4q_xpqp + qp_{xxx} \\ & - q_{xxx}p + q_{xx}p_x - q_xp_{xx}]; \end{aligned}$$

where $\beta = \beta_1 - \beta_2$. Now, we can see the zero-curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r\ge 0, \quad (10)$$

which are the compatibility conditions of the two matrix eigenvalue problems in (3), generate one so-called matrix AKNS integrable hierarchy (see, e.g., Ref. [27]):

$$p_t = i\alpha b^{[r+1]}, \quad q_t = -i\alpha c^{[r+1]}, \quad r\ge 0, \quad (11)$$

which possesses a bi-Hamiltonian structure. The nonlinear integrable models with $r = 2$ and $r = 4$ in the hierarchy give us the matrix NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \ q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq), \quad (12)$$

and the matrix fourth-order NLS equations:

$$\begin{aligned} p_t = & \frac{\beta}{\alpha^4} i[p_{xxxx} + 4pqp_{xx} + 2(3p_xq + pq_x)p_x \\ & + 2(2p_{xx}q + p_xq_x + pq_{xx} + 3pqpq)p], \\ q_t = & -\frac{\beta}{\alpha^4} i[q_{xxxx} + 4q_{xx}pq + 2q_x(3pq_x + p_xq) \\ & + 2q(2pq_{xx} + p_xq_x + p_{xx}q + 3pqpq)], \end{aligned} \quad (13)$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$ are arbitrary constants, and the two matrix potentials, p and q, are defined by (2).

Reduced Matrix AKNS Integrable Hierarchies. We would like to construct a kind of novel reduced matrix integrable AKNS models by taking a kind of group constraints for the matrix AKNS eigenvalue problems in (3) (see Refs. [8,28] for other applications).

Assume that Σ_1 and Σ_2 are a pair of constant invertible Hermitian matrices of sizes m and n, respectively. Let us consider a kind of group constraints for the spectral matrix U:

$$U^\dagger(x,t,\lambda^*) = [U(x,t,\lambda^*)]^\dagger = \Sigma U(x,t,\lambda)\Sigma^{-1}, \quad (14)$$

where $*$ denotes the complex conjugation, as indicated in the introduction, $\dagger$ denotes the Hermitian transpose, and the constant invertible Hermitian matrix Σ is defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}. \quad (15)$$

It is easy to see that every group constraint requires

$$P^{\dagger}(x,t)=\Sigma P(x,t)\Sigma^{-1}, \tag{16}$$

and more precisely, it requires either of the constraints for the two matrix potentials:

$$q(x,t)=\Sigma_2^{-1}p^{\dagger}(x,t)\Sigma_1, \tag{17}$$
$$p(x,t)=\Sigma_1^{-1}q^{\dagger}(x,t)\Sigma_2. \tag{18}$$

Moreover, we observe that under the group constraints in (14), we can have

$$W^{\dagger}(x,t,\lambda)=[W(x,t,\lambda)]^{\dagger}=\Sigma W(x,t,\lambda)\Sigma^{-1}, \tag{19}$$

and this implies that we have

$$V^{[r]\dagger}(x,t,\lambda^*)=[V^{[r]}(x,t,\lambda^*)]^{\dagger}=\Sigma V^{[r]}(x,t,\lambda)\Sigma^{-1}, \tag{20}$$

or equivalently, we have

$$Q^{[r]\dagger}(x,t,\lambda^*)=[Q^{[r]}(x,t,\lambda^*)]^{\dagger}=\Sigma Q^{[r]}(x,t,\lambda)\Sigma^{-1}, \tag{21}$$

where $r\geq 0$.

Consequently, it is direct to see that under one potential constraint defined by Eq. (17) or (18), the integrable matrix AKNS models in (11) reduce to one hierarchy of reduced matrix integrable AKNS models:

$$p_t=i\alpha b^{[r+1]}|_{q=\Sigma_2^{-1}p^{\dagger}\Sigma_1},\quad r\geq 0, \tag{22}$$

where p is an $m\times n$ matrix potential, or

$$q_t=-i\alpha c^{[r+1]}|_{p=\Sigma_1^{-1}q^{\dagger}\Sigma_2},\quad r\geq 0, \tag{23}$$

where q is an $n\times m$ matrix potential. In the above reduced matrix integrable models, Σ_1 and Σ_2 are a pair of arbitrary invertible Hermitian matrices of sizes m and n, respectively. Each reduced equation in the hierarchy (22) or (23) with a fixed integer $r\geq 0$ possesses a Lax pair of the reduced spatial and temporal matrix eigenvalue problems in (3), and infinitely many symmetries and conserved densities from those for the integrable matrix AKNS models in (11).

Reduced Matrix Integrable Fourth-Order NLS Equations. If we take $r=2$ and $r=4$, then the reduced matrix integrable AKNS models in (22) give a kind of matrix integrable second-order NLS equations

$$p_t=-\frac{\beta}{\alpha^2}i(p_{xx}+2p\Sigma_2^{-1}p^{\dagger}\Sigma_1 p), \tag{24}$$

and a kind of matrix integrable matrix fourth-order NLS equations:

$$\begin{aligned}p_t=\frac{\beta}{\alpha^4}i[&p_{xxxx}+4p\Sigma_2^{-1}p^{\dagger}\Sigma_1 p_{xx}+2(3p_x\Sigma_2^{-1}p^{\dagger}\Sigma_1\\&+p\Sigma_2^{-1}p_x^{\dagger}\Sigma_1)p_x+2(2p_{xx}\Sigma_2^{-1}p^{\dagger}\Sigma_1+p_x\Sigma_2^{-1}p_x^{\dagger}\Sigma_1\\&+p\Sigma_2^{-1}p_{xx}^{\dagger}\Sigma_1)p+6(p\Sigma_2^{-1}p^{\dagger}\Sigma_1)^2p],\end{aligned} \tag{25}$$

where p is an $m\times n$ matrix potential, and Σ_1 and Σ_2 are two arbitrary constant invertible Hermitian matrices of sizes m and n, respectively.

In what follows, we are going to compute three illustrative examples of these novel reduced matrix integrable fourth-order NLS equations, by selecting different values for m,n and appropriate choices for Σ.

Let us first consider $m=n=1$. When we take

$$\Sigma_1=1,\qquad \Sigma_2^{-1}=\sigma, \tag{26}$$

we obtain the following scalar integrable fourth-order NLS equation

$$\begin{aligned}p_{1,t}=\frac{\beta}{\alpha^4}i(&p_{1,xxxx}+6\sigma p_1^*p_{1,x}^2+4\sigma p_1|p_{1,x}|^2\\&+8\sigma|p_1|^2p_{1,xx}+2\sigma p_1^2p_{1,xx}^*+6\sigma^2|p_1|^4p_1),\end{aligned} \tag{27}$$

where $p=(p_1)$ and $\sigma\neq 0$ is an arbitrary complex constant. A special case of this equation with $\sigma=1$ has been solved by Darboux transformation.[29]

Let us second consider $m=1$ and $n=2$. When we take

$$\Sigma_1=1,\quad \Sigma_2^{-1}=\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}, \tag{28}$$

we obtain a two-component system of integrable fourth-order NLS equations:

$$\begin{aligned}p_{1,t}=\frac{\beta}{\alpha^4}i[&p_{1,xxxx}+4(2\sigma_1|p_1|^2+\sigma_2|p_2|^2)p_{1,xx}\\&+4\sigma_2p_1p_2^*p_{2,xx}+2\sigma_1p_1^2p_{1,xx}^*+2\sigma_2p_1p_2p_{2,xx}^*\\&+6\sigma_1p_1^*p_{1,x}^2+2(2\sigma_1p_1p_{1,x}^*+3\sigma_2p_2^*p_{2,x}\\&+\sigma_2p_2p_{2,x}^*)p_{1,x}+2\sigma_2|p_{2,x}|^2p_1\\&+6(\sigma_1|p_1|^2+\sigma_2|p_2|^2)^2p_1],\\
p_{2,t}=\frac{\beta}{\alpha^4}i[&p_{2,xxxx}+4\sigma_1p_1^*p_2p_{1,xx}+4(\sigma_1|p_1|^2\\&+2\sigma_2|p_2|^2)p_{2,xx}+2\sigma_1p_1p_2p_{1,xx}^*+2\sigma_2p_2^2p_{2,xx}^*\\&+6\sigma_2p_2^*p_{2,x}^2+2(3\sigma_1p_1^*p_{1,x}+2\sigma_2p_2p_{2,x}^*\\&+\sigma_1p_1p_{1,x}^*)p_{2,x}+2\sigma_1|p_{1,x}|^2p_2\\&+6(\sigma_1|p_1|^2+\sigma_2|p_2|^2)^2p_2],\end{aligned} \tag{29}$$

where $p=(p_1,p_2)$, and σ_1, σ_2 are two arbitrary nonzero complex constants.

Let us third consider $m=1$ and $n=2$. When we take

$$\Sigma_1=1,\ \Sigma_2^{-1}=\begin{bmatrix}0 & \sigma_1\\ \sigma_2 & 0\end{bmatrix}, \tag{30}$$

we obtain another two-component system of integrable fourth-order NLS equations:

$$\begin{aligned}p_{1,t}=\frac{\beta}{\alpha^4}i[&p_{1,xxxx}+4(2\sigma_1p_1p_2^*+\sigma_2p_1^*p_2)p_{1,xx}\\&+4\sigma_2|p_1|^2p_{2,xx}+2\sigma_2p_1p_2p_{1,xx}^*+2\sigma_1p_1^2p_{2,xx}^*\\&+6\sigma_1p_2^*p_{1,x}^2+2(2\sigma_1p_1p_{2,x}^*+3\sigma_2p_1^*p_{2,x}\\&+\sigma_2p_2p_{1,x}^*)p_{1,x}+2\sigma_2p_{1,x}^*p_{2,x}p_1\\&+6(\sigma_1p_1p_2^*+\sigma_2p_1^*p_2)^2p_1],\\
p_{2,t}=\frac{\beta}{\alpha^4}i[&p_{2,xxxx}+4\sigma_1|p_2|^2p_{1,xx}\\&+4(\sigma_1p_1p_2^*+2\sigma_2p_1^*p_2)p_{2,xx}+2\sigma_2p_2^2p_{1,xx}^*\\&+2\sigma_1p_1p_2p_{2,xx}^*+6\sigma_2p_1^*p_{2,x}^2+2(3\sigma_1p_2^*p_{1,x}\\&+2\sigma_2p_2p_{1,x}^*+\sigma_1p_1p_{2,x}^*)p_{2,x}+2\sigma_1p_{1,x}p_{2,x}^*p_2\\&+6(\sigma_1p_1p_2^*+\sigma_2p_1^*p_2)^2p_2],\end{aligned} \tag{31}$$

where $p=(p_1,p_2)$, and σ_1, σ_2 are two arbitrary nonzero complex constants.

Soliton Solutions via Riemann–Hilbert Problems—Properties of Eigenvalue and Adjoint Eigenvalue Problems. It is easy to see that under the group constraints in (14), we can see that λ is an eigenvalue of the matrix eigenvalue problems in (3) if and only if $\hat\lambda=\lambda^*$ is an adjoint eigenvalue, i.e., the adjoint matrix eigenvalue problems hold:

$$i\tilde\phi_x=\tilde\phi U=\tilde\phi U(u,\hat\lambda),\quad i\tilde\phi_t=\tilde\phi V^{[r]}=\tilde\phi V^{[r]}(u,\hat\lambda),\tag{32}$$

where $r\geq 0$.

Moreover, under each group constraint in (14), if $\phi(\lambda)$ presents an eigenfunction of the matrix eigenvalue problems in (3) associated with an eigenvalue λ, then $\phi^\dagger(\lambda^*)\varSigma$ defines an adjoint eigenfunction associated with the same eigenvalue λ.

Solutions to Reflectionless Riemann–Hilbert Problems. We would like to formulate solutions to the corresponding reflectionless Riemann–Hilbert problems.

Let $N\geq 0$ be an arbitrarily given integer. First, we take N eigenvalues λ_k and N adjoint eigenvalues $\hat\lambda_k$:

$$\lambda_k,\ 1\leq k\leq N:\ \mu_1,\ldots,\ \mu_N,\tag{33}$$

and

$$\hat\lambda_k,\ 1\leq k\leq N:\ \mu_1^*,\cdots,\ \mu_N^*,\tag{34}$$

where $\mu_k\in\mathbb{C}$, $1\leq k\leq N$, and v_k denote their corresponding eigenfunctions and adjoint eigenfunctions by

$$v_k,\ 1\leq k\leq N,\ \text{and}\ \hat v_k,\ 1\leq k\leq N,\tag{35}$$

respectively. If we do not assume

$$\{\lambda_k\,|\,1\leq k\leq N\}\cap\{\hat\lambda_k\,|\,1\leq k\leq N\}=\emptyset,$$

then we have to use the following generalized solutions to the reflectionless Riemann–Hilbert problems:

$$\begin{aligned}G^+(\lambda)&=I_{m+n}-\sum_{k,l=1}^N\frac{v_k(M^{-1})_{kl}\hat v_l}{\lambda-\hat\lambda_l},\\(G^-)^{-1}(\lambda)&=I_{m+n}+\sum_{k,l=1}^N\frac{v_k(M^{-1})_{kl}\hat v_l}{\lambda-\lambda_k},\end{aligned}\tag{36}$$

where M is a square matrix of size N, whose entries are given by

$$m_{kl}=\begin{cases}\frac{\hat v_k v_l}{\lambda_l-\hat\lambda_k}, & \text{if}\quad \lambda_l\neq\hat\lambda_k,\\ 0, & \text{if}\quad \lambda_l=\hat\lambda_k,\end{cases}\tag{37}$$

where $1\leq k,\ l\leq N$.

Indeed, as shown in Ref. [17], these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the reflectionless Riemann–Hilbert problem:

$$(G^-)^{-1}(\lambda)G^+(\lambda)=I_{m+n},\ \lambda\in\mathbb{R},\tag{38}$$

provided that we have the orthogonal condition:

$$\hat v_k v_l=0\quad\text{if}\quad\lambda_l=\hat\lambda_k,\tag{39}$$

where $1\leq k,\ l\leq N$.

Soliton Solutions. Let us take zero potentials in the matrix eigenvalue problems in (3). Then, we can obtain

$$v_k=v_k(x,t,\lambda_k)=\mathrm{e}^{i\lambda_k\varLambda x+i\lambda_k^{r+1}\varOmega t}w_k,\ 1\leq k\leq N,\tag{40}$$

and following the above analysis, we can assume to take

$$\begin{aligned}\hat v_k&=\hat v_k(x,t,\hat\lambda_k)=v_k^\dagger(x,t,\lambda_k^*)\varSigma\\&=\hat w_k\mathrm{e}^{-i\hat\lambda_k\varLambda x-i\hat\lambda_k^{r+1}\varOmega t},\\\hat w_k&=w_k^\dagger\varSigma,\qquad 1\leq k\leq N,\end{aligned}\tag{41}$$

where w_k $(1\leq k\leq N)$ are arbitrary constant column vectors. Furthermore, the orthogonal condition (39) becomes

$$w_k^\dagger\varSigma w_l=0\ \text{if}\ \lambda_l=\hat\lambda_k,\tag{42}$$

where $1\leq k,\ l\leq N$.

A standard step to compute soliton solutions is to make an asymptotic expansion

$$G^+(\lambda)=I_{m+n}+\frac{1}{\lambda}G_1^+ +\mathrm{O}\Big(\frac{1}{\lambda^2}\Big),\tag{43}$$

as $\lambda\to\infty$, we obtain

$$G_1^+=-\sum_{k,l=1}^N v_k(M^{-1})_{kl}\hat v_l.\tag{44}$$

A substitution of this into the matrix spatial eigenvalue problems recovers the potential matrix:

$$P=-[\varLambda,G_1^+]=\lim_{\lambda\to\infty}\lambda[G^+(\lambda),\varLambda].\tag{45}$$

This provides us with the N-soliton solutions to the matrix integrable AKNS models in (11):

$$\begin{aligned}p&=\alpha\sum_{k,l=1}^N v_k^1(M^{-1})_{kl}\hat v_l^2,\\q&=-\alpha\sum_{k,l=1}^N v_k^2(M^{-1})_{kl}\hat v_l^1.\end{aligned}\tag{46}$$

In the above solution formulas, we have split $v_k=((v_k^1)^{\mathrm{T}},(v_k^2)^{\mathrm{T}})^{\mathrm{T}}$ and $\hat v_k=(\hat v_k^1,\hat v_k^2)$ for each $1\leq k\leq N$, where v_k^1 and v_k^2 are column vectors of dimensions m and n, respectively, while $\hat v_k^1$ and $\hat v_k^2$ are row vectors of dimensions m and n, respectively.

To compute N-soliton solutions to the reduced matrix integrable AKNS models in (22), we must check if G_1^+ defined by Eq. (44) satisfies the involution property or not:

$$(G_1^+)^{\mathrm{T}}=-\varSigma G_1^+\varSigma^{-1}.\tag{47}$$

This property exactly means that the resulting potential matrix P determined by Eq. (45) will satisfy the group constraint condition in (16). Accordingly, the above N-soliton solutions to the matrix AKNS integrable models in (11) are reduced to the following N-soliton solutions:

$$p=\alpha\sum_{k,l=1}^N v_k^1(M^{-1})_{kl}\hat v_l^2,\tag{48}$$

to the reduced matrix integrable AKNS models in (22). To sum up, if we have the orthogonal condition for w_k, $1\leq k\leq N$, in (42) and the involution property in (47), then the formula (48), together with (36), (37), (40) and

(41), provides N-soliton solutions to the reduced matrix integrable AKNS models in (22), particularly to the matrix integrable NLS equations in (25).

Lastly, we would like to present three examples of one-soliton solutions in the cases of $m = n = 1$ and $m = n/2 = 1$. Let us consider $\lambda_1 = \mu$, $\hat{\lambda}_1 = \mu^*$, where $\mu \in \mathbb{C}$, and define

$$w_1 = (w_{1,1}, w_{1,2})^{\rm T}, w_{1,1}, w_{1,2} \in \mathbb{R}, \quad \text{for } n = 1, \tag{49}$$

and

$$\begin{aligned} &w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^{\rm T}, \\ &w_{1,1}, w_{1,2}, w_{1,3} \in \mathbb{C}, \quad \text{for } n = 2. \end{aligned} \tag{50}$$

Following the above general formulation of soliton solutions, the first situation leads to a class of one-soliton solutions to the integrable fourth-order NLS Eq. (27) with $\sigma = 1$:

$$\begin{aligned} p_1 = &[(\alpha_1 - \alpha_2)(\mu - \mu^*) w_{1,1} w_{1,2}] \\ &\cdot \big[w_{1,1}^2 {\rm e}^{-i(\alpha_1-\alpha_2)\mu^* x - i(\beta_1-\beta_2)\mu^{*4} t} \\ &+ w_{1,2}^2 {\rm e}^{-i(\alpha_1-\alpha_2)\mu x - i(\beta_1-\beta_2)\mu^4 t}\big]^{-1}, \end{aligned} \tag{51}$$

where $\mu \in \mathbb{C}, w_{1,1}, w_{1,2} \in \mathbb{R}$ are arbitrary nonzero constants. The second situation yields the following one-soliton solutions to the integrable fourth-order NLS equations in (29):

$$\begin{aligned} p_1 &= \frac{(\alpha_1 - \alpha_2)(\mu - \mu^*) w_{1,2}^*}{\sigma_1 w_{1,1}^* {\rm e}^{-i(\alpha_1-\alpha_2)\mu^* x - i(\beta_1-\beta_2)\mu^{*4} t}}, \\ p_2 &= \frac{(\alpha_1 - \alpha_2)(\mu - \mu^*) w_{1,3}^*}{\sigma_2 w_{1,1}^* {\rm e}^{-i(\alpha_1-\alpha_2)\mu^* x - i(\beta_1-\beta_2)\mu^{*4} t}}, \end{aligned} \tag{52}$$

where $w_{1,2}, w_{1,3} \in \mathbb{C}$ need to satisfy the condition

$$\sigma_1 |w_{1,}|^2 + \sigma_2 |w_{1,2}|^2 = 0; \tag{53}$$

and the following one-soliton solutions to the integrable fourth-order equations in (31):

$$\begin{aligned} p_1 &= \frac{(\alpha_1 - \alpha_2)(\mu - \mu^*) w_{1,3}^*}{\sigma_1 w_{1,1}^* {\rm e}^{-i(\alpha_1-\alpha_2)\mu^* x - i(\beta_1-\beta_2)\mu^{*4} t}}, \\ p_2 &= \frac{(\alpha_1 - \alpha_2)(\mu - \mu^*) w_{1,2}^*}{\sigma_2 w_{1,1}^* {\rm e}^{-i(\alpha_1-\alpha_2)\mu^* x - i(\beta_1-\beta_2)\mu^{*4} t}}, \end{aligned} \tag{54}$$

where $w_{1,2}, w_{1,3} \in \mathbb{C}$ need to satisfy the condition

$$\sigma_1 w_{1,2} w_{1,3}^* + \sigma_2 w_{1,2}^* w_{1,3} = 0. \tag{55}$$

The conditions in (53) and (55) are just consequences of the involution property in (47).

Concluding Remarks. A kind of reduced matrix local integrable AKNS models, including matrix integrable fourth-order NLS equations, and their soliton solutions have been constructed. The formulation of soliton solutions has been established by using the associated Riemann–Hilbert problems. Three illustrative examples of the resulting matrix integrable fourth-order NLS equations have been worked out, together with their one-soliton solutions.

We point out that there is another kind of group constraints, through which one can create local reduced integrable mKdV equations from the AKNS matrix eigenvalue problems.[9] Therefore, there are more diverse matrix integrable mKdV equations than matrix integrable NLS equations. In the nonlocal case, the situation becomes different.[10]

We also remark that it is very interesting to explore more reduced local integrable models by different kinds of group constraints from other Lax pairs,[30,31] integrable couplings[32] and variable-coefficient integrable models.[33] When conducting group constraints, we can assume to include the shifts of potentials:

$$\begin{aligned} U^\dagger(x + x_0, t + t_0, \lambda^*) &= [U(x + x_0, t + t_0, \lambda^*)]^\dagger \\ &= \varSigma U(x, t, , \lambda) \varSigma^{-1}, \end{aligned} \tag{56}$$

where x_0, x_0', t_0, t_0' are arbitrary real constants, to generate diverse reduced integrable models. It is also surely important to study dynamical properties of exact analytical solutions, including lump and breather wave solutions,[34,35] rogue wave solutions,[36,37] Wronskian solutions,[38,39] algebro-geometric solutions[40,41] and solitonless solutions,[42] from a perspective of the Riemann–Hilbert technique. Such studies will supplement the existing theory on fourth-order NLS equations and their applications.[43–45]

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