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Matrix integrable fifth-order mKdV equations and their soliton solutions

Wen-Xiu Ma(马文秀)[1,2,3,4,†]

[1] *Department of Mathematics, Zhejiang Normal University, Jinhua* 321004, *China*
[2] *Department of Mathematics, King Abdulaziz University, Jeddah* 21589, *Saudi Arabia*
[3] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL* 33620-5700, *USA*
[4] *School of Mathematical and Statistical Sciences, North-West University, Mafikeng Campus, Private Bag X*2046, *Mmabatho* 2735, *South Africa*

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We consider matrix integrable fifth-order mKdV equations via a kind of group reductions of the Ablowitz–Kaup–Newell–Segur matrix spectral problems. Based on properties of eigenvalue and adjoint eigenvalue problems, we solve the corresponding Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues, and construct their soliton solutions, when there are zero reflection coefficients. Illustrative examples of scalar and two-component integrable fifth-order mKdV equations are given.

Keywords: matrix integrable equation, Riemann–Hilbert problem, soliton

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1. Introduction

Integrable equations describe diverse nonlinear phenomena in applied and engineering sciences.[1] Two paradigmatic examples are the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation, both local and nonlocal (see, e.g., Refs. [2,3]). Lax pairs of matrix spectral problems play a fundamental role in solving their Cauchy problems by the inverse scattering transform.

Group reductions of matrix spectral problems can produce reduced integrable equations and keep the integrable structures that the original integrable equations exhibit (see, e.g., Refs. [4–6] for nonlocal reduced integrable equations). When one group reduction is taken, we obtain a kind of local matrix integrable NLS equations and two kinds of local matrix integrable mKdV equations,[2,7] based on two classes of local group reductions; and three kinds of nonlocal matrix integrable NLS equations and two kinds of nonlocal matrix integrable mKdV equations, based on nine classes of nonlocal group reductions.[3,4]

The Riemann–Hilbert technique has been shown to be one of powerful methods to solve integrable equations, and particularly to construct soliton solutions.[8,9] Various kinds of integrable equations have been studied by the associated Riemann–Hilbert problems (see, e.g., Refs. [10–12] and Refs. [6,13–15] for details in the local and nonlocal cases, respectively). On the other hand, higher-order integrable equations are introduced to describe nonlinear dispersive waves in optical fibers and water waves (see, e.g., Refs. [16,17]). In this paper, we would like to construct a kind of reduced matrix mKdV hierarchies, including reduced matrix integrable fifth-order mKdV equations. Riemann–Hilbert problems will be used to present soliton solutions to the resulting reduced matrix integrable mKdV equations of odd order.

The rest of this paper is organized as follows. In Section 2, we make a kind of group reductions of the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems to generate reduced matrix integrable mKdV hierarchies. In Section 3, based on properties of eigenvalue and adjoint eigenvalue problems, we solve the corresponding reflectionless Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues, and compute soliton solutions to the resulting reduced matrix integrable odd-order mKdV equations, particularly to the scalar integrable fifth-order mKdV equation and a two-component system of integrable fifth-order mKdV equations. The final section gives a conclusion and a few concluding remarks.

2. Reduced matrix integrable mKdV hierarchies

2.1. The matrix AKNS integrable hierarchies revisited

Let us first recall the AKNS hierarchies of matrix integrable equations, which will be used in the subsequent analysis. As normal, we assume that p and q are two matrix potentials:

$$\begin{aligned} p &= p(x,t) = (p_{jk})_{m\times n}, \\ q &= q(x,t) = (q_{kj})_{n\times m}, \end{aligned} \tag{1}$$

where $m,n \geq 1$ are two given integers. We take the matrix AKNS spectral problems as follows:

$$\begin{aligned} -\mathrm{i}\phi_x &= U\phi = U(u,\lambda)\phi = (\lambda\Lambda + P)\phi, \\ -\mathrm{i}\phi_t &= V^{[r]}\phi = V^{[r]}(u,\lambda)\phi = (\lambda^r\Omega + Q^{[r]})\phi, \quad r \geq 0, \end{aligned} \tag{2}$$

†Corresponding author. E-mail: mawx@cas.usf.edu

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where λ is the spectral parameter. Moreover, the constant square matrices Λ and Ω are defined by

$$\Lambda=\mathrm{diag}(\alpha_1 I_m,\alpha_2 I_n),\quad \Omega=\mathrm{diag}(\beta_1 I_m,\beta_2 I_n), \tag{3}$$

with I_s being the identity matrix of size s, and α_1,α_2 and β_1,β_2 being two arbitrary pairs of distinct real constants. The other two involved square matrices of size $m+n$ are defined by

$$P=P(u)=\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{4}$$

called the potential matrix, and

$$Q^{[r]}=\sum_{s=0}^{r-1}\lambda^s\begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{5}$$

where $a^{[s]}$, $b^{[s]}$, $c^{[s]}$, and $d^{[s]}$ are defined as follows:

$$b^{[0]}=0,\ c^{[0]}=0,\ a^{[0]}=\beta_1 I_m,\ d^{[0]}=\beta_2 I_n, \tag{6a}$$

$$b^{[s+1]}=\frac{1}{\alpha}(-\mathrm{i}b_x^{[s]}-pd^{[s]}+a^{[s]}p),\ s\ge 0, \tag{6b}$$

$$c^{[s+1]}=\frac{1}{\alpha}(\mathrm{i}c_x^{[s]}+qa^{[s]}-d^{[s]}q),\ s\ge 0, \tag{6c}$$

$$a_x^{[s]}=\mathrm{i}(pc^{[s]}-b^{[s]}q),\ \ d_x^{[s]}=\mathrm{i}(qb^{[s]}-c^{[s]}p),\ s\ge 1, \tag{6d}$$

with $\alpha=\alpha_1-\alpha_2$ and zero constants of integration being selected. The relations in (6) also imply that

$$W=\sum_{s\ge 0}\lambda^{-s}\begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{7}$$

solves the stationary zero curvature equation

$$W_x=\mathrm{i}[U,W], \tag{8}$$

which is crucial in defining an integrable hierarchy. Particularly, we can work out that

$$\begin{aligned}
b^{[0]} &= 0,\ \ c^{[0]}=0,\ \ a^{[0]}=\beta_1 I_m,\ \ d^{[0]}=\beta_2 I_n;\\
b^{[1]} &= \frac{\beta}{\alpha}p,\ \ c^{[1]}=\frac{\beta}{\alpha}q,\ \ a^{[1]}=0,\ \ d^{[1]}=0;\\
b^{[2]} &= -\frac{\beta}{\alpha^2}\mathrm{i}p_{xx},\ \ c^{[2]}=\frac{\beta}{\alpha^2}\mathrm{i}q_{xx},\\
a^{[2]} &= -\frac{\beta}{\alpha^2}pq,\ \ d^{[2]}=\frac{\beta}{\alpha^2}qp;\\
b^{[3]} &= -\frac{\beta}{\alpha^3}(p_{xx}+2pqp),\ \ c^{[3]}=-\frac{\beta}{\alpha^3}(q_{xx}+2qpq),\\
a^{[3]} &= -\frac{\beta}{\alpha^3}\mathrm{i}(pq_x-p_xq),\ \ d^{[3]}=-\frac{\beta}{\alpha^3}\mathrm{i}(qp_x-q_xp);\\
b^{[4]} &= \frac{\beta}{\alpha^4}\mathrm{i}(p_{xxx}+3pqp_x+3p_xqp),\\
c^{[4]} &= -\frac{\beta}{\alpha^4}\mathrm{i}(q_{xxx}+3q_xpq+3qpq_x),\\
a^{[4]} &= \frac{\beta}{\alpha^4}(3pqpq+pq_{xx}-p_xq_x+p_{xx}q),\\
d^{[4]} &= -\frac{\beta}{\alpha^4}(3qpqp+qp_{xx}-q_xp_x+q_{xx}p);\\
b^{[5]} &= \frac{\beta}{\alpha^5}[p_{xxxx}+4pqp_{xx}+(6p_xq+2pq_x)p_x\\
&\quad +(4p_{xx}q+2p_xq_x+2pq_{xx}+6pqpq)p],\\
c^{[5]} &= \frac{\beta}{\alpha^5}[q_{xxxx}+4q_{xx}pq+q_x(6pq_x+2p_xq)\\
&\quad +q(4pq_{xx}+2p_xq_x+2p_{xx}q+6pqpq)],\\
a^{[5]} &= \frac{\beta}{\alpha^5}\mathrm{i}[6pq(pq_x-p_xq)\\
&\quad +pq_{xxx}-p_{xxx}q+p_{xx}q_x-p_xq_{xx}],\\
d^{[5]} &= \frac{\beta}{\alpha^5}\mathrm{i}[2q(p_xq-pq_x)p+4qpqp_x-4q_xpqp\\
&\quad +qp_{xxx}-q_{xxx}p+q_{xx}p_x-q_xp_{xx}];\\
b^{[6]} &= -\frac{\beta}{\alpha^6}\mathrm{i}(p_{5x}+5p_{xxx}qp+5pqp_{xxx}+10p_{xx}qp_x+10p_xqp_{xx}\\
&\quad +5p_{xx}q_xp+5pq_xp_{xx}+5p_xq_{xx}p+5pq_{xx}p_x+10p_xq_xp_x\\
&\quad +18p_xqpqp+2pqp_xqp+10pqpqp_x),\\
c^{[6]} &= \frac{\beta}{\alpha^6}\mathrm{i}(q_{5x}+5q_{xxx}pq+5qpq_{xxx}+10q_{xx}pq_x+10q_xpq_{xx}\\
&\quad +5q_{xx}p_xq+5qp_xq_{xx}+5q_xp_{xx}q+5qp_{xx}q_x+10q_xp_xq_x\\
&\quad +10q_xpqpq+2qpq_xpq+18qpqpq_x),\\
a^{[6]} &= -\frac{\beta}{\alpha^6}[p_{xxxx}q+pq_{xxxx}-p_{xxx}q_x-p_xq_{xxx}+p_{xx}q_{xx}\\
&\quad +10p_{xx}qpq+10pqpq_{xx}+5p_xqp_xq\\
&\quad +5pq_xpq_x+10(pq)^3],\\
d^{[6]} &= \frac{\beta}{\alpha^6}(qp_{xxxx}+q_{xxxx}p-q_xp_{xxx}-q_{xxx}p_x+q_{xx}p_{xx}\\
&\quad +5q_{xx}pqp+5qp_{xx}qp+5qpq_{xx}p\\
&\quad +5qpqp_{xx}+5qp_xqp_x+5q_xpq_xp\\
&\quad +5qp_xq_xp-5q_xpqp_x+10qpqpqp),
\end{aligned}$$

with $\beta=\beta_1-\beta_2$.

The compatibility conditions of the two matrix spectral problems in (2), i.e., the zero curvature equations

$$U_t-V_x^{[r]}+\mathrm{i}[U,V^{[r]}]=0,\ \ r\ge 0, \tag{9}$$

generate one so-called matrix AKNS integrable hierarchy (see, e.g., Ref. [18])

$$p_t=\mathrm{i}\alpha b^{[r+1]},\ \ q_t=-\mathrm{i}\alpha c^{[r+1]},\ r\ge 0, \tag{10}$$

which has a bi-Hamiltonian structure. The nonlinear integrable equations with $r=3$ and $r=5$ in the hierarchy give us the AKNS matrix mKdV equations

$$\begin{aligned}
p_t &= -\frac{\beta}{\alpha^3}(p_{xxx}+3pqp_x+3p_xqp),\\
q_t &= -\frac{\beta}{\alpha^3}(q_{xxx}+3q_xpq+3qpq_x),
\end{aligned} \tag{11}$$

and the AKNS matrix fifth-order mKdV equations

$$\begin{aligned}
p_t &= \frac{\beta}{\alpha^5}(p_{5x}+5p_{xxx}qp+5pqp_{xxx}+10p_{xx}qp_x+10p_xqp_{xx}\\
&\quad +5p_{xx}q_xp+5pq_xp_{xx}+5p_xq_{xx}p+5pq_{xx}p_x+10p_xq_xp_x\\
&\quad +18p_xqpqp+2pqp_xqp+10pqpqp_x),
\end{aligned}$$

$$
\begin{aligned}
q_t = {} & \frac{\beta}{\alpha^5}(q_{5x} + 5q_{xxx}pq + 5qpq_{xxx} + 10q_{xx}pq_x + 10q_xpq_{xx} \\
& + 5q_{xx}p_xq + 5qp_xq_{xx} + 5q_xp_{xx}q + 5qp_{xx}q_x + 10q_xp_xq_x \\
& + 10q_xpqpq + 2qpq_xpq + 18qpqpq_x),
\end{aligned} \tag{12}
$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$ are arbitrary nonzero constants, and the two matrix potentials p and q are defined in Eq. (1).

2.2. Reduced matrix integrable mKdV hierarchies

We would like to construct a kind of novel reduced matrix integrable mKdV equations of odd order by taking one kind of group reductions for the matrix AKNS spectral problems in Eq. (2), which is different from the one in Refs. [2,19].

Assume that Σ_1 and Σ_2 are a pair of constant invertible symmetric matrices of sizes m and n, respectively. Let us consider a kind of group reductions for the spectral matrix U:

$$
U^{\mathrm{T}}(x,t,-\lambda) = (U(x,t,-\lambda))^{\mathrm{T}} = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{13}
$$

where the constant invertible matrix Σ is defined by

$$
\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}. \tag{14}
$$

It is easy to see that every group reduction requires

$$
P^{\mathrm{T}}(x,t) = -\Sigma P(x,t)\Sigma^{-1}, \tag{15}
$$

and more precisely, it requires either of the reductions for the two matrix potentials:

$$
q(x,t) = -\Sigma_2^{-1}p^{\mathrm{T}}(x,t)\Sigma_1, \tag{16}
$$

$$
p(x,t) = -\Sigma_1^{-1}q^{\mathrm{T}}(x,t)\Sigma_2. \tag{17}
$$

Moreover, we observe that under each group reduction defined in Eq. (13), we can have that

$$
W^{\mathrm{T}}(x,t,-\lambda) = (W(x,t,-\lambda))^{\mathrm{T}} = \Sigma W(x,t,\lambda)\Sigma^{-1}, \tag{18}
$$

and this implies that we have

$$
\begin{aligned}
V^{[2s+1]\mathrm{T}}(x,t,-\lambda) &= (V^{[2s+1]}(x,t,-\lambda))^{\mathrm{T}} \\
&= -\Sigma V^{[2s+1]}(x,t,\lambda)\Sigma^{-1},
\end{aligned} \tag{19}
$$

or equivalently, we have

$$
\begin{aligned}
Q^{[2s+1]\mathrm{T}}(x,t,-\lambda) &= (Q^{[2s+1]}(x,t,-\lambda))^{\mathrm{T}} \\
&= -\Sigma Q^{[2s+1]}(x,t,\lambda)\Sigma^{-1},
\end{aligned} \tag{20}
$$

where $s \geq 0$.

Consequently, it is direct to see that under one potential reduction defined by Eq. (16) or Eq. (17), the integrable matrix AKNS equations in (10) with $r = 2s+1,\ s \geq 0$, reduce to one hierarchy of reduced matrix integrable mKdV equations of odd order

$$
p_t = \mathrm{i}\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1}, \quad s \geq 0, \tag{21}
$$

where p is an $m \times n$ matrix potential, or

$$
q_t = -\mathrm{i}\alpha c^{[2s+2]}|_{p=-\Sigma_1^{-1}q^{\mathrm{T}}\Sigma_2}, \quad s \geq 0, \tag{22}
$$

with q being an $n \times m$ matrix potential. In the above reduced matrix integrable equations, Σ_1 and Σ_2 are a pair of arbitrary invertible symmetric matrices of sizes m and n, respectively. Each reduced equation in the hierarchy (21) or (22) with a fixed integer $s \geq 0$ possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in Eq. (2) with $r = 2s+1$, and infinitely many symmetries and conserved densities from those for the integrable matrix AKNS equations in Eq. (10) with $r = 2s+1$.

If we take $s = 2$, namely, $r = 5$, then the reduced matrix integrable mKdV type equations (21) give a kind of reduced matrix integrable fifth-order mKdV equations

$$
\begin{aligned}
p_t = {} & \frac{\beta}{\alpha^5}(p_{5x} - 5p_{xxx}\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p - 5p\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p_{xxx} \\
& - 10p_{xx}\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p_x - 10p_x\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p_{xx} \\
& - 5p_{xx}\Sigma_2^{-1}p_x^{\mathrm{T}}\Sigma_1 p - 5p\Sigma_2^{-1}p_x^{\mathrm{T}}\Sigma_1 p_{xx} - 5p_x\Sigma_2^{-1}p_{xx}^{\mathrm{T}}\Sigma_1 p \\
& - 5p\Sigma_2^{-1}p_{xx}^{\mathrm{T}}\Sigma_1 p_x - 10p_x\Sigma_2^{-1}p_x^{\mathrm{T}}\Sigma_1 p_x \\
& + 18p_x\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p \\
& + 2p\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p_x\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p \\
& + 10p\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p\Sigma_2^{-1}p^{\mathrm{T}}\Sigma_1 p_x),
\end{aligned} \tag{23}
$$

where p is an $m \times n$ matrix potential, and Σ_1 and Σ_2 are two arbitrary constant invertible symmetric matrices of sizes m and n, respectively.

In what follows, we are going to compute two illustrative examples of these novel reduced matrix integrable fifth-order mKdV equations, by selecting different values for m, n and appropriate choices for Σ.

Let us first consider $m = n = 1$. When we set

$$
\Sigma_1 = 1, \quad \Sigma_2^{-1} = -\sigma, \tag{24}
$$

we obtain the scalar integrable fifth-order mKdV equation

$$
\begin{aligned}
p_{1,t} = {} & \frac{\beta}{\alpha^5}(p_{1,5x} + 10\sigma p_1^2 p_{1,xxx} + 40\sigma p_1 p_{1,x} p_{1,xx} \\
& + 10\sigma p_{1,x}^3 + 30\sigma^2 p_1^4 p_{1,x}),
\end{aligned} \tag{25}
$$

where $p = (p_1)$, and $\sigma \neq 0$ is an arbitrary complex constant.

Let us secondly consider $m = 1$ and $n = 2$. When we set

$$
\Sigma_1 = 1, \quad \Sigma_2^{-1} = -\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \tag{26}
$$

we obtain a two-component system of integrable fifth-order mKdV equations

$$\begin{aligned} p_{1,t} &= \frac{\beta}{\alpha^5}[p_{1,5x}+5(2\sigma_1 p_1^2+\sigma_2 p_2^2)p_{1,xxx}+5\sigma_2 p_1 p_2 p_{2,xxx}+5(8\sigma_1 p_1 p_{1,x}+3\sigma_2 p_2 p_{2,x})p_{1,xx} \\ &\quad +5\sigma_2(2p_1 p_{2,x}+3p_2 p_{1,x})p_{2,xx}+10(\sigma_1 p_{1,x}^2+\sigma_2 p_{2,x}^2)p_{1,x}+10(\sigma_1 p_1^2+\sigma_2 p_2^2)(3\sigma_1 p_1^2+\sigma_2 p_2^2)p_{1,x} \\ &\quad +20\sigma_2 p_1 p_2(\sigma_1 p_1^2+\sigma_2 p_2^2)p_{2,x}], \\ p_{2,t} &= \frac{\beta}{\alpha^5}[p_{2,5x}+5\sigma_1 p_1 p_2 p_{1,xxx}+5(\sigma_1 p_1^2+2\sigma_2 p_2^2)p_{2,xxx}+5\sigma_1(3p_1 p_{2,x}+2p_2 p_{1,x})p_{1,xx} \\ &\quad +5(3\sigma_1 p_1 p_{1,x}+8\sigma_2 p_2 p_{2,x})p_{2,xx}+10(\sigma_1 p_{1,x}^2+\sigma_2 p_{2,x}^2)p_{2,x}+20\sigma_1 p_1 p_2(\sigma_1 p_1^2+\sigma_2 p_2^2)p_{1,x} \\ &\quad +10(\sigma_1 p_1^2+\sigma_2 p_2^2)(\sigma_1 p_1^2+3\sigma_2 p_2^2)p_{2,x}], \end{aligned} \tag{27}$$

where $p=(p_1,p_2)$, and σ_1 and σ_2 are two arbitrary nonzero complex constants.

3. Soliton solutions via Riemann–Hilbert problems

3.1. Properties of eigenvalue and adjoint eigenvalue problems

Under the group reductions in Eq. (13), we can see that λ is an eigenvalue of the matrix spectral problems in Eq. (2) if and only if $\hat{\lambda}=-\lambda$ is an adjoint eigenvalue, i.e., the adjoint matrix spectral problems hold:

$$\mathrm{i}\tilde{\phi}_x=\tilde{\phi}U=\tilde{\phi}U(u,\hat{\lambda}),\ \ \mathrm{i}\tilde{\phi}_t=\tilde{\phi}V^{[r]}=\tilde{\phi}V^{[r]}(u,\hat{\lambda}), \tag{28}$$

where $r=2s+1,\ s\geq 0$.

On the other hand, under each group reduction in Eq. (13), if $\phi(\lambda)$ is an eigenfunction of the matrix spectral problems in Eq. (2) associated with an eigenvalue λ, then we can see that $\phi^{\mathrm{T}}(-\lambda)\Sigma$ gives an adjoint eigenfunction associated with the same eigenvalue λ.

3.2. Solutions to reflectionless Riemnn–Hilbert problems

We would like to establish a formulation of solutions to the corresponding reflectionless Riemann–Hilbert problems.

Let $N\geq 0$ be an arbitrary integer. First, let us take N eigenvalues λ_k and N adjoint eigenvalues $\hat{\lambda}_k$ as follows:

$$\lambda_k,\ 1\leq k\leq N:\ \mu_1,\ \ldots,\ \mu_N, \tag{29}$$

$$\hat{\lambda}_k,\ 1\leq k\leq N:\ -\mu_1,\ \ldots,\ -\mu_N, \tag{30}$$

where $\mu_k\in\mathbb{C},\ 1\leq k\leq N$, and assume that their corresponding eigenfunctions and adjoint eigenfunctions are denoted by

$$v_k,\ 1\leq k\leq N,\ \text{ and }\ \hat{v}_k,\ 1\leq k\leq N, \tag{31}$$

respectively. If we do not have the property

$$\{\lambda_k\,|\,1\leq k\leq N\}\cap\{\hat{\lambda}_k\,|\,1\leq k\leq N\}=\emptyset,$$

we need to define the following generalized solutions to reflectionless Riemann–Hilbert problems:

$$G^+(\lambda)=I_{m+n}-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},$$

$$(G^-)^{-1}(\lambda)=I_{m+n}+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_k}, \tag{32}$$

where M is a square matrix of size N, with its entries being defined by

$$m_{kl}=\begin{cases}\frac{\hat{v}_k v_l}{\lambda_l-\hat{\lambda}_k}, & \text{if } \lambda_l\neq\hat{\lambda}_k,\\ 0, & \text{if } \lambda_l=\hat{\lambda}_k,\end{cases}\qquad 1\leq k,l\leq N. \tag{33}$$

As shown in Ref. [14], these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the reflectionless Riemann–Hilbert problem

$$(G^-)^{-1}(\lambda)G^+(\lambda)=I_{m+n},\ \ \lambda\in\mathbb{R}, \tag{34}$$

when we have the orthogonal condition

$$\hat{v}_k v_l=0\ \text{ if }\ \lambda_l=\hat{\lambda}_k, \tag{35}$$

where $1\leq k,\,l\leq N$.

3.3. Soliton solutions

When we take zero potentials in the matrix spectral problems in Eq. (2), we can obtain

$$v_k=v_k(x,t,\lambda_k)=\mathrm{e}^{\mathrm{i}\lambda_k\Lambda x+\mathrm{i}\lambda_k^{2s+1}\Omega t}w_k,\ \ 1\leq k\leq N, \tag{36}$$

and following the preceding analysis, we can take

$$\begin{aligned} \hat{v}_k &= \hat{v}_k(x,t,\hat{\lambda}_k)=v_k^{\mathrm{T}}(x,t,-\lambda_k)\Sigma \\ &= \hat{w}_k\mathrm{e}^{-\mathrm{i}\hat{\lambda}_k\Lambda x-\mathrm{i}\hat{\lambda}_k^{2s+1}\Omega t}, \\ \hat{w}_k &= w_k^{\mathrm{T}}\Sigma,\ \ 1\leq k\leq N, \end{aligned} \tag{37}$$

where $w_k,\ 1\leq k\leq N$, are arbitrary constant column vectors. In this way, the orthogonal condition (35) becomes

$$w_k^{\mathrm{T}}\Sigma w_l=0\ \text{ if }\ \lambda_l=\hat{\lambda}_k, \tag{38}$$

where $1\leq k,\,l\leq N$.

Now, as normal, making an asymptotic expansion

$$G^+(\lambda)=I_{m+n}+\frac{1}{\lambda}G_1^+ +\mathrm{O}\left(\frac{1}{\lambda^2}\right), \tag{39}$$

as $\lambda\to\infty$, we arrive at

$$G_1^+=-\sum_{k,l=1}^{N}v_k(M^{-1})_{kl}\hat{v}_l, \tag{40}$$

and further, substituting this into the matrix spatial spectral problems, we get the potential matrix

$$P=-[\Lambda,G_1^+]=\lim_{\lambda\to\infty}[G^+(\lambda),\Lambda]. \tag{41}$$

This gives rise to the N-soliton solutions to the matrix integrable AKNS equations (10)

$$p=\alpha\sum_{k,l=1}^{N}v_k^1(M^{-1})_{kl}\hat{v}_l^2,\ \ q=-\alpha\sum_{k,l=1}^{N}v_k^2(M^{-1})_{kl}\hat{v}_l^1. \tag{42}$$

In the above expressions, we have split $v_k=((v_k^1)^{\mathrm{T}},(v_k^2)^{\mathrm{T}})^{\mathrm{T}}$ and $\hat{v}_k=(\hat{v}_k^1,\hat{v}_k^2)$ for each $1\le k\le N$, where v_k^1 and v_k^2 are column vectors of dimensions m and n, respectively, while $\hat{v}_k^1$ and $\hat{v}_k^2$ are row vectors of dimensions m and n, respectively.

To present N-soliton solutions for the reduced matrix integrable mKdV equations of odd order in Eq. (21), we need to check whether G_1^+ defined by (40) satisfies the involution property

$$(G_1^+)^{\mathrm{T}}=\Sigma G_1^+\Sigma^{-1}. \tag{43}$$

This means that the resulting potential matrix P defined by Eq. (41) will satisfy the group reduction condition (15). As a consequence, the above N-soliton solutions to the matrix AKNS integrable equations (10) reduce to the following class of N-soliton solutions:

$$p=\alpha\sum_{k,l=1}^{N}v_k^1(M^{-1})_{kl}\hat{v}_l^2, \tag{44}$$

to the reduced matrix integrable mKdV equations of odd order in Eq. (21). To summarize, when the orthogonal condition for w_k, $1\le k\le N$, in Eq. (38) and the involution property in Eq. (43) are satisfied, the formula (44), together with Eqs. (32), (33), (36) and (37), presents N-soliton solutions to the reduced matrix integrable mKdV equations of odd order in Eq. (21).

Finally, let us compute two examples of one-soliton solutions in the cases of $m=n=1$ and $m=n/2=1$. We take $\lambda_1=\mu_1$, $\hat{\lambda}_1=-\mu_1$, where $\mu_1\in\mathbb{C}$, $\mu_1\neq 0$, and choose

$$\begin{aligned} &w_1=(w_{1,1},w_{1,2})^{\mathrm{T}}, \quad n=1;\\ &w_1=(w_{1,1},w_{1,2},w_{1,3})^{\mathrm{T}}, \quad n=2; \end{aligned} \tag{45}$$

where $w_{1,1},w_{1,2},w_{1,3}\in\mathbb{C}$ are arbitrary constants. The first situation yields a class of one-soliton solutions to the integrable fifth-order mKdV equation (25):

$$p_1=-\frac{2(\alpha_1-\alpha_2)\mu_1w_{1,1}w_{1,2}}{\sigma w_{1,1}^2\mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)\mu_1x+\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}-w_{1,2}^2\mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)\mu_1x-\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}}, \tag{46}$$

where μ_1, $w_{1,1}$, $w_{1,2}\in\mathbb{C}$ are arbitrary nonzero constants. The second situation generates the following choice for one-soliton solutions to the integrable fifth-order mKdV equations (27):

$$\begin{aligned} p_1&=-\frac{2\sigma_2(\alpha_1-\alpha_2)\mu_1w_{1,1}w_{1,2}}{\sigma_1\sigma_2w_{1,1}^2\mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)\mu_1x+\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}-(\sigma_1w_{1,3}^2+\sigma_2w_{1,2}^2)\mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)\mu_1x-\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}},\\ p_2&=-\frac{2\sigma_1(\alpha_1-\alpha_2)\mu_1w_{1,1}w_{1,3}}{\sigma_1\sigma_2w_{1,1}^2\mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)\mu_1x+\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}-(\sigma_1w_{1,3}^2+\sigma_2w_{1,2}^2)\mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)\mu_1x-\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}}. \end{aligned} \tag{47}$$

They solve the integrable fifth-order mKdV equations (27) if and only if the condition

$$\sigma_1w_{1,3}^2+\sigma_2w_{1,2}^2=0 \tag{48}$$

is satisfied, which is a consequence of the involution property in Eq. (43). Therefore, the one-soliton solutions of Eq. (27) formulated above read

$$\begin{aligned} p_1&=-\frac{2(\alpha_1-\alpha_2)\mu_1w_{1,2}}{\sigma_1w_{1,1}\mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)\mu_1x+\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}},\\ p_2&=-\frac{2(\alpha_1-\alpha_2)\mu_1w_{1,3}}{\sigma_2w_{1,1}\mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)\mu_1x+\mathrm{i}(\beta_1-\beta_2)\mu_1^5t}}, \end{aligned} \tag{49}$$

where μ_1, $w_{1,1}$, $w_{1,2}$, $w_{1,3}\in\mathbb{C}$ are arbitrary nonzero constants, but $w_{1,2}$ and $w_{1,3}$ need to satisfy Eq. (48).

4. Concluding remarks

A kind of reduced matrix local integrable mKdV equations of odd order and their soliton solutions have been constructed. The construction is based on a new kind of group reductions for the AKNS matrix spectral problems. Two illustrative examples of the resulting matrix integrable fifth-order mKdV equations have been presented.

We remark that it would be interesting to look for other kinds of reduced local integrable equations from different kinds of Lax pairs,[20,21] variable-coefficient integrable equations[22] and integrable couplings.[23] In the presented group reductions, we can also take

$$\begin{aligned} U^{\mathrm{T}}(x+x_0,t+t_0,-\lambda)&=(U(x+x_0,t+t_0,-\lambda))^{\mathrm{T}}\\ &=-\Sigma U(x,t,,\lambda)\Sigma^{-1}, \end{aligned}$$

with the shifted potentials, where x_0,x_0',t_0,t_0' are arbitrary real constants. Another interesting topic is to study dynamical properties of exact solutions, including lump and breather wave solutions,[25–29] rogue wave solutions,[30,31] solitonless solutions[32] and algebro-geometric solutions,[33] from a perspective of Riemann–Hilbert problems. All these will amend the existing mathematical theory on higher-order integrable equations and their applications.[22,34,35]

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