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A generating scheme for conservation laws of discrete zero curvature equations and its application

Wen-Xiu Ma [*]

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China
College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China
International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

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ABSTRACT

A generating scheme for conservation laws of discrete integrable equations, arising as discrete zero curvature equations, is presented from pairs of associated matrix spectral problems. An illustrative application to the Volterra lattice equation is made through a pair of 2×2 matrix spectral problems. The general theory is not limited to 2×2 matrix spectral problems but for arbitrary-order matrix spectral problems.

1. Introduction

It is an important topic in soliton theory to search for conservation laws for nonlinear integrable differential or differential–difference equations. Conservation laws in physics state that particular measurable properties of physical systems do not change as the systems evolve over time. Noether's theorem tells that conservation laws are associated with symmetries for Lagrangian equations [1]. When an equation is not Lagrangian, its conservation laws come from pairs of symmetries and adjoint symmetries [2,3]. Mathematically speaking, conservation laws are helpful in solving nonlinear equations. With the help of conservation laws, we can decrease orders or numbers of differential and differential–difference equations, and particularly, we can often reduce partial differential equations to ordinary differential equations and ordinary differential equations to algebraic equations. Conservation laws play a crucial role in treating equations of mathematical physics, indeed [4].

It is known that integrable equations are generated from zero curvature equations, whose Liouville integrability could be explored usually through applying the trace identities [5,6], or more generally, the variational identities [7,8]. Zero curvature equations are compatibility conditions of pairs of matrix spectral problems, whose Cauchy problems can be solved by the inverse scattering transform [9,10]. The pairs of spectral problems are called Lax pairs [11], and 2×2 matrix Lax pairs have been used to construct conservation laws (see, e.g., [12–15]).

We shall focus on integrable lattice equations within the discrete zero curvature formulation [6,8]. If f be a lattice function or matrix, its shift with E and reverse shift with E^{-1} are given by

$$(Ef)(n) = f(n+1), \ (E^{-1}f)(n) = f(n-1), \ n \in \mathbb{Z}, \tag{1.1}$$

* Correspondence to: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA.
E-mail address: mawx@cas.usf.edu.

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and a general shift or reverse shift is simply denoted by

$$f^{(k)}(n) = (E^k f)(n) = f(n+k),\ \ n, k \in \mathbb{Z}. \tag{1.2}$$

A lattice equation

$$u_t = K(u, u^{(1)}, u^{(-1)}, \dots),\ \ u = u(n,t), \tag{1.3}$$

is said to possess a Lax pair, U and V, which are two square matrices, if it can be expressed as a discrete zero curvature equation (see, e.g., [6,8])

$$U_t = (EV)U - UV. \tag{1.4}$$

This is the compatibility condition of a pair of matrix spectral problems, often also referred as a Lax pair:

$$E\phi = U(u, \lambda)\phi \tag{1.5}$$

and

$$\phi_t = V(u, u^{(1)}, u^{(-1)}, \dots; \lambda)\phi, \tag{1.6}$$

where λ is a spectral parameter, ϕ is an eigenfunction, and U is called a spectral matrix and V, a Lax operator or matrix. In the setting up of the Lax theory, the key is a spatial matrix spectral problem. Generally, zero curvature equations resulting from matrix spectral problems can carry high nonlinearity, but they often possess Abelian Lax operator algebras [16] and conserved densities constructed from the discrete trace identity [6] or discrete variational identity [8].

In this paper, we would like to formulate a general scheme to generate conservation laws of discrete integrable equations from arbitrary-order matrix spectral problems. Such a scheme is more direct than many other methods for constructing conservation laws. To shed light on the general scheme, an illustrative application to the Volterra lattice equation will be made, based on a pair of 2×2 matrix spectral problems. A few concluding remarks will be given in the last section.

2. General scheme for conservation laws

We would like to formulate a general scheme for constructing conservation laws from pairs of arbitrary-order matrix spectral problems. Let us begin with a pair of matrix spectral problems:

$$E\phi = U(u, \lambda)\phi \tag{2.1}$$

and

$$\phi_t = V(u, u^{(1)}, u^{(-1)}, \lambda)\phi, \tag{2.2}$$

where $u = u(n,t)$ and

$$\phi = (\phi_1, \dots, \phi_r)^T,\ \ U = (U_{ij})_{r\times r},\ \ V = (V_{ij})_{r\times r}, \tag{2.3}$$

r being an arbitrary natural number. A discrete integrable equation is generated from the discrete zero curvature equation

$$U_t = (EV)U - UV. \tag{2.4}$$

Introduce the ratios of eigenfunctions:

$$\psi_{ij} = \frac{\phi_i}{\phi_j},\ \ 1 \le i, j \le r, \tag{2.5}$$

and then we have a set of quadratic equations for the ratios to satisfy:

$$\Big(\sum_{k=1}^{r} U_{jk}\psi_{kj}\Big)E\psi_{ij} = \Big(\sum_{k=1}^{r} U_{ik}\psi_{ki}\Big)\psi_{ij},\ \ 1 \le i, j \le r. \tag{2.6}$$

In the continuous case, such quadratic equations are replaced with Riccati type differential equations. If we further define the Laurent expansions for the ratios of eigenfunctions as follows:

$$\psi_{ij} = \lambda^{\alpha} \sum_{k\ge 0} \theta_{ij,k}\lambda^{-k},\ \ 1 \le i, j \le r, \tag{2.7}$$

where α is a constant depending on U, then substituting them into the quadratic equations in (2.6) leads to a system of recursion relations for determining $\theta_{ij,k},\ 1 \le i \ne j \le r,\ k \ge 0$.

Now, based on the pair of matrix spectral problems (2.1) and (2.2), we can compute that

$$(\ln \frac{E\phi_i}{\phi_i})_t = [(E-1)\ln \phi_i]_t = (E-1)\frac{\phi_{i,t}}{\phi_i}$$
$$= (E-1)\frac{\sum_{k=1}^r V_{ik}\phi_k}{\phi_i} = (E-1)\sum_{k=1}^r V_{ik}\psi_{ki},\ 1 \le i \le r,$$

and

$$(\ln \frac{E\phi_i}{\phi_i})_t = (\ln \frac{\sum_{k=1}^r U_{ik}\phi_k}{\phi_i})_t = (\ln \sum_{k=1}^r U_{ik}\psi_{ki})_t,\ 1 \le i \le r.$$

Therefore, upon defining

$$X_i = \sum_{k=1}^r V_{ik}\psi_{ki},\ T_i = \ln \sum_{k=1}^r U_{ik}\psi_{ki},\ 1 \le i \le r, \tag{2.8}$$

we have the following r generating formulas for conservation laws:

$$(T_i)_t = (E-1)X_i,\ 1 \le i \le r. \tag{2.9}$$

Plugging the Laurent expansions of T_i and X_i in terms of λ,

$$T_i = \lambda^\beta \sum_{k\ge 0} T_{i,k}\lambda^{-k},\ X_i = \lambda^\beta \sum_{k \ge 0} X_{i,k}\lambda^{-k},\ 1 \le i \le r, \tag{2.10}$$

where β is a constant depending on U and V, and comparing the coefficients of the same powers of λ generate infinitely many conservation laws

$$(T_{i,k})_t = (E-1)X_{i,k},\ k \ge 0,\ 1 \le i \le r, \tag{2.11}$$

for the zero curvature equation (2.4). Here the conserved densities $T_{i,k}$ and the conserved fluxes $X_{i,k}$ are all functions of $\theta_{i'j',k'}$, $1 \le i' \ne j' \le r,\ k' \ge 0$. Note that in general, for each $1 \le i \le r$, $\ln \phi_i$ has no Laurent expansion of λ, and so $T_{i,k}$, $k \ge 0$, are nontrivial, i.e., are not of form $(E-1)S_{i,k}$, $k \ge 0$, where the $S_{i,k}$'s are differential–difference functions.

Next, let us show that two sequences of conservation laws with different integers $1 \le i, j \le r$ will be equivalent to each other, i.e., the differences of the conserved densities $T_{i,k} - T_{j,k}$, $k \ge 0$, are of form $(E-1)S_{ij,k}$, $k \ge 0$, where the $S_{ij,k}$'s are differential–difference functions.

This could be proved as follows. First, note that we have

$$(E-1)\ln \psi_{ij} = \ln E\psi_{ij} - \ln \psi_{ij}$$
$$= \ln \frac{E\psi_{ij}}{\psi_{ij}} = \ln \frac{\sum_{k=1}^r U_{ik}\psi_{ki}}{\sum_{k=1}^r U_{jk}\psi_{kj}},\ 1 \le i \ne j \le r,$$

where we have used the quadratic equations in (2.6). It then follows that

$$T_i - T_j = (E-1)\ln \psi_{ij},\ 1 \le i \ne j \le r. \tag{2.12}$$

Therefore, T_i is equivalent to T_j for $1 \le i \ne j \le r$.

To conclude, we only need to consider one generating formula with $1 \le i \le r$ to compute nontrivial conservation laws, while dealing with practical problems.

3. Application to the Volterra lattice equation

3.1. Lax pair

Let us consider the Volterra lattice equation [17]:

$$u_t = u(u^{(-1)} - u^{(1)}), \tag{3.1}$$

where $u = u(n,t)$. This Volterra equation has important applications in mathematical ecology [17]. We begin with a spatial 2×2 matrix discrete spectral problem:

$$E\phi = U\phi,\ U = U(u,\lambda) = \begin{bmatrix} \lambda & \lambda u \\ 1 & 0 \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{3.2}$$

where u is the potential and λ is the spectral parameter. This is a scalar multiple of the spectral problem in [16]. It is easy to check (or see, e.g., [16]) that the Volterra lattice equation (3.1) can be expressed as the compatibility condition of the spatial matrix discrete spectral problem (3.2) and the temporal matrix spectral problem

$$\phi_t = V\phi,\ V = V(u,\lambda) = \begin{bmatrix} \frac{1}{2}\lambda - u & \lambda u \\ 1 & -\frac{1}{2}\lambda - u^{(-1)} \end{bmatrix}. \tag{3.3}$$

Associated with the Lax pair in (3.2) and (3.3), a Darboux transformation has been presented for the Volterra lattice equation (3.1) in [18].

3.2. Conservation laws

For the Volterra lattice equation, we have a quadratic equation

$$(\lambda + \lambda u\psi_{21})E\psi_{21} = 1,\ \psi_{21} = \frac{\phi_2}{\phi_1}. \tag{3.4}$$

Expand ψ_{21} as

$$\psi_{21} = \sum_{k\ge 1} \theta_k \lambda^{-k}, \tag{3.5}$$

and then from (3.4), we have

$$\theta_1 = 1,\ \theta_k = -u^{(-1)} \sum_{j=1}^{k-1} \theta_j^{(-1)}\theta_{k-j},\ k \ge 2. \tag{3.6}$$

This particularly tells

$$\begin{cases} \theta_2 = -u^{(-1)}, \\ \theta_3 = u^{(-1)}(u^{(-1)} + u^{(-2)}), \\ \theta_4 = -u^{(-1)}[(u^{(-1)})^2 + 2u^{(-1)}u^{(-2)} + (u^{(-2)})^2 + u^{(-2)}u^{(-3)}]. \end{cases} \tag{3.7}$$

Now, on one hand, we have

$$\begin{aligned} T_{1,t} &= [\ln(U_{11} + U_{12}\psi_{21})]_t = [\ln(\lambda + \lambda u\psi_{21})]_t \\ &= [\ln(1 + u\psi_{21})]_t = [\ln(1 + \sum_{k\ge 1} u\theta_k\lambda^{-k})]_t \\ &= \Big\{ \sum_{k\ge 1} \Big[\sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} u^j \sum_{i_1+\cdots+i_j=k,\, i_1,\ldots,i_j\ge 1} \theta_{i_1}\cdots\theta_{i_j} \Big] \lambda^{-k} \Big\}_t . \end{aligned}$$

In the above computation, we have used

$$\ln(1+x) = \sum_{k\ge 1} \frac{(-1)^{k-1}}{k} x^k,$$

which holds for small values of x: $|x| < 1$.

On the other hand, we have

$$\begin{aligned} (E-1)X_1 &= (E-1)(V_{11} + V_{12}\psi_{21}) \\ &= (E-1)(\frac{1}{2}\lambda - u + \lambda u\psi_{21}) \\ &= (E-1)(\sum_{k\ge 1} u\theta_{k+1}\lambda^{-k}). \end{aligned}$$

Therefore, we obtain infinitely many conservation laws

$$(T_{1,k})_t = (E-1)X_{1,k},\ k \ge 1, \tag{3.8}$$

where

$$T_{1,k} = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} u^j \sum_{i_1+\cdots+i_j=k,\, i_1,\ldots,i_j\ge 1} \theta_{i_1}\cdots\theta_{i_j},\ X_{1,k} = u\theta_{k+1},\ k \ge 1. \tag{3.9}$$

The first three conservation laws above can be worked out as follows:

$$\begin{cases} T_{1,1} = u\theta_1 = u,\ X_{1,1} = u\theta_2 = -uu^{(-1)}; \\ T_{1,2} = u\theta_2 - \frac{1}{2}(u\theta_1)^2 = -uu^{(-1)} - \frac{1}{2}u^2,\ X_{1,2} = u\theta_3 = uu^{(-1)}(u^{(-1)} + u^{(-2)}); \\ T_{1,3} = u\theta_3 - (u\theta_1)(u\theta_2) + \frac{1}{3}(u\theta_1)^3 = uu^{(-1)}(u^{(-1)} + u^{(-2)}) + u^2u^{(-1)} + \frac{1}{3}u^3, \\ X_{1,3} = u\theta_4 = -uu^{(-1)}[(u^{(-1)})^2 + 2u^{(-1)}u^{(-2)} + (u^{(-2)})^2 + u^{(-2)}u^{(-3)}]. \end{cases} \tag{3.10}$$

3.3. Equivalence

In the subsequent analysis, we explicitly explore that T_1 and T_2 are equivalent. That is to say, the other generating formula

$$(\ln\psi_{12})_t = (E-1)(\psi_{12} - \frac{1}{2}\lambda - u^{(-1)}) \tag{3.11}$$

leads to equivalent conservation laws.

To prepare, let us set

$$\frac{1}{\sum_{k\geq 1}\theta_k\lambda^{-k}} = \sum_{k\geq 0}\eta_k\lambda^{-k+1},$$

and this generates

$$\eta_0 = 1,\ \eta_k = -\sum_{j=2}^{k+1}\theta_j\eta_{k-j+1},\ k\geq 1, \tag{3.12}$$

the first three of which are

$$\eta_1 = -\theta_2,\ \eta_2 = \theta_2^2 - \theta_3,\ \eta_3 = -\theta_2^3 + 2\theta_2\theta_3 - \theta_4. \tag{3.13}$$

On one hand, we have

$$\begin{aligned} T_2 - T_1 &= \ln\frac{E\psi_{21}}{\psi_{21}} = \ln\frac{\sum_{k\geq 1}\theta_k^{(1)}\lambda^{-k}}{\sum_{k\geq 1}\theta_k\lambda^{-k}} \\ &= \ln\sum_{k\geq 1}\theta_k^{(1)}\lambda^{-k}\sum_{l\geq 0}\eta_l\lambda^{-l+1} = \ln(1 + \sum_{k\geq 1}\xi_k\lambda^{-k}) \\ &= \sum_{k\geq 1}\Big[\sum_{j=1}^{k}\frac{(-1)^{j-1}}{j}\sum_{i_1+\cdots+i_j=k,\, i_1,\ldots,i_j\geq 1}\xi_{i_1}\cdots\xi_{i_j}\Big]\lambda^{-k}, \end{aligned}$$

where

$$\xi_k = \sum_{j=1}^{k+1}\theta_j^{(1)}\eta_{k-j+1},\ k\geq 1, \tag{3.14}$$

which engenders

$$\begin{cases} \xi_1 = \theta_1^{(1)}\eta_1 + \theta_2^{(1)}\eta_0 = -\theta_2 + \theta_2^{(1)}, \\ \xi_2 = \theta_3^{(1)}\eta_0 + \theta_2^{(1)}\eta_1 + \theta_1^{(1)}\eta_2 = \theta_3^{(1)} - \theta_2\theta_2^{(1)} + \theta_2^2 - \theta_3. \end{cases} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} X_2 - X_1 &= [V_{21}(\psi_{21})^{-1} + V_{22}] - (V_{11} + V_{12}\psi_{21}) \\ &= (\psi_{21})^{-1} - \frac{1}{2}\lambda - u^{(-1)} - (\frac{1}{2}\lambda - u) - \lambda u\psi_{21} \\ &= (\psi_{21})^{-1} - \lambda - u^{(-1)} + u - \lambda u\psi_{21} \\ &= \lambda\sum_{k\geq 0}\eta_k\lambda^{-k} - \lambda - u^{(-1)} + u - \lambda u\sum_{k\geq 1}\theta_k\lambda^{-k} \\ &= \sum_{k\geq 1}(\eta_{k+1} - u\theta_{k+1})\lambda^{-k}. \end{aligned}$$

It follows now that the corresponding trivial conservation laws read

$$(T_{21,k})_t = (E-1)X_{21,k},\ \ k \geq 1, \tag{3.16}$$

where the conserved densities and the conserved fluxes are given by

$$T_{21,k} = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \sum_{i_1+\cdots+i_j=k,\, i_1,\ldots,i_j\geq 1} \xi_{i_1}\cdots\xi_{i_j},\ \ X_{21,k} = \eta_{k+1} - u\theta_{k+1},\ \ k\geq 1. \tag{3.17}$$

Let us now observe the equivalence between T_1 and T_2, or the trivialness of the conservation laws in (3.16), precisely:

$$\begin{aligned} T_2 - T_1 &= (E-1)\ln\psi_{21} \\ &= (E-1)\ln\Big(1+\sum_{k\geq 1}\theta_{k+1}\lambda^{-k}\Big) \\ &= (E-1)\Big[\sum_{k\geq 1}\Big(\sum_{j=1}^{k}\frac{(-1)^{j-1}}{j}\sum_{i_1+\cdots+i_j=k,\, i_1,\ldots,i_j\geq 1}\theta_{i_1+1}\cdots\theta_{i_j+1}\Big)\lambda^{-k}\Big]. \end{aligned}$$

This exactly tells us the required equivalence:

$$T_{2,k} - T_{1,k} = T_{21,k} = (E-1)S_{21,k},\ \ k\geq 1, \tag{3.18}$$

where

$$S_{21,k} = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \sum_{i_1+\cdots+i_j=k,\, i_1,\ldots,i_j\geq 1} \theta_{i_1+1}\cdots\theta_{i_j+1},\ \ k\geq 1, \tag{3.19}$$

the first two of which are

$$S_{21,1} = \theta_2 = -u^{(-1)},\ \ S_{21,2} = \theta_3 - \frac{1}{2}\theta_2^2 = \frac{1}{2}(u^{(-1)})^2 + u^{(-1)}u^{(-2)}. \tag{3.20}$$

Indeed, we can see clearly that the first two pairs of conservation laws in (3.16) behave trivially:

$$\begin{cases} T_{21,1} = \xi_1 = -\theta_2 + \theta_2^{(1)} = u^{(-1)} - u = (E-1)S_{21,1}, \\ T_{21,2} = \xi_2 - \frac{1}{2}\xi_1^2 = \theta_3^{(1)} - \theta_3 + \frac{1}{2}\theta_2^2 - \frac{1}{2}(\theta_2^{(1)})^2 \\ \qquad = \frac{1}{2}(u+u^{(-1)})^2 - u^{(-1)}(u^{(-1)}+u^{(-2)}) = (E-1)S_{21,2}, \end{cases} \tag{3.21}$$

and

$$\begin{cases} X_{21,1} = \eta_2 - u\theta_2 = \theta_2^2 - \theta_3 - u\theta_2 = u^{(-1)}(u - u^{(-2)}) = (S_{21,1})_t, \\ X_{21,2} = \eta_3 - u\theta_3 = -\theta_4 - \theta_2^3 + 2\theta_2\theta_3 - u\theta_3 \\ \qquad = u^{(-1)}u^{(-2)}(u^{(-2)}+u^{(-3)}) - uu^{(-1)}(u^{(-1)}+u^{(-2)}) = (S_{21,2})_t. \end{cases} \tag{3.22}$$

4. Concluding remarks

We have formulated a general scheme for conservation laws, based on Lax pairs of discrete integrable equations. The success is to use the Laurent expansions of the ratios of eigenfunctions of matrix spectral problems. An illustrative application has been made for the Volterra lattice equation through a pair of 2×2 matrix spectral problems. By irreducible representations of matrix Lie algebras, we can present different matrix spectral problems for the same zero curvature equation [19], and thus, more conservation laws. The generating scheme can also be generalized to compute conservation laws in the case of multiple discrete variables.

It is worth noting that there are recent studies on lump solutions to continuous integrable equations, particularly in (2+1)-dimensions and in (3+1)-dimensions (see, e.g., [20–26]). There are even more general solutions—interaction solutions between lumps and other kinds of exact solutions (see, e.g., [27–30] for lump–kink interaction solutions and [31–33] for lump–soliton interaction solutions). This is a characterization of complete integrability, and from such interaction solutions, symmetries and conserved quantities could be derived as well.

It will also be definitely interesting and important to construct soliton and interaction solutions and study their patterns for integrable lattice equations. More importantly, we need to identify nonlinear lattice equations that admit infinitely many conservation laws and possess soliton and interaction solutions. All such studies will enrich the existing theories of soliton solutions and dromion-type solutions, through different approaches, including the Hirota perturbation technique, symmetry reductions, symmetry constraints and the Riemann–Hilbert technique (see, e.g., [34–46]).

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