Commutativity of the extended KP flows

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Abstract

The commutativity problem of the extended KP hierarchy is analyzed. The compatibility equation of two extended KP flows is constructed, together with its Lax representations involving two extended Lax operators. The resulting theory shows that the extended KP hierarchy is a natural generalization of the KP flows, but does not commute unlike the constrained KP hierarchy. A few particular examples are computed, along with their Lax pairs.

1. Introduction

Soliton equations including soliton equations with self-consistent sources [1, 2], source soliton equations [3] and constrained soliton equations [4–7] arise in many fields of applied science such as nonlinear optics, hydrodynamics, solid state physics, and plasma physics, and mathematical physics. Among the extended KP and mKP hierarchies are the KdV equation, the KP equation, the NLS equation, the Boussinesq equation and the Davey–Stewartson equation with self-consistent sources as examples. There are different approaches to solve those nonlinear systems of soliton equations [2, 8–10] and Darboux transformations yield complexiton solutions besides soliton and positon solutions [11]. Non-Lie symmetries expressed in terms of squared eigenfunctions play a key role in separating, constraining or extending soliton equations within the Lax formulation (see, e.g., [12, 13]). There are also various other generalized KP hierarchies, for example, the ones generated by proper combinations of the additional symmetry generators [14], from the nonlocal \( \partial \)-problem for the wave function [15], by introducing fractional-order pseudo-differential operators [16], and by including a set of evolution equations in the Moyal deformation parameters [17].

An extended KP hierarchy and an extended \( q \)-deformed KP hierarchy were presented recently by using the dressing operator and its corresponding wave function [18, 19]. The dressing technique (see, e.g., [20]) was used to solve the extended KP hierarchy and the extended mKP hierarchy. Sato’s theory (see, e.g., [21]) was also extended to construct Wronskian solutions of the extended KP and mKP flows [22]. Those results generalize integrable theories of the KP and \( q \)-deformed KP equations with self-consistent sources and the constrained KP and \( q \)-deformed KP equations.

All extended KP flows can commute with the KP flows [18]. With this in mind, a natural question is whether the extended KP flows themselves can be commutative or not. Any study on such a problem will help establish a \( \tau \)-function theory, important in soliton mathematics, for an extended integrable hierarchy. In this paper, we will discuss this question and answer the...
2. Compatibility equations

2.1. The extended KP flows

Let us recall the construction of the extended KP hierarchy by the technique of pseudo-differential operators [18]. We start with the algebra $g$ of pseudo-differential operators

$$ P = \sum_{i=-n}^{\infty} a_i \partial^{-i}, \quad n \in \mathbb{Z}, \tag{2.1} $$

where $\partial = \frac{d}{dx}$ and $a_i, i \geq -n$, are differential functions of $x$. A pseudo-differential operator is the analogous Laurent series obtained by admitting negative powers of $\partial$. We treat $\partial^{-1}$ as the inverse of $\partial$ and so $\partial^{-1} \partial = \partial \partial^{-1} = 1$. We will use the general Leibniz rule

$$ \partial^m f = f \partial^m + \sum_{i=1}^{\infty} C^m_i f \partial^{m-i}, \quad C^m_i = \frac{m(m-1)\cdots(m-i+1)}{i!}, \quad m \in \mathbb{Z}, \tag{2.2} $$

where $f^{(i)} = \partial^i f, \ i \geq 0$. For $P \in g$ in (2.1), its adjoint pseudo-differential operator $P^*$ is defined by

$$ P^* = \sum_{i=-n}^{\infty} (-1)^i \partial^{-i} a_i, \tag{2.3} $$

and its decomposition is taken as

$$ P = P_+ + P_-, \quad P_+ = P_{\geq 0} = \sum_{i=0}^{\infty} a_i \partial^{-i}, \quad P_- = P_{\leq 0} = \sum_{i=-1}^{\infty} a_i \partial^{-i}, \tag{2.4} $$

which leads to an $r$-matrix structure on the algebra $g$ [23].

It is well-known that associated with the pseudo-differential operator $L$:

$$ L = \partial + u \partial^{-1} + v \partial^{-2} + w \partial^{-3} + u \partial^{-4} + \cdots, \tag{2.5} $$

the KP hierarchy is determined by

$$ L_n = [L_n, L], \quad n \geq 1, \tag{2.6} $$

where the differential operators $B_n$’s are given by

$$ B_n = (L^n)_+, \quad n \geq 1. \tag{2.7} $$

In the above expression of $L$, we assumed that $u = u_1, v = u_2$ and $w = u_3$ for convenience. Now a direct computation tells us

$$ B_2 = \partial^2 + 2u, $$
$$ B_3 = \partial^3 + 3u \partial + 3(u_2 + v), $$
$$ B_4 = \partial^4 + 4u_3 \partial^2 + 2(3u_2 + 2v) \partial + 2(2u_{1x} + 3u_2 + 3v_2 + 2w). $$

The readers may refer to Sato’s theory for exact solutions to the KP equations by the so-called $\tau$-function [21].

For $k \geq 1$, let a set of $N$ pairs of new functions, $q_i$ and $r_i, 1 \leq i \leq N$, be determined by

$$ q_i = B_i q_i, \quad r_i = -B_i^* r_i, \quad 1 \leq i \leq N, \tag{2.8} $$

where $P^*$ is defined by (2.3). If the functions $q_i$’s and $r_i$’s also satisfy

$$ L q_i = \lambda q_i, \quad L^* r_i = \lambda r_i, \quad 1 \leq i \leq N, $$

then they are eigenfunctions and adjoint eigenfunctions of the spectral problems:

$$ L \phi = \lambda \phi, \quad \phi_{q_i} = B_i \phi, $$

and

$$ L^* \phi = \lambda \phi, \quad \phi_{r_i} = -B_i^* \phi, $$

each of which yields the Lax equation $L_n = [B_n, L]$. 

question by constructing the compatibility equations of the extended KP flows. The resulting compatibility equations have Lax presentations with Lax pairs involving two extended operators. We will illustrate the compatibility equations by two particular examples. The compatibility equations of the $k$-constrained extended KP hierarchy will also be analyzed, together with two examples in this reduced case. Conclusions and a more general question are given in the last section.

Now, the pseudo-differential operators
\[
\tilde{B}_k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, \quad k \geq 1,
\] (2.9)
lead to the so-called extended KP hierarchy
\[
L_{\tau_2} = [\tilde{B}_k, L], \quad k \geq 1.
\] (2.10)

This hierarchy is compatible with the KP flows in (2.6) and examples of the extended KP flows are given in [18].

We remark that the processes of constructing the extended flows and the constrained flows look very similar. In practice, the extended flows \( L_{\tau_2} = [\tilde{B}_k, L], \) \( k \geq 1, \) are generated from a standard pseudo-differential spectral operator \( L = P(\partial) + \sum_{i=1}^{\infty} u_i \partial^{-i} \) and non-standard Lax operators \( \tilde{B}_k = (\tilde{L}^k)_{\tau_2} + \sum_{i=1}^{N} q_i \partial^{-1} r_i, \) but the constrained flows \( L_{\tau_2} = [B_{n, L}], \) \( n \geq 1, \) are generated from a non-standard pseudo-differential spectral operator \( L = P(\partial) + \sum_{i=1}^{N} q_i \partial^{-1} r_i \) and standard Lax operators \( B_n = (\tilde{L}^k)_{\tau_2} \) where \( P \) is a polynomial of order \( m. \) Therefore, the resulting extended flows and constrained flows have different characters.

2.2. The compatibility equations

The question for us here is whether a pair of two extended KP flows in (2.10) commute with each other or not. In what follows, we will discuss the commutativity problem of the extended KP hierarchy (2.10), to see what the compatibility equations will be.

To analyze the commutativity problem, we will use a basic formula:
\[
[B_n, q^l \partial^{-1} r]_+ = (B_n q) \partial^{-1} r - q^l \partial^{-1} (B_n r),
\] (2.11)
where the differential operator \( B_n \) is defined by (2.7) and \( P_\tau = P_\partial \) is defined as in (2.4). Actually the formula (2.11) works for all differential operators (not only for \( B_n). \) This feature provides insight into the role of the pseudo-differential operator \( q^l \partial^{-1} r. \) We provide an answer to the commutativity question as follows.

**Theorem 2.1.** Let the Lax operator \( B_k \) and the extended Lax operator \( \tilde{B}_k \) be defined by (2.7) and (2.9). Then for \( k, l \geq 1, \) under the conditions
\[
q_{l,t_k} = B_k q_l, \quad r_{i,t_k} = -B_k r_i, \quad 1 \leq i \leq N,
\] (2.12)
and
\[
q_{l,t_\tau} = B_k q_l, \quad r_{i,t_\tau} = -B_k r_i, \quad 1 \leq i \leq N,
\] (2.13)
the \( \tau_\tau \)-flow by \( L_{\tau_2} = [\tilde{B}_k, L] \) and the \( \tau_\tau \)-flow by \( L_{\tau_1} = [B_{l, L}] \) in the extended KP hierarchy (2.10) commute if and only if the following differential relation
\[
B_{k,t_\tau} - B_{t_\tau} + [B_k, B_l] + \left[ B_k, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_+ + \left[ \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_l \right]_+ = 0
\] (2.14)
holds.

**Proof.** Obviously, the pseudo-differential zero curvature equation
\[
\tilde{B}_{k,t_\tau} - \tilde{B}_{t_\tau} + [\tilde{B}_k, \tilde{B}_l] = 0
\] (2.15)
implies that
\[
[\tilde{B}_{k,t_\tau} - \tilde{B}_{t_\tau} + [\tilde{B}_k, \tilde{B}_l], L] = 0.
\]
This tells that \( L_{\tau_1} = L_{t_\tau}, \) and thus, the two Lax equations
\[
L_{\tau_2} = [\tilde{B}_k, L], \quad L_{\tau_1} = [\tilde{B}_l, L].
\] (2.16)
are compatible with each other and the corresponding flows commute.

In what follows, we are going to verify that the zero curvature (2.15) is necessary to guarantee the commutativity of the \( \tau_\tau \)-flow and the \( \tau_\tau \)-flow in the hierarchy (2.10). For brevity, we only focus on the case of \( N = 1, \) and rewrite \( q_1 \) and \( r_1 \) as \( q \) and \( r, \) respectively. Based on the Lax equations in (2.16), and noting that
\[
[q^l \partial^{-1} r, L^+]_+ = 0,
\]
we can compute that
\[
\tilde{B}_k t_i = B_k t_i + (q \partial^{-1} r)_t i = [B_k + q \partial^{-1} r, L^k]_+ + (q \partial^{-1} r)_{t i} = [B_k, L^k]_+ + [q \partial^{-1} r, L^k]_+ + q \partial^{-1} r + q \partial^{-1} r_{t i}
\]
\[
= [B_k, L^k]_+ + [q \partial^{-1} r, B_k]_+ + (B_k q) \partial^{-1} r - q \partial^{-1} (B_k r),
\]
and similarly, we have
\[
\tilde{B}_l t_i = [B_k, L^l]_+ + [q \partial^{-1} r, B_l]_+ + (B_k q) \partial^{-1} r - q \partial^{-1} (B_k r).
\]
Moreover, on one hand, we have
\[
[B_k, q \partial^{-1} r] = [B_k, q \partial^{-1} r]_+ + [B_k, q \partial^{-1} r]_- = [B_k, q \partial^{-1} r]_+ + (B_k q) \partial^{-1} r - q \partial^{-1} (B_k r),
\]
and similarly,
\[
[q \partial^{-1} r, B] = [q \partial^{-1} r, B]_+ - (B q) \partial^{-1} r + q \partial^{-1} (B r),
\]
where (2.11) was used.

Now since
\[
[B_k, \tilde{B}_l] = [B_k, B_l] + [B_k, q \partial^{-1} r] + [q \partial^{-1} r, B_l],
\]
we can see from (2.17)–(2.21) that the zero curvature Eq. (2.15) holds.

Finally based on the formula (2.11), the introduction of \( q_i \) and \( r_i \), \( 1 \leq i \leq N \), by (2.12) and (2.13) allows us conclude that the zero curvature Eq. (2.15) is equivalent to the differential relation (2.14). This completes the proof of the theorem.

The compatibility equation of the \( \tau_k \)-flow and the \( \tau_l \)-flow is so given by
\[
\left( B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)_{t_j} - \left( B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)_{t_k} + \left[ B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_l + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] = 0.
\]
(2.22a)
\[
q_{l, t_i} = B_k q_i, \quad r_{l, t_i} = -B_k r_i, \quad 1 \leq i \leq N,
\]
(2.22b)
\[
q_{l, t_i} = B_k q_i, \quad r_{l, t_i} = -B_k r_i, \quad 1 \leq i \leq N,
\]
(2.22c)
where \( k, l, N > 1 \). Under (2.22b) and (2.22c), (2.22a) has the Lax representation:
\[
\psi_{t_j} = \left( B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \Psi, \quad \psi_{t_k} = \left( B_l + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \Psi.
\]
(2.23)

We want to point out that (2.22a), equivalently, (2.14), is a differential operator equation. It generalizes the triple \( L-A-B \) representation of integrable equations, which has nice algebraic structures [24].

In order to compute examples of the compatibility equations of the extended KP hierarchy, we list some useful expressions in the following proposition.

**Proposition 2.1.** The following equalities hold:
\[
[a \partial^2, q \partial^{-1} r]_+ = 2a(q r)_x + a q r,
\]
\[
[a \partial^3, q \partial^{-1} r]_+ = 3[a(q r)_x + a q r]_x + 3a(q r)_x - a_{xx} q r - 2a_q r_x,
\]
\[
[a \partial^4, q \partial^{-1} r]_+ = 4[a(q r)_x + a q r]_x^2 + [6a(q r)_x - 4a(q r)_x - a_{xx} q r - 2a_q r_x]_x + 4a(q r)_{xxx} - 6a(q r)_x + 4a(q r)_x + a_{xx} q r + 3a_{xx} q r_x + 3a_{xq r}_x,
\]
\[
[B_2, B_3] = 3(u_{xx} + 2 v) \partial + u_{xxx} - 6u_x + 3 v x_x,
\]
\[
[B_2, B_4] = 4(u_{xx} + 2 v) \partial^2 + 2(u_{xxx} + 4 u_x + 8 v_x + 4 w_x) \partial + 2(u_{xxxx} + 2 u_{xx} - 4 u_x v + 3 v_{xxx} + 2 w_x),
\]
where \( P = P > 0 \) is defined as in (2.4).

The proof of this proposition just needs some direct calculation, and the proposition itself shows the non-symmetric feature of the eigenfunctions \( q_i \)'s and the adjoint eigenfunctions \( r_i \)'s. Now the following two examples of the compatibility equation (2.22) of the extended KP hierarchy can easily be presented.

**Example 2.1.** Let us first take \( k = 2 \) and \( l = 3 \), and set \( y = \tau_2 \) and \( t = \tau_3 \). Then the compatibility equation (2.22) becomes
\[
B_{2x} - B_{3y} + [B_2, B_3] + \left[ B_2, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_+ + \left[ \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_3 \right] = 0.
\]
(2.24a)
2.3. The $k$-constraint

Under (2.27d), (2.27e) and (2.27f), this $(2+1)$-dimensional nonlinear system has the Lax representation:

Example 2.2. Let us second take $k = 2$ and $l = 4$, and set $y = \tau_2$ and $t = \tau_4$. Then the compatibility equation (2.22) becomes

$$B_{2l} - B_{4y} + [B_2, B_4] + \left[ B_{2}, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] + \left[ \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_4 \right] = 0.$$  

Under (2.25c) and (2.25d), this $(2+1)$-dimensional nonlinear system has the Lax representation:

$$\psi_y = \left( \partial^2 + 2u + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \psi,$$

$$\psi_t = \left[ \partial^4 + 3u \partial + 3(u_x + v) + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] \psi.$$
which presents a holonomic constraint when \( k = 1 \) and a non-holonomic constraint when \( k \geq 2 \). We require that the eigenfunctions and adjoint eigenfunctions satisfy

\[
B_k q_i = \lambda_k q_i, \quad B_k^* r_i = \lambda_k r_i, \quad 1 \leq i \leq N,
\]

where \( \lambda_k, 1 \leq i \leq N \), are constants, to avoid using the evolution law of the \( q_i \)'s and \( r_i \)'s with respect to \( \tau_k \). Then on the sub-manifold determined by the \( k \)-constraint (2.28), we can have

\[
(L^k)_{\tau_1} = [\tilde{B}_k, L^k],
\]

\[
B_{k, \tau_1} = (L^k)_{\tau_1} = [\tilde{B}_k, L^k]_+, \nonumber
\]

\[
\left( \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)_{\tau_1} - \sum_{i=1}^{N} \left( q_i \partial^{-1} (B^*_k r_i) - (B_k q_i) \partial^{-1} r_i \right) = \left[ \tilde{B}_k, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] = \left[ \tilde{B}_k, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_-, \nonumber
\]

\[
= [\tilde{B}_k, L^k]_+ - [\tilde{B}_k, B_k]_+ = [\tilde{B}_k, L^k]_+ - [\tilde{B}_k, B_k]_+ = [\tilde{B}_k, L^k]_- , \nonumber
\]

where (2.11) was used for the first time, because we have

\[
[\tilde{B}_k, B_k]_+ = [N_{i=1}^{N} q_i \partial^{-1} r_i, B_k]_+ = \sum_{i=1}^{N} \left( q_i \partial^{-1} (B^*_k r_i) - (B_k q_i) \partial^{-1} r_i \right) = \sum_{i=1}^{N} \left[ q_i \partial^{-1} (\lambda_k r_i) - (\lambda_k q_i) \partial^{-1} r_i \right] = 0, \nonumber
\]

where (2.11) was used for the second time. It then follows that

\[
(L^k)_{\tau_1} = B_{k, \tau_1} + \left( \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)_{\tau_1}. \nonumber
\]

Therefore, the sub-manifold determined by the \( k \)-constraint (2.28) is invariant under the \( \tau_k \)-flow.

We constrain the \( \tau_k \)-flow on this sub-manifold, and so, we have \( L_{\tau_1} = [B_k, L] = [L^k, L] = 0 \) and further \( B_{k, \tau_1} = (L^k)_{\tau_1} = 0 \). It is now direct to show that under

\[
B_k q_i = \lambda_k q_i, \quad B_k^* r_i = \lambda_k r_i, \quad 1 \leq i \leq N, \nonumber
\]

\[
q_i \partial r_i = B_k q_i, \quad r_i \partial r_i = -B_k^* r_i, \quad 1 \leq i \leq N, \nonumber
\]

the pseudo-differential zero curvature Eq. (2.14) becomes

\[
B_{k, \tau_1} + [B_k, B_k] + \left[ B_k, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_+ + \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_k \] = 0. \nonumber
\]

Therefore, the compatibility equation under the \( k \)-constraint reads

\[
B_{k, \tau_1} + [B_k, B_k] + \left[ B_k, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_+ + \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_k \] = 0, \nonumber
\]

\[
B_k q_i = \lambda_k q_i, \quad B_k^* r_i = \lambda_k r_i, \quad 1 \leq i \leq N, \nonumber
\]

\[
q_i \partial r_i = B_k q_i, \quad r_i \partial r_i = -B_k^* r_i, \quad 1 \leq i \leq N. \nonumber
\]

This is the \( k \)-constrained hierarchy of the compatibility equations with self-consistent sources. With (2.31b) and (2.31c), it has the Lax representation:

\[
\left( B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \psi = \mu \psi, \quad \psi_{\tau_1} = \left( B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \psi, \nonumber
\]

where \( \mu \) is a spectral parameter.

**Example 2.3.** Similarly, let us first take \( k = 2 \) and \( l = 3 \), and set \( y = \tau_2 \) and \( t = \tau_3 \). Then the compatibility equation (2.31) becomes

\[
B_{2, \tau_3} + [B_2, B_3] + \left[ B_2, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_+ + \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_3 \] = 0, \nonumber
\]

\[
B_2 q_i = \lambda_2 q_i, \quad B_2^* r_i = \lambda_2 r_i, \quad 1 \leq i \leq N, \nonumber
\]

\[
q_i \partial r_i = B_2 q_i, \quad r_i \partial r_i = -B_2^* r_i, \quad 1 \leq i \leq N. \nonumber
\]
On the basis of Proposition (2.1), the nonlinear system above reads
\begin{align}
    u_{xx} + 2v_x - \sum_{i=1}^{N} (q_ir_i)_x &= 0, \quad (2.34a) \\
    2u_t + u_{xxx} - 6uu_x + 3v_{xx} + \sum_{i=1}^{N} [2(q_ir_i)_x - 3(q_{ir_i})_x] &= 0, \quad (2.34b) \\
    qi_{xx} + 2uqi &= \lambda_2 q_i, \quad r_{i,xx} + 2uri = \lambda_2 ri, \quad 1 \leq i \leq N, \quad (2.34c) \\
    qi_{tt} = qi_{xxx} + 3u(q_{ux} + v)qi, \quad ri_{tt} = ri_{xxx} + 3u ri_{xx} - 3vri, \quad 1 \leq i \leq N. \quad (2.34d)
\end{align}

Under (2.34c) and (2.34d), this (2+1)-dimensional nonlinear system has the Lax representation:

\[
    \left( \partial^2 + 2u + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \psi = \mu \psi, \quad \psi_t = \left( \partial^3 + 3u \partial + 3(u_x + v) + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \psi.
\]

Example 2.4. Similarly, let us second take \( k = 4 \) and \( l = 2 \), and set \( y = \tau_4 \) and \( t = \tau_2 \). Now the compatibility equation (2.31) becomes

\[
    - B_{2,y} + [B_4, B_2] + \left[ B_4, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] + \left[ \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_2 \right] = 0, \quad (2.35a)
\]

\[
    B_4 q_i = \lambda_4 q_i, \quad B_4 r_i = \lambda_4 r_i, \quad 1 \leq i \leq N, \quad (2.35b)
\]

\[
    qi_{tt} = qi_{xxx} + 2uqi, \quad ri_{tt} = B_4 r_i, \quad 1 \leq i \leq N. \quad (2.35c)
\]

On the basis of Proposition (2.1), the nonlinear system above reads

\begin{align}
    u_{xx} + 2v_x - \sum_{i=1}^{N} (q_ir_i)_x &= 0, \quad (2.36a) \\
    u_{xxx} + 4uu_x + 8v_{xx} + 4w_x - \sum_{i=1}^{N} [3(q_ir_i)_x - 2(q_{ir_i})_x] &= 0, \quad (2.36b) \\
    u_y + u_{xxx} + 2uu_x - 4u_x v + 3v_{xxx} + 2w_{xx} + \sum_{i=1}^{N} [(q_ir_i)_x - 2(q_{ir_i})_x + 3(q_{r_ix})_x - 2(q_{r_ix})_x - 4u(q_ir_i)_x - 2u q_ir_i] &= 0, \quad (2.36c) \\
    qi_{xxx} + 4uq_{ir_i} + 2(3u_x + 2v)q_{ix} + 2(2u_{xx} + 3u^2 + 3v_x + 2w)q_i &= \lambda_4 q_i, \quad 1 \leq i \leq N, \quad (2.36d) \\
    r_{i,xxx} + 4ur_{ix} + 2(u_x - 2v)r_{ix} + 2(ux + 3u^2 + v_x + 2w)r_i &= \lambda_4 ri, \quad 1 \leq i \leq N, \quad (2.36e) \\
    qi_{tt} = qi_{xxx} + 2uqi, \quad ri_{tt} = -r_{i,xx} - 2ur_i, \quad 1 \leq i \leq N. \quad (2.36f)
\end{align}

Under (2.36d), (2.36e) and (2.36f), this (2+1)-dimensional nonlinear system has the Lax representation:

\[
    \left[ \partial^4 + 4u \partial^2 + 2(3u_x + 2v) \partial + 2(2u_{xx} + 3u^2 + 3v_x + 2w) + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] \psi = \mu \psi,
\]

\[
    \psi_t = \left( \partial^3 + 2u + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) \psi.
\]

3. Conclusions and remarks

We considered the commutativity problem of the extended KP flows and constructed the compatibility equation of two extended KP flows. The \( k \)-constraint problem was discussed and the resulting hierarchy of the compatibility equations reduces to the compatibility equations between the extended KP hierarchy and the \( k \)-constrained KP hierarchy. A few particular examples in each of two cases were computed. Since the (2+1)-dimensional KP hierarchy can be reduced to (1+1)-
dimensional soliton hierarchies, many (1+1)-dimensional compatibility equations of extended soliton hierarchies could be generated.

Theorem 2.1 exposes that unlike the constrained KP flows, the extended KP flows don’t commute automatically, since the differential relation (2.14) brings new conditions – the compatibility equations. The presented examples show us that those compatibility equations could be complicated nonlinear partial differential equations. It is still open to us how to solve the compatibility equations analytically.

There should also exist complexon solutions besides solitons and positions, to the compatibility equation hierarchy. Inverse scattering technique [9] and Darboux transformation method [10] could be helpful in solving the compatibility equation hierarchy.

Theorem 2.1 exposes that unlike the constrained KP flows, the extended KP flows don’t commute automatically, since the differential relation (2.14) brings new conditions – the compatibility equations. The presented examples show us that those compatibility equations could be complicated nonlinear partial differential equations. It is still open to us how to solve the compatibility equations analytically.

To guarantee the commutativity of the \( X \)-flow and the \( Y \)-flow defined by

\[ L_i = \left[ \tilde{B}_k, L \right], \quad L_k = \left[ \tilde{B}_i, L \right], \]

we need to compute a more general pseudo-differential zero curvature equation

\[ \tilde{B}_i = \Lambda_i + \hat{B}_i = 0. \]

Based on the formula (2.11), this equation is equivalent to

\[ \tilde{B}_i = \Lambda_i + \hat{B}_i + \left[ B_k, \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j \right] + \left[ \sum_{i=1}^{N} \partial^{1-N} r_i, B_i \right] = 0, \]

and

\[ \sum_{i=1}^{N} q_i \partial^{-1} r_i + \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j = 0. \]

The corresponding compatibility equations will be much more general than the ones of the extended KP hierarchy (2.22). However, the last equation above generates infinitely many differential constraints on the eigenfunctions and adjoint eigenfunctions: \( q_i \) and \( r_i \), \( 1 \leq i \leq N \), and \( \alpha_j \) and \( \beta_j \), \( 1 \leq j \leq M \). It is interesting how to satisfy these constraints generally to guarantee the commutativity of the \( X \)-flow and the \( Y \)-flow. One possibility was given in this paper: \( M = N, q_i = \alpha_i, r_i = \beta_i, 1 \leq i \leq N \); and another possibility is to take one set of zero eigenfunctions and adjoint eigenfunctions: \( q_i = r_i = 0, 1 \leq i \leq N \); or \( \alpha_j = \beta_j = 0, 1 \leq j \leq M \), which gives the extended KP flows [18]. How about the general case?

Acknowledgements:

The work was supported in part by the Established Researcher Grant and the CAS faculty development grant and the CAS Dean research grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, and the State Administration of Foreign Experts Affairs of China.

References