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Research paper

Four-component integrable hierarchies and their Hamiltonian structures

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ABSTRACT

We aim to construct four-component integrable hierarchies from a kind of matrix spectral problems within the zero curvature formulation. The Liouville integrability of the resulting hierarchies are guaranteed through establishing Hamiltonian structures by the trace identity. Illustrative examples include novel four-component nonlinear Schrödinger type equations and modified Korteweg–de Vries type equations.

1. Introduction

Zero curvature equations play an crucial role in studying integrable equations in soliton theory [1–3]. It is an essential step to formulate appropriate matrix spectral problems. Let us consider a q-dimensional potential: $u = (u_1, \ldots, u_q)^T$ and assume that λ is the spectral parameter. The starting point is to use loop algebras to determine spectral matrices of the form:

$$U = U(u, \lambda) = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_q e_q(\lambda), \tag{1.1}$$

where $e_1, \ldots, e_q$ are linear independent and e_0 is a pseudo-regular element in a loop algebra $\tilde{g}$:

$$\mathrm{Ker}\,\mathrm{ad}_{e_0} \oplus \mathrm{Im}\,\mathrm{ad}_{e_0} = \tilde{g}, \text{ and } \mathrm{Ker}\,\mathrm{ad}_{e_0} \text{ is commutative.}$$

This property ensures that there is a Laurent series solution $W = \sum_{i\geq 0} \lambda^{-i} W_i$ to the stationary zero curvature equation:

$$W_x = i[U, W]. \tag{1.2}$$

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Then, an integrable hierarchy can be presented through zero curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{r}] = 0,\ \ r \geq 0, \tag{1.3}$$

which are the compatibility conditions between the spatial and temporal matrix spectral problems:

$$i\phi_x = U\phi,\ \ i\phi_t = V^{[r]}\phi,\ \ r \geq 0. \tag{1.4}$$

Hamiltonian structures of the resulting integrable equations could be worked out [4,5] by applying the trace identity:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big), \tag{1.5}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u and γ is the constant:

$$\gamma = -\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\ln|\mathrm{tr}(W^2)|.$$

Many integrable hierarchies are generated in this way, based on the special linear algebras (see, e.g., [2,6–10]), and the special orthogonal algebras (see, e.g., [11–13]). Usually, bi-Hamiltonian structures can be furnished, which show the Liouville integrability of the associated zero curvature equations [14]. There are many integrable hierarchies with two components, p and q, and four such well-known integrable hierarchies are associated with the following spectral matrices:

$$U = \begin{bmatrix} \lambda p & \\ q & -\lambda \end{bmatrix},\ U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix},\ U = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix},\ U = \begin{bmatrix} \lambda r & \lambda p \\ \lambda q & -\lambda r \end{bmatrix},$$

where $pq + r^2 = 1$. The corresponding integrable hierarchies are the Ablowitz–Kaup–Newell–Segur hierarchy [2], the Kaup–Newell hierarchy [15], the Wadati–Konno–Ichikawa hierarchy [16] and the Heisenberg hierarchy [17], respectively.

This paper aims to construct integrable hierarchies of four-component equations within the zero curvature formulation. Motivated by recent studies on group reductions of matrix spectral problems (see, e.g., [13,18]), we take a special kind of reduced spatial spectral matrices in the construction process. By applying the trace identity, we furnish Hamiltonian structures for the resulting integrable hierarchies. Two illustrative examples are four-component nonlinear Schrödinger type equations and four-component modified Korteweg–de Vries type equations. The final section provides a conclusion, together with some concluding remarks.

2. An integrable Hamiltonian hierarchy with four components

Within the zero curvature formulation, let us consider a matrix spectral problem of the form:

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ \ U = \begin{bmatrix} -\lambda & p_1 & p_2 & p_2 & p_1 & 0 \\ q_1 & 0 & 0 & 0 & 0 & p_1 \\ q_2 & 0 & 0 & 0 & 0 & p_2 \\ q_2 & 0 & 0 & 0 & 0 & p_2 \\ q_1 & 0 & 0 & 0 & 0 & p_1 \\ 0 & q_1 & q_2 & q_2 & q_1 & \lambda \end{bmatrix}, \tag{2.1}$$

where u is the four-dimensional potential

$$u = u(x,t) = (p_1, p_2, q_1, q_2)^T. \tag{2.2}$$

This spectral problem cannot be reduced from the matrix Ablowitz–Kaup–Newell–Segur spectral problem (see, e.g., [18]).

In order to construct an associated integrable hierarchy, we first solve the stationary zero curvature equation (1.2) by searching for a Laurent series solution:

$$W = \begin{bmatrix} -a & b_1 & b_2 & b_2 & b_1 & 0 \\ c_1 & 0 & d & d & 0 & b_1 \\ c_2 & -d & 0 & 0 & -d & b_2 \\ c_2 & -d & 0 & 0 & -d & b_2 \\ c_1 & 0 & d & d & 0 & b_1 \\ 0 & c_1 & c_2 & c_2 & c_1 & a \end{bmatrix} = \sum_{s\geq 0}\lambda^{-s}W^{[s]}, \tag{2.3}$$

with

$$W^{[s]} = \begin{bmatrix} -a^{[s]} & b_1^{[s]} & b_2^{[s]} & b_2^{[s]} & b_1^{[s]} & 0 \\ c_1^{[s]} & 0 & d^{[s]} & d^{[s]} & 0 & b_1^{[s]} \\ c_2^{[s]} & -d^{[s]} & 0 & 0 & -d^{[s]} & b_2^{[s]} \\ c_2^{[s]} & -d^{[s]} & 0 & 0 & -d^{[s]} & b_2^{[s]} \\ c_1^{[s]} & 0 & d^{[s]} & d^{[s]} & 0 & b_1^{[s]} \\ 0 & c_1^{[s]} & c_2^{[s]} & c_2^{[s]} & c_1^{[s]} & a^{[s]} \end{bmatrix}. \tag{2.4}$$

Obviously, the corresponding stationary zero curvature equation yields the initial conditions:

$$a_x^{[0]}=0,\ b_1^{[0]}=b_2^{[0]}=c_1^{[0]}=c_2^{[0]}=0,\ d_x^{[0]}=0, \tag{2.5}$$

and the recursion relation:

$$\begin{cases} b_1^{[s+1]}=ib_{1,x}^{[s]}+p_1a^{[s]}-2p_2d^{[s]},\\ b_2^{[s+1]}=ib_{2,x}^{[s]}+p_2a^{[s]}+2p_1d^{[s]}, \end{cases} \tag{2.6}$$

$$\begin{cases} c_1^{[s+1]}=-ic_{1,x}^{[s]}+q_1a^{[s]}+2q_2d^{[s]},\\ c_2^{[s+1]}=-ic_{2,x}^{[s]}+q_2a^{[s]}-2q_1d^{[s]}, \end{cases} \tag{2.7}$$

and

$$\begin{cases} d_x^{[s+1]}=i(q_1b_2^{[s+1]}-q_2b_1^{[s+1]}+p_1c_2^{[s+1]}-p_2c_1^{[s+1]}),\\ a_x^{[s+1]}=i(2q_1b_1^{[s+1]}+2q_2b_2^{[s+1]}-2p_1c_1^{[s+1]}-2p_2c_2^{[s+1]})\\ \qquad =-2(q_1b_{1,x}^{[s]}+q_2b_{2,x}^{[s]}+p_1c_{1,x}^{[s]}+p_2c_{2,x}^{[s]}), \end{cases} \tag{2.8}$$

where $s\geq 0$. We take the initial values,

$$a^{[0]}=1,\ d^{[0]}=0, \tag{2.9}$$

and choose the constant of integration as zero,

$$a^{[s]}|_{u=0}=0,\ d^{[s]}|_{u=0}=0,\ s\geq 1. \tag{2.10}$$

Then, we can work out

$$b_1^{[1]}=p_1,\ b_2^{[1]}=p_2,\ c_1^{[1]}=q_1,\ c_2^{[1]}=q_2,\ a^{[1]}=0,\ d^{[1]}=0;$$

$$\begin{cases} b_1^{[2]}=ip_{1,x},\ b_2^{[2]}=ip_{2,x},\ c_1^{[2]}=-iq_{1,x},\ c_2^{[2]}=-iq_{2,x},\\ a^{[2]}=-2p_1q_1-2p_2q_2,\ d^{[2]}=p_1q_2-p_2q_1; \end{cases}$$

$$\begin{cases} b_1^{[3]}=-p_{1,xx}-2p_1^2q_1-4p_1p_2q_2+2p_2^2q_1,\\ b_2^{[3]}=-p_{2,xx}+2p_1^2q_2-4p_1p_2q_1-2p_2^2q_2, \end{cases}$$

$$\begin{cases} c_1^{[3]}=-q_{1,xx}-2p_1q_1^2+2p_1q_2^2-4p_2q_1q_2,\\ c_2^{[3]}=-q_{2,xx}-4p_1q_1q_2+2p_2q_1^2-2p_2q_2^2, \end{cases}$$

$$\begin{cases} a^{[3]}=2i(p_1q_{1,x}-p_{1,x}q_1+p_2q_{2,x}-p_{2,x}q_2),\\ d^{[3]}=-i(p_1q_{2,x}-p_2q_{1,x}-p_{1,x}q_2+p_{2,x}q_1); \end{cases}$$

and

$$\begin{cases} b_1^{[4]}=-i(p_{1,xxx}+6p_1p_{1,x}q_1+6p_1p_{2,x}q_2-6p_2p_{2,x}q_1+6p_{1,x}p_2q_2),\\ b_2^{[4]}=-i(p_{2,xxx}+6p_1p_{2,x}q_1-6p_1p_{1,x}q_2+6p_{1,x}p_2q_1+6p_2p_{2,x}q_2), \end{cases}$$

$$\begin{cases} c_1^{[4]}=i(q_{1,xxx}+6p_1q_1q_{1,x}-6p_1q_2q_{2,x}+6p_2q_1q_{2,x}+6p_2q_{1,x}q_2),\\ c_2^{[4]}=i(q_{2,xxx}+6p_1q_1q_{2,x}+6p_1q_{1,x}q_2-6p_2q_1q_{1,x}+6p_2q_2q_{2,x}), \end{cases}$$

$$\begin{cases} a^{[4]}=6p_1^2q_1^2-6p_1^2q_2^2+24p_1p_2q_1q_2-6p_2^2q_1^2+6p_2^2q_2^2\\ \qquad +2p_1q_{1,xx}+2p_{1,x,x}q_1+2p_2q_{2,xx}+2p_{2,xx}q_2-2p_{1,x}q_{1,x}-2p_{2,x}q_{2,x},\\ d^{[4]}=-6(p_1q_1+p_2q_2)(p_1q_2-p_2q_1)-p_{1,xx}q_2+p_{2,xx}q_1\\ \qquad +p_2q_{1,xx}-p_1q_{2,xx}+p_{1,x}q_{2,x}-p_{2,x}q_{1,x}. \end{cases}$$

At this moment, we can see that the temporal matrix spectral problems:

$$-i\phi_t=V^{[r]}\phi=V^{[r]}(u,\lambda)\phi,\ V^{[r]}=(\lambda^rW)_+=\sum_{s=0}^{r}\lambda^sW^{[r-s]},\ r\geq 0, \tag{2.11}$$

are the other parts of Lax pairs of matrix spectral problems in the zero curvature formulation. The compatibility conditions of the spatial and temporal matrix spectral problems in (2.1) and (2.11) are the zero curvature equations in (1.3). Those equations generate a four-component integrable hierarchy:

$$u_{t_r}=K^{[r]}=(ib_1^{[r+1]},ib_2^{[r+1]},-ic_1^{[r+1]},-ic_2^{[r+1]})^T,\ r\geq 0, \tag{2.12}$$

or more concretely,

$$p_{1,t_r} = ib_1^{[r+1]},\ p_{2,t_r} = ib_2^{[r+1]},\ q_{1,t_r} = -ic_1^{[r+1]},\ q_{2,t_r} = -ic_2^{[r+1]},\ r \geq 0. \tag{2.13}$$

The first two nonlinear examples in this integrable hierarchy are the nonlinear Schrödinger type equations

$$\begin{cases} ip_{1,t_2} = p_{1,xx} + 2p_1^2 q_1 + 4p_1 p_2 q_2 - 2p_2^2 q_1, \\ ip_{2,t_2} = p_{2,xx} - 2p_1^2 q_2 + 4p_1 p_2 q_1 + 2p_2^2 q_2, \\ iq_{1,t_2} = -q_{1,xx} - 2p_1 q_1^2 + 2p_1 q_2^2 - 4p_2 q_1 q_2, \\ iq_{2,t_2} = -q_{2,xx} - 4p_1 q_1 q_2 + 2p_2 q_1^2 - 2p_2 q_2^2, \end{cases} \tag{2.14}$$

and the modified Korteweg–de Vries type equations

$$\begin{cases} p_{1,t_3} = p_{1,xxx} + 6p_1 p_{1,x} q_1 + 6p_1 p_{2,x} q_2 - 6p_2 p_{2,x} q_1 + 6p_{1,x} p_2 q_2, \\ p_{2,t_3} = p_{2,xxx} + 6p_1 p_{2,x} q_1 - 6p_1 p_{1,x} q_2 + 6p_{1,x} p_2 q_1 + 6p_2 p_{2,x} q_2, \\ q_{1,t_3} = -q_{1,xxx} - 6p_1 q_1 q_{1,x} + 6p_1 q_2 q_{2,x} - 6p_2 q_1 q_{2,x} - 6p_2 q_{1,x} q_2, \\ q_{2,t_3} = -q_{2,xxx} - 6p_1 q_1 q_{2,x} - 6p_1 q_{1,x} q_2 + 6p_2 q_1 q_{1,x} - 6p_2 q_2 q_{2,x}. \end{cases} \tag{2.15}$$

They add to the class of integrable nonlinear Schrödinger equations and modified Korteweg–de Vries equations.

3. Hamiltonian structures

To establish Hamiltonian structures for the integrable hierarchy (2.13), we apply the trace identity (1.5) to the matrix spectral problem (2.1). Noting that the solution W is given by (2.3), we can directly compute

$$\mathrm{tr}\big(W\frac{\partial U}{\partial \lambda}\big) = 2a,\ \mathrm{tr}\big(W\frac{\partial U}{\partial u}\big) = 4(c_1, c_2, b_1, b_2)^T,$$

and hence, we have

$$\frac{\delta}{\delta u}\int a^{[s+1]}\lambda^{-s-1}\, dx = 2\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma-s}(c_1^{[s]}, c_2^{[s]}, b_1^{[s]}, b_2^{[s]})^T,\ s \geq 0.$$

Upon considering the case with $s = 2$, we know $\gamma = 0$, and therefore, we obtain

$$\frac{\delta \mathcal{H}^{[s]}}{\delta u} = \frac{\delta}{\delta u}\int H^{[s]}\, dx = 2(c_1^{[s+1]}, c_2^{[s+1]}, b_1^{[s+1]}, b_2^{[s+1]})^T,\ s \geq 0, \tag{3.1}$$

where the Hamiltonian functionals are given by

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+2]}}{s+1}\, dx,\ s \geq 0. \tag{3.2}$$

This allows us to establish the Hamiltonian structures for the integrable hierarchy (2.13):

$$u_{t_r} = K_r = J\frac{\delta \mathcal{H}^{[r]}}{\delta u},\ J = \left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & \frac{1}{2}i \\ \hline -\frac{1}{2}i & 0 & 0 & 0 \\ 0 & -\frac{1}{2}i & 0 & 0 \end{array}\right],\ r \geq 0, \tag{3.3}$$

where J is a Hamiltonian operator and the Hamiltonian functionals $\mathcal{H}^{[r]}$ are defined by (3.2). These Hamiltonian structures show a relation $S = J\frac{\delta \mathcal{H}}{\delta u}$ from a conserved functional $\mathcal{H}$ to a symmetry S. The commuting characteristic of those symmetries:

$$[\![K_{s_1}, K_{s_2}]\!] = K'_{s_1}(u)[K_{s_2}] - K'_{s_2}(u)[K_{s_1}] = 0,\ s_1, s_2 \geq 0, \tag{3.4}$$

is guaranteed by exploring a Lax operator algebra:

$$[\![V^{[s_1]}, V^{[s_2]}]\!] = V^{[s_1]\prime}(u)[K^{[s_2]}] - V^{[s_2]\prime}(u)[K^{[s_1]}] + [V^{[s_1]}, V^{[s_2]}] = 0,\ s_1, s_2 \geq 0, \tag{3.5}$$

which is a consequence of the isospectral zero curvature equations (see [19] for details). It further follows from the Hamiltonian structures that the conserved functionals also commute under the corresponding Poisson bracket:

$$\{\mathcal{H}_{s_1}, \mathcal{H}_{s_2}\}_J = \int \big(\frac{\delta \mathcal{H}^{[s_1]}}{\delta u}\big)^T J \frac{\delta \mathcal{H}^{[s_2]}}{\delta u}\, dx = 0,\ s_1, s_2 \geq 0. \tag{3.6}$$

By combining J with a recursion operator [20], generated from K_s determined by (2.6), (2.7) and (2.8), bi-Hamiltonian structures [14] can also be established for the integrable equations in the hierarchy (2.13).

4. Generalized integrable hierarchies

Let $n \geq 1$ be an arbitrary natural number. If we take a generalization of the matrix spectral problem (2.1):

$$i\phi_x = U\phi,\ U = \left[\begin{array}{c|ccccc|c} -\lambda & p_1 & p_2 & \cdots & p_2 & p_1 & 0 \\ \hline q_1 & & & & & & p_1 \\ q_2 & & & & & & p_2 \\ \vdots & & & 0 & & & \vdots \\ q_2 & & & & & & p_2 \\ q_1 & & & & & & p_1 \\ \hline 0 & q_1 & q_2 & \cdots & q_2 & q_1 & \lambda \end{array}\right]_{(n+4)\times(n+4)}, \tag{4.1}$$

and a Laurent series solution

$$W = \begin{bmatrix} -a & b_1 & b_2 & \cdots & b_2 & b_1 & 0 \\ c_1 & 0 & d & \cdots & d & 0 & b_1 \\ c_2 & -d & 0 & \cdots & 0 & -d & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_2 & -d & 0 & \cdots & 0 & -d & b_2 \\ c_1 & 0 & d & \cdots & d & 0 & b_1 \\ 0 & c_1 & c_2 & \cdots & c_2 & c_1 & a \end{bmatrix}_{(n+4)\times(n+4)} = \sum_{s\geq 0} \lambda^{-s} W^{[s]},$$

with $W^{[s]}$ being defined by

$$W^{[s]} = \begin{bmatrix} a^{[s]} & b_1^{[s]} & b_2^{[s]} & \cdots & b_2^{[s]} & b_1^{[s]} & 0 \\ c_1^{[s]} & 0 & d^{[s]} & \cdots & d^{[s]} & 0 & b_1^{[s]} \\ c_2^{[s]} & -d^{[s]} & 0 & \cdots & 0 & -d^{[s]} & b_2^{[s]} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_2^{[s]} & -d^{[s]} & 0 & \cdots & 0 & -d^{[s]} & b_2^{[s]} \\ c_1^{[s]} & 0 & d^{[s]} & \cdots & d^{[s]} & 0 & b_1^{[s]} \\ 0 & c_1^{[s]} & c_2^{[s]} & \cdots & c_2^{[s]} & c_1^{[s]} & -a^{[s]} \end{bmatrix}_{(n+4)\times(n+4)},$$

to the stationary zero curvature equation (1.2), then we have

$$\begin{cases} b_{1,x} = i(p_1 a - np_2 d - \lambda b_1), \\ b_{2,x} = i(p_2 a + 2p_1 d - \lambda b_2), \\ c_{1,x} = -i(q_1 a + nq_2 d - \lambda c_1), \\ c_{2,x} = -i(q_2 a - 2q_1 d - \lambda c_2), \\ d_x = i(q_1 b_2 - q_2 b_1 + p_1 c_2 - p_2 c_1), \\ a_x = i(2q_1 b_1 + nq_2 b_2 - 2p_1 c_1 - np_2 c_2) \\ \quad = -(2q_1 b_{1,x} + nq_2 b_{2,x} + 2p_1 c_{1,x} + np_2 c_{2,x}), \end{cases}$$

and

$$\frac{\delta}{\delta u}\int a\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(2c_1, nc_2, 2b_1, nb_2)^T.$$

Therefore, we obtain the Hamiltonian structures of those associated integrable equations:

$$u_{t_r} = K_r = (ib_1^{[r+1]}, ib_2^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]})^T = J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\ r \geq 0, \tag{4.2}$$

where

$$J = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \frac{1}{2}i & 0 \\ & & 0 & \frac{1}{n}i \\ \hline -\frac{1}{2}i & 0 & \multicolumn{2}{c}{0} \\ 0 & -\frac{1}{n}i & & \end{array}\right],\ \mathcal{H}^{[r]} = -\int \frac{a^{[r+2]}}{r+1},\ r \geq 0. \tag{4.3}$$

If we take the initial values in (2.9) and zero constants of integration, then we can work out the first two nonlinear examples in those generalized hierarchies, which are the nonlinear Schrödinger type equations

$$\begin{cases} ip_{1,t_2} = p_{1,xx} + 2p_1^2q_1 + 2np_1p_2q_2 - np_2^2q_1, \\ ip_{2,t_2} = p_{2,xx} - 2p_1^2q_2 + 4p_1p_2q_1 + np_2^2q_2, \\ iq_{1,t_2} = -q_{1,xx} - 2p_1q_1^2 + np_1q_2^2 - 2np_2q_1q_2, \\ iq_{2,t_2} = -q_{2,xx} - 4p_1q_1q_2 + 2p_2q_1^2 - np_2q_2^2, \end{cases} \tag{4.4}$$

and the modified Korteweg–de Vries type equations

$$\begin{cases} p_{1,t_3} = p_{1,xxx} + 6p_1p_{1,x}q_1 + 3np_1p_{2,x}q_2 - 3np_2p_{2,x}q_1 + 3np_{1,x}p_2q_2, \\ p_{2,t_3} = p_{2,xxx} + 6p_1p_{2,x}q_1 - 6p_1p_{1,x}q_2 + 6p_{1,x}p_2q_1 + 3np_2p_{2,x}q_2, \\ q_{1,t_3} = -q_{1,xxx} - 6p_1q_1q_{1,x} + 3np_1q_2q_{2,x} - 3np_2q_1q_{2,x} - 3np_2q_{1,x}q_2, \\ q_{2,t_3} = -q_{2,xxx} - 6p_1q_1q_{2,x} - 6p_1q_{1,x}q_2 + 6p_2q_1q_{1,x} - 3np_2q_2q_{2,x}. \end{cases} \tag{4.5}$$

where n is an arbitrary natural number.

5. Concluding remarks

A few integrable hierarchies of Hamiltonian equations with four components have been presented from a kind of special matrix spectral problems within the zero curvature formulation. One crucial step is to compute a Laurent series solution to the corresponding stationary zero curvature equations. The resulting integrable equations possess Hamiltonian structures, established by an application of the trace identity to the underlying matrix spectral problems.

The considered matrix spectral problems could be generalized further by taking more copies of p_1 as did for p_2 (see, e.g., [21]). Of course, we can also involve more dependent variables in matrix spectral problems to generate bigger systems of integrable equations.

It should be interesting to explore structures of soliton solutions to the resulting integrable equations by solution approaches in soliton theory, such as the Riemann–Hilbert technique [22], the Zakharov–Shabat dressing method [23], the Darboux transformation [24,25] and the determinant approach [26]. If we start from the infinite-dimensional algebra $\mathrm{gl}(\infty)$, then we can have soliton solutions presented by a τ-function theory. There are many other types of interesting solutions (see, e.g., [27–31]), which can be computed by taking wave number reductions of soliton solutions. Upon conducting nonlocal group reductions for matrix spectral problems, nonlocal reduced integrable equations can also be presented (see, e.g., [32–35]). It needs a further investigation how to formulate soliton solutions to the resulting integrable equations and their associated nonlocal reduced integrable equations.

CRediT authorship contribution statement

Wen-Xiu Ma: Conceptualization, Methodology, Writing – original draft, Visualization, Investigation, Supervision, Validation, Writing – review & editing.

Declaration of competing interest

The author declares that there is no known competing interest that could have appeared to influence this work.

Data availability

No data was used for the research described in the article.

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