Commun Nonlinear Sci Numer Simulat 16 (2011) 2663–2666

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Comment on the 3+1 dimensional Kadomtsev–Petviashvili equations

Wen-Xiu Ma *

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

ARTICLE INFO

Article history:
Received 22 May 2010
Accepted 2 October 2010
Available online 8 October 2010

Keywords:
Traveling wave solution
Rational solution
Hirota's bilinear form
Three-soliton condition

ABSTRACT

We comment on traveling wave solutions and rational solutions to the 3+1 dimensional Kadomtsev–Petviashvili (KP) equations: $(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0$. We also show that both of the 3+1 dimensional KP equations do not possess the three-soliton solution. This suggests that none of the 3+1 dimensional KP equations should be integrable, and partially explains why they do not pass the Painlevé test. As by-products, the one-soliton and two-soliton solutions and four classes of specific three-soliton solutions are explicitly presented.

The 3+1 dimensional Kadomtsev–Petviashvili (KP) equations read

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0, \tag{1}$$

which describe three-dimensional solitons in weakly dispersive media [1], particularly in fluid dynamics and plasma physics [2,3]. Like the 2 + 1 dimensional KP equations, the 3+1 dimensional KP equations with the negative sign "−" and the positive sign "+" are called the 3+1 dimensional KP-I and KP-II equations, respectively. Several classes of exact traveling wave solutions to the 3+1 dimensional KP-II equation were presented by various authors (see, e.g., [4–6]). Very recently, traveling wave solutions and rational solutions were discussed and generated in [7,8] for the 3+1 dimensional KP-I equation, based on the homogeneous balance method and under help of a Riccati equation.

In this note, on one hand, we would like to explain that more general traveling wave and rational solutions, including the ones presented in [4–8], can be constructed under transformations of dependent and independent variables. Three exact and explicit solutions to a Riccati equation will help in generating those solutions. This also contributes to identifying and correcting one common error in finding exact solutions to nonlinear wave equations: not using sufficiently general classes of solutions to ordinary differential equations [9]. On the other hand, using the Hirota bilinear method, we would like to analyze the existence condition of the three-soliton solution, and present the one-soliton and two-soliton solutions and four classes of specific three-soliton solutions to both of the 3+1 dimensional KP equations.

It is known that many direct methods are nowadays available for constructing exact traveling wave solutions to nonlinear differential equations. The Riccati equation

$$\phi_\xi = \alpha\phi^2 + \beta \quad (\alpha \neq 0) \tag{2}$$

plays a crucial role in manipulating nonlinear equations to get exact solutions by the homogeneous balance method. This equation has the following three exact solutions [10]:

* Tel.: +1 813 9749563; fax: +1 813 9742700.
E-mail addresses: mawx@cas.usf.edu, mawx@math.usf.edu

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2010.10.003

$$\phi = -\frac{1}{\alpha\xi + \xi_0}, \quad \xi_0 = \text{const.}, \tag{3}$$

when $\beta = 0$;

$$\phi = -\frac{2\varepsilon\sqrt{-\alpha\beta}}{\alpha}\frac{1}{\xi_0 \exp(-2\varepsilon\sqrt{-\alpha\beta}\xi) + 1} + \frac{\varepsilon\sqrt{-\alpha\beta}}{\alpha} = \begin{cases} \frac{\varepsilon\sqrt{-\alpha\beta}}{\alpha}, & \text{for } \xi_0 = 0, \\ -\frac{\sqrt{-\alpha\beta}}{\alpha}\tanh(\sqrt{-\alpha\beta}\,\xi - \frac{\varepsilon \ln \xi_0}{2}), & \text{for } \xi_0 > 0, (\varepsilon = \pm 1) \\ -\frac{\sqrt{-\alpha\beta}}{\alpha}\coth(\sqrt{-\alpha\beta}\xi - \frac{\varepsilon \ln(-\xi_0)}{2}), & \text{for } \xi_0 < 0, \end{cases} \tag{4}$$

when $\alpha\beta < 0$; and

$$\phi = \frac{\sqrt{\alpha\beta}}{\alpha}\tan\left(\sqrt{\alpha\beta}\xi + \xi_0\right), \quad \xi_0 = \text{const.}, \tag{5}$$

when $\alpha\beta > 0$. These help generate various traveling wave solutions, including periodic traveling wave solutions, in elementary functions, and the traveling wave solutions obtained in [7,8] corresponds to a sub-case of the solution (4).

It is difficult and even impossible in most cases to determine the solution set of a nonlinear differential equation, and so, the concept of general solutions is introduced only in the field of linear differential equations. Nonlinear differential equations possess diverse solutions, indeed [11]. For example, there exist soliton solutions, positon solutions and complexiton solutions to many integrable equations (see, e.g., [12] for the KdV case). For a generalized differential equation

$$(u_t + K(u, u_x, \ldots))_x + au_{yy} + bu_{zz} = 0, \quad a, b = \text{consts.},$$

a kind of traveling wave solutions

$$u = f\left(kx + ly + mz + \omega t - \frac{al^2 + bm^2}{k}t\right), \tag{6}$$

where l and m are arbitrary constants, can be generated [13] from a known traveling wave solution $u = f(kx + \omega t)$ $(k \neq 0)$ to the original differential equation

$$u_t + K(u, u_x, \ldots) = 0,$$

which is assumed to be invariant under the translation of independent variables. The traveling wave solutions presented for the 3+1 dimensional KP-I equation in [7] and for the 3+1 dimensional KP-II equation in [4–6] are among such a class of exact solutions. A general practical algorithm for generating traveling wave solutions was given in [14], which unifies many existing methods such as the tanh-function method, the homogeneous balance method and the exp-function method.

Upon taking the transformation $r = x + cz + dt$, where c and d are constants, the 3+1 dimensional KP-II equation by (1) can be reduced to the good Boussinesq equation

$$(6uu_r + u_{rrr})_r + (d + 3c^2)u_{rr} + 3u_{yy} = 0.$$

This equation can be transformed into the standard good Boussinesq equation

$$(v^2)_r + v_{rrrr} + v_{yy} = 0, \tag{7}$$

under a transformation

$$u(r, y) = -\frac{d + 3c^2}{6} + \frac{1}{3}v\left(r, \frac{\sqrt{3}}{3}y\right).$$

A Wronskian formulation to get rational solutions to the standard good Boussinesq equation was given in [15], and it can be used to present rational solutions to the 3+1 dimensional KP-II equation defined by (1). An interesting open question for us is: Is there any other rational solution to (7) not in the Wronskian form presented in [15]?

Let us now analyze the existence condition of the three-soliton solution. It is direct to see that under the transformation of dependent variables

$$u = 2(\ln f)_{xx}, \tag{8}$$

the 3+1 dimensional KP equations defined by (1) can be cast into

$$P_\pm(D_x, D_y, D_z, D_t)f \cdot f = 0, \tag{9}$$

where D_x, D_y, D_z and D_t are Hirota's differential operators [16] and two polynomials $P_\pm$ are defined by

$$P_\pm(x, y, z, t) = x^4 + xt \pm 3y^2 \pm 3z^2. \tag{10}$$

Those bilinear equations in (9) exactly give

$$f_{xxxx}f - 4f_xf_{xxx} + 3f_{xx}^2 + ff_{xt} - f_xf_t \pm 3\left(ff_{yy} - f_y^2 + ff_{zz} - f_z^2\right) = 0,$$

for which a sufficient condition is

$$\begin{cases} f_{xxxx} + f_{xt} \pm 3f_{yy} \pm 3f_{zz} = 0, \\ -4f_xf_{xxx} + 3f_{xx}^2 - f_xf_t \mp 3f_y^2 \mp 3f_z^2 = 0. \end{cases} \tag{11}$$

In attempting to find the three-soliton solution, we always introduce three wave variables:

$$\eta_i = k_ix + l_iy + m_iz + \omega_it + \eta_{i,0}, \quad 1 \leqslant i \leqslant 3, \tag{12}$$

where k_i, l_i, m_i, ω_i, $1 \leqslant i \leqslant 3$, are constants to be determined, and $\eta_{i,0}$, $1 \leqslant i \leqslant 3$, are arbitrary constant shifts; and define a set of prominent constants:

$$A_{ij} = -\frac{P_\pm(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j)}{P_\pm(k_i + k_j, l_i + l_j, m_i + m_j, \omega_i + \omega_j)}, \quad 1 \leqslant i,\ j \leqslant 3. \tag{13}$$

Following Hirota's bilinear theory [16], we know that under the dispersion relations

$$k_i^4 + k_i\omega_i \pm 3l_i^2 \pm 3m_i^2 = 0, \quad 1 \leqslant i \leqslant 3, \tag{14}$$

the 3+1 dimensional KP equations by (1) have the one-soliton and two-soliton solutions:

$$f = 1 + \varepsilon e^{\eta_1}, f = 1 + \varepsilon(e^{\eta_1} + e^{\eta_2}) + \varepsilon^2 A_{12} e^{\eta_1 + \eta_2}, \tag{15}$$

where ε is an arbitrary perturbation parameter. Moreover, under (14), the 3+1 dimensional KP equations by (1) have the three-soliton solution

$$f = 1 + \varepsilon(e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) + \varepsilon^2(A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3}) + \varepsilon^3 A_{123} e^{\eta_1+\eta_2+\eta_3}, \tag{16}$$

where $A_{123} = A_{12}A_{13}A_{23}$ and ε is an arbitrary perturbation parameter, if and only if the corresponding three-soliton conditions [17,18]:

$$\sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} P_\pm(\sigma_1\bar{p}_1 + \sigma_2\bar{p}_2 + \sigma_3\bar{p}_3)P_\pm(\sigma_1\bar{p}_1 - \sigma_2\bar{p}_2)P_\pm(\sigma_2\bar{p}_2 - \sigma_3\bar{p}_3)P_\pm(\sigma_1\bar{p}_1 - \sigma_3\bar{p}_3) = 0 \tag{17}$$

are satisfied, where $\bar{p}_i = (k_i, l_i, m_i, \omega_i), 1 \leqslant i \leqslant 3$. More general exact solutions than the two-soliton solution can be computed by adopting an ansatz:

$$f = 1 + g(\eta_1)e^{\eta_2},$$

where g has many choices determined by (11) [19].

The existence of the three-solution solutions usually implies the integrability [20] of the considered equations. Noting that the even property of the polynomials $P_\pm$, a direct computation can show that

$$\begin{aligned} &\sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} P_\pm(\sigma_1\bar{p}_1 + \sigma_2\bar{p}_2 + \sigma_3\bar{p}_3)P_\pm(\sigma_1\bar{p}_1 - \sigma_2\bar{p}_2)P_\pm(\sigma_2\bar{p}_2 - \sigma_3\bar{p}_3)P_\pm(\sigma_1\bar{p}_1 - \sigma_3\bar{p}_3) \\ &= 2\sum_{(\sigma_1,\sigma_2,\sigma_3)\in S} P_\pm(\sigma_1\bar{p}_1 + \sigma_2\bar{p}_2 + \sigma_3\bar{p}_3)P_\pm(\sigma_1\bar{p}_1 - \sigma_2\bar{p}_2)P_\pm(\sigma_2\bar{p}_2 - \sigma_3\bar{p}_3)P_\pm(\sigma_1\bar{p}_1 - \sigma_3\bar{p}_3) \\ &= 10368k_1^2k_2^2k_3^2\begin{vmatrix} k_1 & l_1 & m_1 \\ k_2 & l_2 & m_2 \\ k_3 & l_3 & m_3 \end{vmatrix}^2, \end{aligned} \tag{18}$$

where $S = \{(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\}$. Since the last term cannot equal to zero automatically, the 3+1 dimensional bilinear KP equations by (9) do not have the three-soliton solution. This is in agreement with that the equations by (9) do not pass the Painlevé test [21]. For more generalized KP equations, their existence and non-existence cases for localized solitary waves were classified according to the sign of the transverse dispersion coefficients and to the nonlinearity in [22]. The non-existence of the three-soliton solution also suggests that none of the 3+1 dimensional KP equations by (1) should be integrable. Nevertheless, the formula in (18) allows us to conclude that under (14), the function f defined by (16) gives rise to an exact three-wave solution provide that one of the following two conditions holds:

(a) one of three wave numbers k_i, $1 \leqslant i \leqslant 3$, is zero;
(b) any two of three vectors (k_1,k_2,k_3), (l_1,l_2,l_3) and (m_1,m_2,m_3) are parallel.

The condition (a) presents one class of specific three-wave solutions, due to a cyclic characteristic, and the condition (b) presents three classes of specific three-wave solutions. The last three classes actually correspond to the three-soliton solutions of three dimensional reductions of the 3+1 dimensional KP equations.

To summarize, we have discussed traveling wave solutions, including periodic traveling wave solutions, and rational solutions to both of the 3+1 dimensional KP equations defined by (1), using transformations of independent and dependent variables and three exact solutions to the Riccati Eq. (2). We have also showed that both of the 3+1 dimensional KP equations by (1) do not pass the three-soliton solution test. This suggests that none of the 3+1 dimensional KP equations by (1) should be integrable, and partially explains why they do not possess the Painlevé property. The one-soliton and two-soliton solutions and four classes of specific three-soliton solutions are presented as by-products.

Acknowledgements

The work was supported in part by the Established Researcher Grant, the CAS faculty development grant and the CAS Dean research grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, and the State Administration of Foreign Experts Affairs of China.

References

[1] Kuznetsov EA, Turitsyn SK. Soviet Phys JETP 1982;82:1457.
[2] Ablowitz MJ, Segur H. J Fluid Mech 1979;92:691.
[3] Infeld E, Rowlands G. Acta Phys Polon A 1979;56:329.
[4] Senthilvelan M. Appl Math Comput 2001;123:381.
[5] Chen Y, Yan ZY, Zhang HQ. Phys Lett A 2003;307:107.
[6] El-Sayed SM, Kaya D. Appl Math Comput 2004;157:523.
[7] Khalfallah M. Commun Nonlinear Sci Numer Simul 2009;14:1169.
[8] Sinelshchikov DI. Commun Nonlinear Sci Numer Simul 2010;15:3235.
[9] Kudryashov NA. Commun Nonlinear Sci Numer Simul 2009;14:3507.
[10] Ma WX, Fuchssteiner B. Int J Non-Linear Mech 1996;31:329.
[11] Ma WX. Phys Lett A 2003;319:325.
[12] Ma WX, You YC. Trans Am Math Soc 2005;357:1753.
[13] Ma WX. Phys Lett A 1993;180:221.
[14] Ma WX, Lee JH. Chaos Solitons Fract 2009;42:1356.
[15] Ma WX, Li CX, He JS. Nonlinear Anal 2009;70:4245.
[16] Hirota R. The Direct Method in Soliton Theory. Cambridge University Press; 2004.
[17] Hietarinta J. J Math Phys 1987;28:1732.
[18] Hietarinta J. J Math Phys 1987;28:2094.
[19] Wang LY, Lou SY. Commun Theor Phys 2000;33:683.
[20] Ma WX. In: Scott A, editor. Encyclopedia of nonlinear science. New York: Taylor & Francis; 2005. p. 450–3.
[21] Ruan HY, Lou SY, Chen YX. J Phys A: Math Gen 1999;32:2719.
[22] de Bouard A, Saut JC. Ann Inst H Poincaré Anal Non Linéaire 1997;14:211.