Commun Nonlinear Sci Numer Simulat 16 (2011) 722–730

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Commutativity of the extended KP flows

Wen-Xiu Ma *

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
College of Information Science and Technology, Shandong University of Science and Technology, Shandong 266510, PR China

ARTICLE INFO

Article history:
Received 17 December 2009
Received in revised form 14 May 2010
Accepted 17 May 2010
Available online 24 May 2010

Keywords:
The KP hierarchy
Sato's theory
Lax representation
Pseudo-differential operator
Constrained soliton hierarchy

ABSTRACT

The commutativity problem of the extended KP hierarchy is analyzed. The compatibility equation of two extended KP flows is constructed, together with its Lax representations involving two extended Lax operators. The resulting theory shows that the extended KP hierarchy is a natural generalization of the KP flows, but does not commute unlike the constrained KP hierarchy. A few particular examples are computed, along with their Lax pairs.

1. Introduction

Soliton equations including soliton equations with self-consistent sources [1,2], source soliton equations [3] and constrained soliton equations [4–7] arise in many fields of applied science such as nonlinear optics, hydrodynamics, solid state physics, and plasma physics, mechanics and mathematical physics. Among the extended KP and mKP hierarchies are the KdV equation, the KP equation, the NLS equation, the Boussinesq equation and the Davey–Stewartson equation with self-consistent sources as examples. There are different approaches to solve those nonlinear systems of soliton equations [2,8–10] and Darboux transformations yield complexiton solutions besides soliton and positon solutions [11]. Non-Lie symmetries expressed in terms of squared eigenfunctions play a key role in separating, constraining or extending soliton equations within the Lax formulation (see, e.g., [12,13]). There are also various other generalized KP hierarchies, for example, the ones generated by proper combinations of the additional symmetry generators [14], from the nonlocal $\bar{\partial}$-problem for the wave function [15], by introducing fractional-order pseudo-differential operators [16], and by including a set of evolution equations in the Moyal deformation parameters [17].

An extended KP hierarchy and an extended q-deformed KP hierarchy were presented recently by using the dressing operator and its corresponding wave function [18,19]. The dressing technique (see, e.g., [20]) was used to solve the extended KP hierarchy and the extended mKP hierarchy. Sato's theory (see, e.g., [21]) was also extended to construct Wronskian solutions of the extended KP and mKP flows [22]. Those results generalize integrable theories of the KP and q-deformed KP equations with self-consistent sources and the constrained KP and q-deformed KP equations.

All extended KP flows can commute with the KP flows [18]. With this in mind, a natural question is whether the extended KP flows themselves can be commutative or not. Any study on such a problem will help establish a τ-function theory, important in soliton mathematics, for an extended integrable hierarchy. In this paper, we will discuss this question and answer the

* Address: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.
E-mail address: mawx@cas.usf.edu

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2010.05.016

question by constructing the compatibility equations of the extended KP flows. The resulting compatibility equations have Lax presentations with Lax pairs involving two extended operators. We will illustrate the compatibility equations by two particular examples. The compatibility equations of the k-constrained extended KP hierarchy will also be analyzed, together with two examples in this reduced case. Conclusions and a more general question are given in the last section.

2. Compatibility equations

2.1. The extended KP flows

Let us recall the construction of the extended KP hierarchy by the technique of pseudo-differential operators [18]. We start with the algebra g of pseudo-differential operators

$$P = \sum_{i=-n}^{\infty} a_i \partial^{-i}, \quad n \in \mathbb{Z}, \tag{2.1}$$

where $\partial = \frac{\partial}{\partial x}$ and a_i, $i \geqslant -n$, are differential functions of x. A pseudo-differential operator is the analogous Laurent series obtained by admitting negative powers of ∂. We treat ∂^{-1} as the inverse of ∂ and so $\partial^{-1}\partial = \partial\partial^{-1} = 1$. We will use the general Leibniz rule

$$\partial^m f = f\partial^m + \sum_{i=1}^{\infty} C_m^i f^{(i)} \partial^{m-i}, \quad C_m^i = \frac{m(m-1)\cdots(m-i+1)}{i!}, \quad m \in \mathbb{Z}, \tag{2.2}$$

where $f^{(i)} = \partial^i f$, $i \geqslant 0$. For $P \in g$ in (2.1), its adjoint pseudo-differential operator P^* is defined by

$$P^* = \sum_{i=-n}^{\infty} (-1)^i \partial^{-i} a_i, \tag{2.3}$$

and its decomposition is taken as

$$P = P_+ + P_-, \quad P_+ = P_{\geqslant 0} = \sum_{i=-n}^{0} a_i \partial^{-i}, \quad P_- = P_{<0} = \sum_{i=1}^{\infty} a_i \partial^{-i}, \tag{2.4}$$

which leads to an r-matrix structure on the algebra g [23].

It is well-known that associated with the pseudo-differential operator L:

$$L = \partial + u\partial^{-1} + v\partial^{-2} + w\partial^{-3} + u_4\partial^{-4} + \cdots, \tag{2.5}$$

the KP hierarchy is determined by

$$L_{t_n} = [B_n, L], \quad n \geqslant 1, \tag{2.6}$$

where the differential operators B_n's are given by

$$B_n = (L^n)_+ = (L^n)_{\geqslant 0}, \quad n \geqslant 1. \tag{2.7}$$

In the above expression of L, we assumed that $u = u_1$, $v = u_2$ and $w = u_3$ for convenience. Now a direct computation tells us

$$\begin{aligned} B_2 &= \partial^2 + 2u, \\ B_3 &= \partial^3 + 3u\partial + 3(u_x + v), \\ B_4 &= \partial^4 + 4u\partial^2 + 2(3u_x + 2v)\partial + 2(2u_{xx} + 3u^2 + 3v_x + 2w). \end{aligned}$$

The readers may refer to Sato's theory for exact solutions to the KP equations by the so-called τ-function [21].

For $k \geqslant 1$, let a set of N pairs of new functions, q_i and r_i, $1 \leqslant i \leqslant N$, be determined by

$$q_{i,\tau_k} = B_k q_i, \quad r_{i,\tau_k} = -B_k^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.8}$$

where P^* is defined by (2.3). If the functions q_i's and r_i's also satisfy

$$Lq_i = \lambda_i q_i, \quad L^* r_i = \lambda_i r_i, \quad 1 \leqslant i \leqslant N,$$

then they are eigenfunctions and adjoint eigenfunctions of the spectral problems:

$$L\phi = \lambda\phi, \quad \phi_{\tau_k} = B_k\phi,$$

and

$$L^*\phi = \lambda\phi, \quad \phi_{\tau_k} = -B_k^*\phi,$$

each of which yields the Lax equation $L_{\tau_k} = [B_k, L]$.

Now, the pseudo-differential operators

$$\widetilde{B}_k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, \quad k \geqslant 1, \tag{2.9}$$

lead to the so-called extended KP hierarchy

$$L_{\tau_k} = [\widetilde{B}_k, L], \quad k \geqslant 1. \tag{2.10}$$

This hierarchy is compatible with the KP flows in (2.6) and examples of the extended KP flows are given in [18].

We remark that the processes of constructing the extended flows and the constrained flows look very similar. In practice, the extended flows $L_{\tau_k} = [\widetilde{B}_k, L]$, $k \geqslant 1$, are generated from a standard pseudo-differential spectral operator $L = P(\partial) + \sum_{i=1}^{\infty} u_i \partial^{-i}$ and non-standard Lax operators $\widetilde{B}_k = \left(L^{\frac{k}{m}}\right)_+ + \sum_{i=1}^{N} q_i \partial^{-1} r_i$, but the constrained flows $L_{t_n} = [B_n, L]$, $n \geqslant 1$, are generated from a non-standard pseudo-differential spectral operator $L = P(\partial) + \sum_{i=1}^{N} q_i \partial^{-1} r_i$ and standard Lax operators $B_n = \left(L^{\frac{n}{m}}\right)_+$, where P is a polynomial of order m. Therefore, the resulting extended flows and constrained flows have different characters.

2.2. The compatibility equations

The question for us here is whether a pair of two extended KP flows in (2.10) commute with each other or not. In what follows, we will discuss the commutativity problem of the extended KP hierarchy (2.10), to see what the compatibility equations will be.

To analyze the commutativity problem, we will use a basic formula:

$$\left[B_n, q\partial^{-1} r\right]_- = (B_n q)\partial^{-1} r - q\partial^{-1}(B_n^* r), \tag{2.11}$$

where the differential operator B_n is defined by (2.7) and $P_- = P_{<0}$ is defined as in (2.4). Actually the formula (2.11) works for all differential operators (not only for B_n). This feature provides insight into the role of the pseudo-differential operator $q\partial^{-1} r$. We provide an answer to the commutativity question as follows.

Theorem 2.1. *Let the Lax operator B_k and the extended Lax operator $\widetilde{B}_k$ be defined by (2.7) and (2.9). Then for $k, l \geqslant 1$, under the conditions*

$$q_{i,\tau_k} = B_k q_i, \quad r_{i,\tau_k} = -B_k^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.12}$$

and

$$q_{i,\tau_l} = B_l q_i, \quad r_{i,\tau_l} = -B_l^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.13}$$

the τ_k-flow by $L_{\tau_k} = [\widetilde{B}_k, L]$ and the τ_l-flow by $L_{\tau_l} = [\widetilde{B}_l, L]$ in the extended KP hierarchy (2.10) commute if and only if the following differential relation

$$B_{k,\tau_l} - B_{l,\tau_k} + [B_k, B_l] + \left[B_k, \sum_{i=1}^{N} q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i, B_l\right]_+ = 0 \tag{2.14}$$

holds.

Proof. Obviously, the pseudo-differential zero curvature equation

$$\widetilde{B}_{k,\tau_l} - \widetilde{B}_{l,\tau_k} + \left[\widetilde{B}_k, \widetilde{B}_l\right] = 0 \tag{2.15}$$

implies that

$$\left[\widetilde{B}_{k,\tau_l} - \widetilde{B}_{l,\tau_k} + [\widetilde{B}_k, \widetilde{B}_l], L\right] = 0.$$

This tells that $L_{\tau_k,\tau_l} = L_{\tau_l,\tau_k}$, and thus, the two Lax equations

$$L_{\tau_k} = [\widetilde{B}_k, L], \quad L_{\tau_l} = [\widetilde{B}_l, L], \tag{2.16}$$

are compatible with each other and the corresponding flows commute.

In what follows, we are going to verify that the zero curvature (2.15) is necessary to guarantee the commutativity of the τ_k-flow and the τ_l-flow in the hierarchy (2.10). For brevity, we only focus on the case of $N = 1$, and rewrite q_1 and r_1 as q and r, respectively. Based on the Lax equations in (2.16), and noting that

$$[q\partial^{-1} r, L_-^n]_+ = 0,$$

we can compute that

$$\widetilde{B}_{k,\tau_l} = B_{k,\tau_l} + (q\partial^{-1}r)_{\tau_l} = \left[B_l + q\partial^{-1}r, L^k\right]_+ + (q\partial^{-1}r)_{\tau_l} = [B_l, L^k]_+ + [q\partial^{-1}r, L^k]_+ + q_{\tau_l}\partial^{-1}r + q\partial^{-1}r_{\tau_l}$$
$$= [B_l, L^k]_+ + [q\partial^{-1}r, B_k]_+ + (B_l q)\partial^{-1}r - q\partial^{-1}(B_l^* r), \tag{2.17}$$

and similarly, we have

$$\widetilde{B}_{l,\tau_k} = [B_k, L^l]_+ + [q\partial^{-1}r, B_l]_+ + (B_k q)\partial^{-1}r - q\partial^{-1}(B_k^* r). \tag{2.18}$$

Moreover, on one hand, we have

$$[B_k, B_l] = \left[L^k - L^k_-, L^l - L^l_-\right]_+ = -[L^k_-, L^l]_+ - [L^k, L^l_-]_+ = [B_k, L^l]_+ - [L^k, B_l]_+, \tag{2.19}$$

and on the other hand, we have

$$[B_k, q\partial^{-1}r] = [B_k, q\partial^{-1}r]_+ + [B_k, q\partial^{-1}r]_- = [B_k, q\partial^{-1}r]_+ + (B_k q)\partial^{-1}r - q\partial^{-1}(B_k^* r), \tag{2.20}$$

and similarly,

$$[q\partial^{-1}r, B_l] = [q\partial^{-1}r, B_l]_+ - (B_l q)\partial^{-1}r + q\partial^{-1}(B_l^* r), \tag{2.21}$$

where (2.11) was used.

Now since

$$[\widetilde{B}_k, \widetilde{B}_l] = [B_k, B_l] + [B_k, q\partial^{-1}r] + [q\partial^{-1}r, B_l],$$

we can see from (2.17)–(2.21) that the zero curvature Eq. (2.15) holds.

Finally based on the formula (2.11), the introduction of q_i and r_i, $1 \leqslant i \leqslant N$, by (2.12) and (2.13) allows us conclude that the zero curvature Eq. (2.15) is equivalent to the differential relation (2.14). This completes the proof of the theorem. □

The compatibility equation of the τ_k-flow and the τ_l-flow is so given by

$$\left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\right)_{\tau_l} - \left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\right)_{\tau_k} + \left[B_k + \sum_{i=1}^N q_i\partial^{-1}r_i, B_l + \sum_{i=1}^N q_i\partial^{-1}r_i\right] = 0, \tag{2.22a}$$

$$q_{i,\tau_k} = B_k q_i, \quad r_{i,\tau_k} = -B_k^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.22b}$$

$$q_{i,\tau_l} = B_l q_i, \quad r_{i,\tau_l} = -B_l^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.22c}$$

where k, l, $N \geqslant 1$. Under (2.22b) and (2.22c), (2.22a) has the Lax representation:

$$\psi_{\tau_k} = \left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\right)\psi, \quad \psi_{\tau_l} = \left(B_l + \sum_{i=1}^N q_i\partial^{-1}r_i\right)\psi. \tag{2.23}$$

We want to point out that (2.22a), equivalently, (2.14), is a differential operator equation. It generalizes the triple L–A–B representation of integrable equations, which has nice algebraic structures [24].

In order to compute examples of the compatibility equations of the extended KP hierarchy, we list some useful expressions in the following proposition.

Proposition 2.1. *The following equalities hold:*

$$[a\partial^2, q\partial^{-1}r]_+ = 2a(qr)_x + a_x qr,$$
$$[a\partial^3, q\partial^{-1}r]_+ = [3a(qr)_x + a_x qr]\partial + 3a(q_x r)_x - a_{xx}qr - 2a_x qr_x,$$

$$[a\partial^4, q\partial^{-1}r]_+ = [4a(qr)_x + a_x qr]\partial^2 + \left[6a(qr)_{xx} - 4a(qr_x)_x - a_{xx}qr - 2a_x qr_x\right]\partial + 4a(qr)_{xxx} - 6a(qr_x)_{xx} + 4a(qr_{xx})_x + a_{xxx}qr$$
$$+ 3a_{xx}qr_x + 3a_x qr_{xx},$$

$$[B_2, B_3] = 3(u_{xx} + 2v_x)\partial + u_{xxx} - 6uu_x + 3v_{xx},$$

$$[B_2, B_4] = 4(u_{xx} + 2v_x)\partial^2 + 2(u_{xxx} + 4uu_x + 8v_{xx} + 4w_x)\partial + 2(u_{xxxx} + 2uu_{xx} - 4u_x v + 3v_{xxx} + 2w_{xx}),$$

where $P_+ = P_{\geqslant 0}$ is defined as in (2.4).

The proof of this proposition just needs some direct calculation, and the proposition itself shows the non-symmetric feature of the eigenfunctions q_i's and the adjoint eigenfunctions r_i's. Now the following two examples of the compatibility equation (2.22) of the extended KP hierarchy can easily be presented.

Example 2.1. Let us first take $k = 2$ and $l = 3$, and set $y = \tau_2$ and $t = \tau_3$. Then the compatibility equation (2.22) becomes

$$B_{2,t} - B_{3,y} + [B_2, B_3] + \left[B_2, \sum_{i=1}^N q_i\partial^{-1}r_i\right]_+ + \left[\sum_{i=1}^N q_i\partial^{-1}r_i, B_3\right]_+ = 0, \tag{2.24a}$$

$$q_{i,y} = B_2 q_i, \quad r_{i,y} = -B_2^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.24b}$$
$$q_{i,t} = B_3 q_i, \quad r_{i,t} = -B_3^* r_i, \quad 1 \leqslant i \leqslant N. \tag{2.24c}$$

On the basis of Proposition (2.1), the nonlinear system above reads

$$u_y - u_{xx} - 2v_x + \sum_{i=1}^N (q_i r_i)_x = 0, \tag{2.25a}$$
$$2u_t - 3(u_x + v)_y + u_{xxx} - 6uu_x + 3v_{xx} + \sum_{i=1}^N [2(q_i r_i)_x - 3(q_{i,x} r_i)_x] = 0, \tag{2.25b}$$
$$q_{i,y} = q_{i,xx} + 2uq_i, \quad r_{i,y} = -r_{i,xx} - 2ur_i, \quad 1 \leqslant i \leqslant N, \tag{2.25c}$$
$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + 3(u_x + v)q_i, \quad r_{i,t} = r_{i,xxx} + 3ur_{i,x} - 3vr_i, \quad 1 \leqslant i \leqslant N. \tag{2.25d}$$

Under (2.25c) and (2.25d), this (2+1)-dimensional nonlinear system has the Lax representation:

$$\psi_y = \left(\partial^2 + 2u + \sum_{i=1}^N q_i \partial^{-1} r_i\right)\psi,$$
$$\psi_t = \left[\partial^3 + 3u\partial + 3(u_x + v) + \sum_{i=1}^N q_i \partial^{-1} r_i\right]\psi.$$

Example 2.2. Let us second take $k = 2$ and $l = 4$, and set $y = \tau_2$ and $t = \tau_4$. Then the compatibility equation (2.22) becomes

$$B_{2,t} - B_{4,y} + [B_2, B_4] + \left[B_2, \sum_{i=1}^N q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^N q_i \partial^{-1} r_i, B_4\right]_+ = 0, \tag{2.26a}$$
$$q_{i,y} = B_2 q_i, \quad r_{i,y} = -B_2^* r_i, \quad 1 \leqslant i \leqslant N, \tag{2.26b}$$
$$q_{i,t} = B_4 q_i, \quad r_{i,t} = -B_4^* r_i, \quad 1 \leqslant i \leqslant N. \tag{2.26c}$$

On the basis of Proposition (2.1), the nonlinear system above reads

$$u_y - u_{xx} - 2v_x + \sum_{i=1}^N (q_i r_i)_x = 0, \tag{2.27a}$$
$$-(3u_x + 2v)_y + u_{xxx} + 4uu_x + 8v_{xx} + 4w_x - \sum_{i=1}^N [3(q_i r_i)_{xx} - 2(q_i r_{i,x})_x] = 0, \tag{2.27b}$$
$$u_t - (2u_{xx} + 3v_x + 2w)_y + u_{xxxx} + 2uu_{xx} - 4u_x v + 3v_{xxx} + 2w_{xx}$$
$$+ \sum_{i=1}^N \left[(q_i r_i)_x - 2(q_i r_i)_{xxx} + 3(q_i r_{i,x})_{xx} - 2(q_i r_{i,xx})_x - 4u(q_i r_i)_x - 2u_x q_i r_i\right] = 0, \tag{2.27c}$$
$$q_{i,y} = q_{i,xx} + 2uq_i, \quad r_{i,y} = -r_{i,xx} - 2ur_i, \quad 1 \leqslant i \leqslant N, \tag{2.27d}$$
$$q_{i,t} = q_{i,xxxx} + 4uq_{i,xx} + 2(3u_x + 2v)q_{i,x} + 2(2u_{xx} + 3u^2 + 3v_x + 2w)q_i, \quad 1 \leqslant i \leqslant N, \tag{2.27e}$$
$$r_{i,t} = -r_{i,xxxx} - 4ur_{i,xx} - 2(u_x - 2v)r_{i,x} - 2(u_{xx} + 3u^2 + v_x + 2w)r_i, \quad 1 \leqslant i \leqslant N. \tag{2.27f}$$

Under (2.27d), (2.27e) and (2.27f), this (2+1)-dimensional nonlinear system has the Lax representation:

$$\psi_t = \left(\partial^2 + 2u + \sum_{i=1}^N q_i \partial^{-1} r_i\right)\psi,$$
$$\psi_y = \left[\partial^4 + 4u\partial^2 + 2(3u_x + 2v)\partial + 2(2u_{xx} + 3u^2 + 3v_x + 2w) + \sum_{i=1}^N q_i \partial^{-1} r_i\right]\psi.$$

2.3. The *k*-constraint

Let us now consider the *k*-constraint

$$L^k = \widetilde{B}_k = B_k + \sum_{i=1}^N q_i \partial^{-1} r_i, \tag{2.28}$$

which presents a holonomic constraint when $k = 1$ and a non-holonomic constraint when $k \geqslant 2$. We require that the eigenfunctions and adjoint eigenfunctions satisfy

$$B_k q_i = \lambda_{k,i} q_i, \quad B_k^* r_i = \lambda_{k,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{2.29}$$

where $\lambda_{k,i}$, $1 \leqslant i \leqslant N$, are constants, to avoid using the evolution law of the q_i's and r_i's with respect to τ_k. Then on the sub-manifold determined by the k-constraint (2.28), we can have

$$\begin{aligned}
&(L^k)_{\tau_l} = [\widetilde{B}_l, L^k],\\
&B_{k,\tau_l} = (L^k_{\tau_l})_+ = [\widetilde{B}_l, L^k]_+,\\
&\left(\sum_{i=1}^N q_i \partial^{-1} r_i\right)_{\tau_l} = \sum_{i=1}^N \left(q_{i,\tau_l}\partial^{-1} r_i + q_i \partial^{-1} r_{i,\tau_l}\right) = \sum_{i=1}^N \left[(B_l q_i)\partial^{-1} r_i - q_i \partial^{-1}(B_l^* r_i)\right] = \left[B_l, \sum_{i=1}^N q_i \partial^{-1} r_i\right]_- = \left[\widetilde{B}_l, \sum_{i=1}^N q_i \partial^{-1} r_i\right]_-\\
&\qquad = [\widetilde{B}_l, L^k - B_k]_- = [\widetilde{B}_l, L^k]_- - [\widetilde{B}_l, B_k]_- = [\widetilde{B}_l, L^k]_-,
\end{aligned}$$

where (2.11) was used for the first time, because we have

$$[\widetilde{B}_l, B_k]_- = \left[\sum_{i=1}^N q_i \partial^{-1} r_i, B_k\right]_- = \sum_{i=1}^N \left[q_i \partial^{-1}(B_k^* r_i) - (B_k q_i)\partial^{-1} r_i\right] = \sum_{i=1}^N \left[q_i \partial^{-1}(\lambda_{k,i} r_i) - (\lambda_{k,i} q_i)\partial^{-1} r_i\right] = 0,$$

where (2.11) was used for the second time. It then follows that

$$(L^k)_{\tau_l} = B_{k,\tau_l} + \left(\sum_{i=1}^N q_i \partial^{-1} r_i\right)_{\tau_l}.$$

Therefore, the sub-manifold determined by the k-constraint (2.28) is invariant under the τ_l-flow.

We constrain the τ_l-flow on this sub-manifold, and so, we have $L_{\tau_k} = [\widetilde{B}_k, L] = [L^k, L] = 0$ and further $B_{l,\tau_k} = (L^l_{\tau_k})_+ = 0$. It is now direct to show that under

$$\begin{aligned}
&B_k q_i = \lambda_{k,i} q_i, \quad B_k^* r_i = \lambda_{k,i} r_i, \quad 1 \leqslant i \leqslant N,\\
&q_{i,\tau_l} = B_l q_i, \quad r_{i,\tau_l} = -B_l^* r_i, \quad 1 \leqslant i \leqslant N,
\end{aligned}$$

the pseudo-differential zero curvature Eq. (2.14) becomes

$$B_{k,\tau_l} + [B_k, B_l] + \left[B_k, \sum_{i=1}^N q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^N q_i \partial^{-1} r_i, B_l\right]_+ = 0. \tag{2.30}$$

Therefore, the compatibility equation under the k-constraint reads

$$B_{k,\tau_l} + [B_k, B_l] + \left[B_k, \sum_{i=1}^N q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^N q_i \partial^{-1} r_i, B_l\right]_+ = 0, \tag{2.31a}$$

$$B_k q_i = \lambda_{k,i} q_i, \quad B_k^* r_i = \lambda_{k,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{2.31b}$$

$$q_{i,\tau_l} = B_l q_i, \quad r_{i,\tau_l} = -B_l^* r_i, \quad 1 \leqslant i \leqslant N. \tag{2.31c}$$

This is the k-constrained hierarchy of the compatibility equations with self-consistent sources. With (2.31b) and (2.31c), it has the Lax representation:

$$\left(B_k + \sum_{i=1}^N q_i \partial^{-1} r_i\right)\psi = \mu\psi, \quad \psi_{\tau_l} = \left(B_l + \sum_{i=1}^N q_i \partial^{-1} r_i\right)\psi, \tag{2.32}$$

where μ is a spectral parameter.

Example 2.3. Similarly, let us first take $k = 2$ and $l = 3$, and set $y = \tau_2$ and $t = \tau_3$. Then the compatibility equation (2.31) becomes

$$B_{2,t} + [B_2, B_3] + \left[B_2, \sum_{i=1}^N q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^N q_i \partial^{-1} r_i, B_3\right]_+ = 0, \tag{2.33a}$$

$$B_2 q_i = \lambda_{2,i} q_i, \quad B_2^* r_i = \lambda_{2,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{2.33b}$$

$$q_{i,t} = B_3 q_i, \quad r_{i,t} = -B_3^* r_i, \quad 1 \leqslant i \leqslant N. \tag{2.33c}$$

On the basis of Proposition (2.1), the nonlinear system above reads

$$u_{xx} + 2v_x - \sum_{i=1}^{N}(q_i r_i)_x = 0, \tag{2.34a}$$

$$2u_t + u_{xxx} - 6uu_x + 3v_{xx} + \sum_{i=1}^{N}\left[2(q_i r_i)_x - 3(q_{i,x} r_i)_x\right] = 0, \tag{2.34b}$$

$$q_{i,xx} + 2uq_i = \lambda_{2,i} q_i, \quad r_{i,xx} + 2ur_i = \lambda_{2,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{2.34c}$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + 3(u_x + v)q_i, \quad r_{i,t} = r_{i,xxx} + 3ur_{i,x} - 3vr_i, \quad 1 \leqslant i \leqslant N. \tag{2.34d}$$

Under (2.34c) and (2.34d), this (2+1)-dimensional nonlinear system has the Lax representation:

$$\left(\partial^2 + 2u + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right)\psi = \mu\psi,$$

$$\psi_t = \left[\partial^3 + 3u\partial + 3(u_x + v) + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right]\psi.$$

Example 2.4. Similarly, let us second take $k = 4$ and $l = 2$, and set $y = \tau_4$ and $t = \tau_2$. Now the compatibility equation (2.31) becomes

$$-B_{2,y} + [B_4, B_2] + \left[B_4, \sum_{i=1}^{N} q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i, B_2\right]_+ = 0, \tag{2.35a}$$

$$B_4 q_i = \lambda_{4,i} q_i, \quad B_4^* r_i = \lambda_{4,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{2.35b}$$

$$q_{i,t} = B_2 q_i, \quad r_{i,t} = -B_2^* r_i, \quad 1 \leqslant i \leqslant N. \tag{2.35c}$$

On the basis of Proposition (2.1), the nonlinear system above reads

$$u_{xx} + 2v_x - \sum_{i=1}^{N}(q_i r_i)_x = 0, \tag{2.36a}$$

$$u_{xxx} + 4uu_x + 8v_{xx} + 4w_x - \sum_{i=1}^{N}\left[3(q_i r_i)_{xx} - 2(q_i r_{i,x})_x\right] = 0, \tag{2.36b}$$

$$u_y + u_{xxxx} + 2uu_{xx} - 4u_x v + 3v_{xxx} + 2w_{xx} + \sum_{i=1}^{N}\left[(q_i r_i)_x - 2(q_i r_i)_{xxx} + 3(q_i r_{i,x})_{xx} - 2(q_i r_{i,xx})_x - 4u(q_i r_i)_x - 2u_x q_i r_i\right] = 0, \tag{2.36c}$$

$$q_{i,xxxx} + 4uq_{i,xx} + 2(3u_x + 2v)q_{i,x} + 2\left(2u_{xx} + 3u^2 + 3v_x + 2w\right)q_i = \lambda_{4,i} q_i, \quad 1 \leqslant i \leqslant N, \tag{2.36d}$$

$$r_{i,xxxx} + 4ur_{i,xx} + 2(u_x - 2v)r_{i,x} + 2\left(u_{xx} + 3u^2 + v_x + 2w\right)r_i = \lambda_{4,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{2.36e}$$

$$q_{i,t} = q_{i,xx} + 2uq_i, \quad r_{i,t} = -r_{i,xx} - 2ur_i, \quad 1 \leqslant i \leqslant N. \tag{2.36f}$$

Under (2.36d), (2.36e) and (2.36f), this (2+1)-dimensional nonlinear system has the Lax representation:

$$\left[\partial^4 + 4u\partial^2 + 2(3u_x + 2v)\partial + 2(2u_{xx} + 3u^2 + 3v_x + 2w) + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right]\psi = \mu\psi,$$

$$\psi_t = \left(\partial^2 + 2u + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right)\psi.$$

3. Conclusions and remarks

We considered the commutativity problem of the extended KP flows and constructed the compatibility equation of two extended KP flows. The k-constraint problem was discussed and the resulting hierarchy of the compatibility equations reduces to the compatibility equations between the extended KP hierarchy and the k-constrained KP hierarchy. A few particular examples in each of two cases were computed. Since the (2+1)-dimensional KP hierarchy can be reduced to (1+1)-

dimensional soliton hierarchies, many (1+1)-dimensional compatibility equations of extended soliton hierarchies could be generated.

Theorem 2.1 exposes that unlike the constrained KP flows, the extended KP flows don't commute automatically, since the differential relation (2.14) brings new conditions – the compatibility equations. The presented examples show us that those compatibility equations could be complicated nonlinear partial differential equations. It is still open to us how to solve the compatibility equations analytically.

There should also exist complexiton solutions, besides solitons and postions, to the compatibility equation hierarchy. Inverse scattering technique [9] and Darboux transformation method [10] could be helpful in solving the compatibility equation hierarchy. Soliton type solutions could be expressed through the Wronskian formulation [22,25,26], and big solution subspaces of the compatibility equation hierarchy might be constructed.

We remark that there are analogous commutativity problems for the extended Harry Dym hierarchy and the extended non-commutative KP hierarchy, discussed in [27] and [28] respectively. A more general commutativity problem is as follows. Assume that a pair of extended Lax operators is defined by

$$\widetilde{B}_k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, \quad \widetilde{B}_l = B_l + \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j,$$

where M and N are natural numbers, and two sets of eigenfunctions and adjoint eigenfunctions are required to satisfy

$$q_{i,\tau_l} = B_l q_i, \quad r_{i,\tau_l} = -B_l^* r_i, \quad 1 \leqslant i \leqslant N,$$

and

$$\alpha_{j,\tau_k} = B_k \alpha_j, \quad \beta_{j,\tau_k} = -B_k^* \beta_j, \quad 1 \leqslant j \leqslant M.$$

To guarantee the commutativity of the τ_k-flow and the τ_l-flow defined by

$$L_{\tau_k} = [\widetilde{B}_k, L], \quad L_{\tau_l} = [\widetilde{B}_l, L],$$

we need to compute a more general pseudo-differential zero curvature equation

$$\widetilde{B}_{k,\tau_l} - \widetilde{B}_{l,\tau_k} + [\widetilde{B}_k, \widetilde{B}_l] = 0.$$

Based on the formula (2.11), this equation is equivalent to

$$B_{k,\tau_l} - B_{l,\tau_k} + [B_k, B_l] + \left[B_k, \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j \right]_+ + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i, B_l \right]_+ = 0,$$

and

$$\left[\sum_{i=1}^{N} q_i \partial^{-1} r_i, \ \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j \right] = 0.$$

The corresponding compatibility equations will be much more general than the ones of the extended KP hierarchy (2.22). However, the last equation above generates infinitely many differential constraints on the eigenfunctions and adjoint eigenfunctions: q_i and r_i, $1 \leqslant i \leqslant N$, and α_j and β_j, $1 \leqslant j \leqslant M$. It is interesting how to satisfy these constraints generally to guarantee the commutativity of the τ_k-flow and the τ_l-flow. One possibility was given in this paper: $M = N$, $q_i = \alpha_i$, $r_i = \beta_i$, $1 \leqslant i \leqslant N$; and another possibility is to take one set of zero eigenfunctions and adjoint eigenfunctions: $q_i = r_i = 0$, $1 \leqslant i \leqslant N$, or $\alpha_j = \beta_j = 0$, $1 \leqslant j \leqslant M$, which gives the extended KP flows [18]. How about the general case?

Acknowledgements:

The work was supported in part by the Established Researcher Grant and the CAS faculty development grant and the CAS Dean research grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, and the State Administration of Foreign Experts Affairs of China.

References

[1] Mel'nikov VK. Lett Math Phys 1983;7:129.
[2] Mel'nikov VK. Phys Lett A 1988;133:493.
[3] Ma WX. J Phys Soc Jpn 2003;72:3017.
[4] Cheng Y, Li YS. Phys Lett A 1991;157:22.
[5] Konopelchenko B, Sidorenko J, Strampp W. Phys Lett A 1991;157:17.
[6] Oevel W, Strampp W. Comm Math Phys 1993;157:51.
[7] Aratyn H, Nissimov E, Pacheva S. Internat J Modern Phys A 1997;12:1265.
[8] Oevel W, Strampp W. J Math Phys 1996;37:6213.
[9] Lin RL, Zeng YB, Ma WX. Phys A 2001;291:287.

[10] Zeng YZ, Ma WX, Shao YJ. J Math Phys 2001;42:2113.
[11] Ma WX. Chaos Solitons & Fractals 2005;26:1453.
[12] Ma WX, Zeng YB. ANZIAM J. 2002;44:129.
[13] Ma WX, Zhou RG. Chin Annals Math B 2002;23:373.
[14] Xiong CS. Lett Math Phys 1996;36:223.
[15] Bogdanov LV, Konopelchenko BG. J Math Phys 1998;39:4683.
[16] Kamata M, Nakamula A. J Phys A: Math Gen 2002;35:9657.
[17] Dimakis A, Müller-Hoissen F. J Phys A: Math Gen 2004;37:10899.
[18] Liu XJ, Zeng YB, Lin RL. Phys Lett A 2008;372:3819.
[19] Lin RL, Liu XJ, Zeng YB. J Nonlinear Math Phys 2008;15:333.
[20] Dickey LA. Soliton equations and Hamiltonian systems. 2nd ed. River Edge, NJ: World Scientific; 2003.
[21] Ohta Y, Satsuma J, Takahashi D, Tokihiro T. Prog Theoret Phys Suppl 1988;94:210.
[22] Liu XJ, Lin RL, Jin B, Zeng YB. J Math Phys 2009;50:053506.
[23] Konopelchenko B, Oevel W. Publ Res Inst Math Sci 1993;29:581.
[24] Ma WX. Chin Sci Bull 1992;37:1249.
[25] Ma WX, You Y. Trans Amer Math Soc 2005;357:1753.
[26] Ma WX, Li CX, He JS. Nonlinear Anal 2009;70:4245.
[27] Ma WX. J Phys A: Math Theor 2010;43:165202.
[28] Ma WX. J Math Phys in press;51.