Comment on the 3+1 dimensional Kadomtsev–Petviashvili equations

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ABSTRACT

We comment on traveling wave solutions and rational solutions to the 3+1 dimensional Kadomtsev–Petviashvili (KP) equations: 
\[(u_t + 6uu_x + u_{xxx})_x \pm 3u_yy \pm 3u_{zz} = 0.\]
We also show that both of the 3+1 dimensional KP equations do not possess the three-soliton solution. This suggests that none of the 3+1 dimensional KP equations should be integrable, and partially explains why they do not pass the Painlevé test. As by-products, the one-soliton and two-soliton solutions and four classes of specific three-soliton solutions are explicitly presented.

The 3+1 dimensional Kadomtsev–Petviashvili (KP) equations read
\[u_t + 6u u_x + u_{xxx} = 0,\]
which describe three-dimensional solitons in weakly dispersive media [1], particularly in fluid dynamics and plasma physics [2,3]. Like the 2 + 1 dimensional KP equations, the 3+1 dimensional KP equations with the negative sign “−” and the positive sign “+” are called the 3+1 dimensional KP-I and KP-II equations, respectively. Several classes of exact traveling wave solutions to the 3+1 dimensional KP-II equation were presented by various authors (see, e.g., [4–6]). Very recently, traveling wave solutions and rational solutions were discussed and generated in [7,8] for the 3+1 dimensional KP-I equation, based on the homogeneous balance method and under help of a Riccati equation.

In this note, on one hand, we would like to explain that more general traveling wave and rational solutions, including the ones presented in [4–8], can be constructed under transformations of dependent and independent variables. Three exact and explicit solutions to a Riccati equation will help in generating those solutions. This also contributes to identifying and correcting one common error in finding exact solutions to nonlinear wave equations: not using sufficiently general classes of solutions to ordinary differential equations [9]. On the other hand, using the Hirota bilinear method, we would like to analyze the existence condition of the three-soliton solution, and present the one-soliton and two-soliton solutions and four classes of specific three-soliton solutions to both of the 3+1 dimensional KP equations.

It is known that many direct methods are nowadays available for constructing exact traveling wave solutions to nonlinear differential equations. The Riccati equation
\[\phi_t = \alpha \phi^2 + \beta \quad (\alpha \neq 0)\]
plays a crucial role in manipulating nonlinear equations to get exact solutions by the homogeneous balance method. This equation has the following three exact solutions [10]:

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\[ \phi = -\frac{1}{\zeta^2 + \zeta_0}, \quad \zeta_0 = \text{const.}, \]  

when \( \beta = 0 \):

\[ \phi = -\frac{2e^{\sqrt{-\alpha \beta}}}{\alpha} \zeta_0 \exp(-2e^{\sqrt{-\alpha \beta}}) + \frac{1}{\alpha} + \frac{e^{\sqrt{-\alpha \beta}}}{\alpha} = \begin{cases} \frac{e^{\sqrt{-\alpha \beta}}}{\alpha}, & \text{for } \zeta_0 = 0. \\ -\frac{\sqrt{\alpha \beta}}{\alpha} \tanh(\sqrt{-\alpha \beta} \zeta - \frac{\ln(\zeta_0)}{2}), & \text{for } \zeta_0 > 0, (\epsilon = \pm 1) \\ -\frac{\sqrt{\alpha \beta}}{\alpha} \coth(\sqrt{-\alpha \beta} \zeta - \frac{\ln(\zeta_0)}{2}), & \text{for } \zeta_0 < 0, \end{cases} \]  

when \( \alpha \beta < 0 \); and

\[ \phi = \frac{\sqrt{\alpha \beta}}{\alpha} \tan \left( \sqrt{-\alpha \beta} \zeta + \zeta_0 \right), \quad \zeta_0 = \text{const.}, \]  

when \( \alpha \beta > 0 \). These help generate various traveling wave solutions, including periodic traveling wave solutions, in elementary functions, and the traveling wave solutions obtained in [7,8] corresponds to a sub-case of the solution (4).

It is difficult and even impossible in most cases to determine the solution set of a nonlinear differential equation, and so, the concept of general solutions is introduced only in the field of linear differential equations. Nonlinear differential equations possess diverse solutions, indeed [11]. For example, there exist soliton solutions, positon solutions and complexiton solutions to many integrable equations (see, e.g., [12] for the KdV case). For a generalized differential equation

\[ (u_t + K(u, u_x, \ldots))_x + au_{yy} + bu_{xx} = 0, \quad a, b = \text{consts.}, \]  
a kind of traveling wave solutions

\[ u = f \left( kx + ly + mz + \omega t - \frac{al^2 + bm^2}{k} t \right), \]  

where \( l \) and \( m \) are arbitrary constants, can be generated [13] from a known traveling wave solution \( u = f(kx + \omega t) (k \neq 0) \) to the original differential equation

\[ u_t + K(u, u_x) = 0, \]  

which is assumed to be invariant under the translation of independent variables. The traveling wave solutions presented for the 3+1 dimensional KP-I equation in [7] and for the 3+1 dimensional KP-II equation in [4–6] are among such a class of exact solutions. A general practical algorithm for generating traveling wave solutions was given in [14], which unifies many existing methods such as the tanh-function method, the homogeneous balance method and the exp-function method.

Upon taking the transformation \( r = x + cz + dt \), where \( c \) and \( d \) are constants, the 3+1 dimensional KP-II equation by (1) can be reduced to the good Boussinesq equation

\[ (6uu_{tt} + uu_{rr})_r + (d + 3c^2)u_{rr} + 3u_{yy} = 0. \]  

This equation can be transformed into the standard good Boussinesq equation

\[ (u^2)_r + u_{rr} + u_{yy} = 0, \]  

under a transformation

\[ u(r, y) = -\frac{d + 3c^2}{6} + \frac{1}{3} v \left( r, \sqrt{\frac{3}{3}} y \right). \]  

A Wronskian formulation to get rational solutions to the standard good Boussinesq equation was given in [15], and it can be used to present rational solutions to the 3+1 dimensional KP-II equation defined by (1). An interesting open question for us is: Is there any other rational solution to (7) not in the Wronskian form presented in [15]?

Let us now analyze the existence condition of the three-soliton solution. It is direct to see that under the transformation of dependent variables

\[ u = 2(\ln f)_x, \]  

the 3+1 dimensional KP equations defined by (1) can be cast into

\[ P_\alpha(D_x, D_y, D_z, D_t)f \cdot f = 0, \]  

where \( D_x, D_y, D_z \) and \( D_t \) are Hirota’s differential operators [16] and two polynomials \( P_\alpha \) are defined by

\[ P_\alpha(x, y, z, t) = x^4 + xt \pm 3y^2 \pm 3z^2. \]  

Those bilinear equations in (9) exactly give
that the even property of the polynomials by adopting an ansatz:

\[ f_{xxxx} - 4f_{xxx} + 3f_x^2 + ff_x - f_x + 3 \left( ff_{yy} - f_y^2 + ff_{zz} - f_z^2 \right) = 0, \]

for which a sufficient condition is

\[
\begin{align*}
&\begin{cases}
    f_{xxxx} + f_x \pm 3f_{yy} \pm 3f_{zz} = 0, \\
    -4f_{xxx} + 3f_x^2 + 3f_x^2 \pm 3f_x^2 = 0.
\end{cases}
\end{align*}
\tag{11}
\]

In attempting to find the three-soliton solution, we always introduce three wave variables:

\[ \eta_i = k_i x + l_i y + m_i z + \omega_i t + \eta_{i0}, \quad 1 \leq i \leq 3, \]

where \( k_i, l_i, m_i, \omega_i, 1 \leq i \leq 3, \) are constants to be determined, and \( \eta_{i0}, 1 \leq i \leq 3, \) are arbitrary constant shifts; and define a set of prominent constants:

\[ A_{ij} = \frac{P_x (k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j)}{P_x (k_i + k_j, l_i + l_j, m_i + m_j, \omega_i + \omega_j)}, \quad 1 \leq i, j \leq 3. \]

Following Hirota’s bilinear theory \([16]\), we know that under the dispersion relations

\[ k_i^4 + k_i \omega_i \pm 3l_i^2 \pm 3m_i^2 = 0, \quad 1 \leq i \leq 3, \]

the 3+1 dimensional KP equations by (1) have the one-soliton and two-soliton solutions:

\[ f = 1 + \varepsilon \eta_1, \quad f = 1 + \varepsilon (\eta_1 + \eta_2) + \varepsilon^2 A_{12} \varepsilon \eta_1 \eta_2, \]

where \( \varepsilon \) is an arbitrary perturbation parameter. Moreover, under (14), the 3+1 dimensional KP equations by (1) have the three-soliton solution

\[ f = 1 + \varepsilon (\eta_1 + \eta_2 + \eta_3) + \varepsilon^2 (A_{12} \varepsilon \eta_1 \eta_2 + A_{13} \varepsilon \eta_1 \eta_3 + A_{23} \varepsilon \eta_2 \eta_3) + \varepsilon^3 A_{123} \varepsilon \eta_1 \eta_2 \eta_3, \]

where \( A_{123} = A_{12} A_{13} A_{23} \) and \( \varepsilon \) is an arbitrary perturbation parameter, if and only if the corresponding three-soliton conditions

\[ \sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} P_\sigma (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) P_\sigma (\sigma_1 p_1 - \sigma_2 p_2) P_\sigma (\sigma_2 p_2 - \sigma_3 p_3) P_\sigma (\sigma_1 p_1 - \sigma_3 p_3) = 0 \]

are satisfied, where \( p_i = (k_i, l_i, m_i, \omega_i), 1 \leq i \leq 3. \) More general exact solutions than the two-soliton solution can be computed by adopting an ansatz:

\[ f = 1 + g(\eta_1) e^{\eta_2}, \]

where \( g \) has many choices determined by (11) \([19]\).

The existence of the three-solution solutions usually implies the integrability \([20]\) of the considered equations. Noting that the even property of the polynomials \( P_x \), a direct computation can show that

\[
\begin{align*}
&\sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} P_\sigma (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) P_\sigma (\sigma_1 p_1 - \sigma_2 p_2) P_\sigma (\sigma_2 p_2 - \sigma_3 p_3) P_\sigma (\sigma_1 p_1 - \sigma_3 p_3) = 2 \sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} P_\sigma (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) P_\sigma (\sigma_1 p_1 - \sigma_2 p_2) P_\sigma (\sigma_2 p_2 - \sigma_3 p_3) P_\sigma (\sigma_1 p_1 - \sigma_3 p_3) \\
&= 10368k_1^2 k_2^2 k_3^2 k_1 l_1 m_1^2 k_2 l_2 m_2^2 \left| \begin{array}{ccc}
    k_1 & l_1 & m_1 \\
    k_2 & l_2 & m_2 \\
    k_3 & l_3 & m_3 
\end{array} \right|. 
\end{align*}
\tag{18}
\]

where \( S = \{(1,1,1),(1,1,1),(1,1,1),(1,1,1)\} \). Since the last term cannot equal to zero automatically, the 3+1 dimensional bilinear KP equations by (9) do not have the three-soliton solution. This is in agreement with that the equations by (9) do not pass the Painlevé test \([21]\). For more generalized KP equations, their existence and non-existence cases for localized solitary waves were classified according to the sign of the transverse dispersion coefficients and to the nonlinearity in \([22]\). The non-existence of the three-soliton solution also suggests that none of the 3+1 dimensional KP equations by (1) should be integrable. Nevertheless, the formula in (18) allows us to conclude that under (14), the function \( f \) defined by (16) gives rise to an exact three-wave solution provide that one of the following two conditions holds:

(a) one of three wave numbers \( k_i, 1 \leq i \leq 3, \) is zero;

(b) any two of three vectors \((k_1, k_2, k_3), (l_1, l_2, l_3) \) and \((m_1, m_2, m_3) \) are parallel.

The condition (a) presents one class of specific three-wave solutions, due to a cyclic characteristic, and the condition (b) presents three classes of specific three-wave solutions. The last three classes actually correspond to the three-soliton solutions of three dimensional reductions of the 3+1 dimensional KP equations.
To summarize, we have discussed traveling wave solutions, including periodic traveling wave solutions, and rational solutions to both of the 3+1 dimensional KP equations defined by (1), using transformations of independent and dependent variables and three exact solutions to the Riccati Eq. (2). We have also showed that both of the 3+1 dimensional KP equations by (1) do not pass the three-soliton solution test. This suggests that none of the 3+1 dimensional KP equations by (1) should be integrable, and partially explains why they do not possess the Painlevé property. The one-soliton and two-soliton solutions and four classes of specific three-soliton solutions are presented as by-products.

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