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Novel Liouville integrable Hamiltonian models with six components and three signs

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ABSTRACT

This paper aims at constructing a six-component integrable hierarchy associated with a matrix spatial spectral problem with six potentials and three signs. The zero curvature formulation and the trace identity are used to generate integrable models and their Hamiltonian structures, respectively. Two expository examples of integrable models of lower orders are six-component integrable coupled nonlinear Schrödinger (NLS) equations and modified Korteweg–de Vries (mKdV) equations. The motivation of this study is to explore typical integrable coupled NLS equations and mKdV equations, and the innovative idea and main advance is to introduce a specific matrix spectral problem involving three signs to construct integrable coupled equations.

1. Introduction

The zero curvature formulation is a powerful and fundamental tool for generating integrable models, and has played a crucial role in the development of soliton theory and nonlinear wave phenomena. The key idea behind the zero curvature formulation is that the compatibility conditions of matrix spectral problems, producing a hierarchy of nonlinear models, ensure that the associated Lax pairs [1] generate an infinite sequence of conserved quantities, which can be used to show the Liouville integrability of the hierarchy. The corresponding inverse scattering transform allows for the construction of solutions to Cauchy problems, particularly solitons [2].

Let us assume that an n-dimensional potential reads $u=(u_1,\dots,u_n)^T$ and λ denotes the spectral parameter. To construct integrable models within the zero curvature formulation, one first needs to take a loop algebra $\tilde{g}$ to introduce a spatial spectral matrix:

$$\mathcal{M}=\mathcal{M}(u,\lambda)=f_0(\lambda)+u_1f_1(\lambda)+\cdots+u_nf_n(\lambda), \tag{1.1}$$

where $f_1,\dots,f_n$ are linearly independent elements in $\tilde{g}$ and f_0 is a pseudo-regular element in $\tilde{g}$:

$$[\mathrm{Ker}\,\mathrm{ad}_{f_0},\mathrm{Ker}\,\mathrm{ad}_{f_0}]=0,\ \ \mathrm{Ker}\,\mathrm{ad}_{f_0}\oplus\mathrm{Im}\,\mathrm{ad}_{f_0}=\tilde{g}.$$

This characteristic property guarantees that there exists a Laurent series solution $Z=\sum_{s=0}^{\infty}\lambda^{-s}Z^{[s]}$ to the stationary zero curvature equation:

$$Z_x=i[\mathcal{M},Z]=i(\mathcal{M}Z-Z\mathcal{M}). \tag{1.2}$$

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Then, take the temporal matrix spectral matrices:

$$\mathcal{N}^{[r]} = (\lambda^r Z)_+ + \Delta_r = \sum_{s=0}^{r} \lambda^{r-s} Z^{[s]} + \Delta_r,\ r \geq 0, \tag{1.3}$$

where $\Delta_r \in \tilde{g},\ r \geq 0$, to generate an integrable hierarchy through the zero curvature equations:

$$\mathcal{M}_{t_r} - \mathcal{N}^{[r]}_x + i[\mathcal{M}, \mathcal{N}^{[r]}] = 0,\ r \geq 0. \tag{1.4}$$

These equations are the compatibility conditions of the spatial and temporal matrix spectral problems:

$$-i\phi_x = \mathcal{M}\phi,\ -i\phi_{t_r} = \mathcal{N}^{[r]}\phi,\ r \geq 0. \tag{1.5}$$

Their Hamiltonian structures can be furnished by using the trace identity [3,4] :

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(Z\frac{\partial \mathcal{M}}{\partial \lambda}\big)\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\big(Z\frac{\partial \mathcal{M}}{\partial u}\big), \tag{1.6}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u and γ is the constant given by

$$\gamma = -\frac{\lambda}{2}\frac{\partial}{\partial \lambda}\ln|\mathrm{tr}(Z^2)|.$$

Hamiltonian structures ensure the Liouville integrability of the associated zero curvature equations.

Various existing integrable hierarchies are generated through the zero curvature formulation. The adopted loop algebras are formed from the special linear algebras (see, e.g., [5–13]), and the special orthogonal algebras (see, e.g., [14–18]). Integrable hierarchies with two components, let us say p and q, are of great importance. The four well-known such integrable hierarchies are the Ablowitz–Kaup–Newell–Segur hierarchy [5], the Kaup–Newell hierarchy [19], the Wadati–Konno–Ichikawa hierarchy [20] and the Heisenberg hierarchy [21]. Their associated spatial spectral matrices read

$$\mathcal{M} = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} \lambda v & \lambda p \\ \lambda q & -\lambda v \end{bmatrix}, \tag{1.7}$$

where $pq + v^2 = 1$, respectively. The four counterparts of spatial spectral matrices associated with so $(3, \mathbb{R})$ are

$$\mathcal{M} = \begin{bmatrix} 0 & q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix}, \tag{1.8}$$

and

$$\mathcal{M} = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} 0 & -\lambda q & -\lambda v \\ \lambda q & 0 & -\lambda p \\ \lambda v & \lambda p & 0 \end{bmatrix}, \tag{1.9}$$

where $p^2 + q^2 + v^2 = 1$, which were introduced and discussed in [22–25], respectively.

This paper aims at constructing a Liouville integrable hierarchy of six-component Hamiltonian equations, through the zero curvature formulation. The main contribution is to introduce a specific spatial matrix spectral problem with six potentials and three signs, yielding the integrable hierarchy. The trace identity will be used to furnish Hamiltonian structures for all models of the hierarchy. Two illustrative examples of lower orders are six-component integrable coupled nonlinear Schrödinger equations and six-component integrable coupled modified Korteweg–de Vries equations, whose coefficients involve three signs and depend on numbers of copies of six potentials appearing in the spatial spectral matrix. The final section provides a conclusion and some concluding remarks.

2. An integrable hierarchy with six potentials and three signs

Let n_1, n_2 and n_3 be three arbitrary natural numbers, and $\delta_j,\ 1 \leq j \leq 3$, be three signs, i.e., $\delta_1, \delta_2, \delta_3 \in \{1, -1\}$.

It is known that hereditary operators can be used to generate integrable hierarchies [26]. In what follows, we will construct integrable models through the zero curvature formulation. To begin with, we introduce a spatial matrix spectral problem of the form:

$$-i\phi_x = \mathcal{M}\phi = \mathcal{M}(u,\lambda)\phi,\ \mathcal{M} = \begin{bmatrix} \lambda & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & 0 \\ \mathbf{q}_1 & 0 & 0 & 0 & \delta_1\mathbf{p}_1^T \\ \mathbf{q}_2 & 0 & 0 & 0 & \delta_2\mathbf{p}_2^T \\ \mathbf{q}_3 & 0 & 0 & 0 & \delta_3\mathbf{p}_3^T \\ 0 & \delta_1\mathbf{q}_1^T & \delta_2\mathbf{q}_2^T & \delta_3\mathbf{q}_3^T & -\lambda \end{bmatrix}, \tag{2.1}$$

where λ is again the spectral parameter, u is the potential with six components:

$$u = u(x,t) = (p_1, p_2, p_3, q_1, q_2, q_3)^T, \tag{2.2}$$

and

$$\mathbf{p}_j = (\underbrace{p_j, \dots, p_j}_{n_j}),\ \mathbf{q}_j = (\underbrace{q_j, \dots, q_j}_{n_j})^T,\ 1 \le j \le 3. \tag{2.3}$$

The above spectral problem is different from the matrix Ablowitz–Kaup–Newell–Segur spectral problem and its various meaningful reductions (see, e.g., [27,28]).

As usual, to construct an associated integrable hierarchy, we first solve the stationary zero curvature Eq. (1.2) among Laurent series matrices of the form

$$Z = \begin{bmatrix} a & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & 0 \\ \mathbf{c}_1 & 0 & d_1 E_{n_1,n_2} & d_2 E_{n_1,n_3} & \delta_1 \mathbf{b}_1^T \\ \mathbf{c}_2 & -\delta_1\delta_2 d_1 E_{n_2,n_1} & 0 & d_3 E_{n_2,n_3} & \delta_2 \mathbf{b}_2^T \\ \mathbf{c}_3 & -\delta_1\delta_3 d_2 E_{n_3,n_1} & -\delta_2\delta_3 d_3 E_{n_3,n_2} & 0 & \delta_3 \mathbf{b}_3^T \\ 0 & \delta_1 \mathbf{c}_1^T & \delta_2 \mathbf{c}_2^T & \delta_3 \mathbf{c}_3^T & -a \end{bmatrix} = \sum_{s=0}^{\infty} \lambda^{-s} Z^{[s]}, \tag{2.4}$$

where

$$\mathbf{b}_j = (\underbrace{b_j, \dots, b_j}_{n_j}),\ \mathbf{c}_j = (\underbrace{c_j, \dots, c_j}_{n_j})^T,\ 1 \le j \le 3, \tag{2.5}$$

E_{n_j,n_k} is an $n_j \times n_k$ matrix with all entries being one, and we take Laurent expansions

$$a = \sum_{s=0}^{\infty} \lambda^{-s} a^{[s]},\ b_j = \sum_{s=0}^{\infty} \lambda^{-s} b_j^{[s]},\ c_j = \sum_{s=0}^{\infty} \lambda^{-s} c_j^{[s]},\ d_j = \sum_{s=0}^{\infty} \lambda^{-s} d_j^{[s]},\ 1 \le j \le 3. \tag{2.6}$$

Such a form of Laurent series solutions has been determined by a symbolic computation process.

Now, one can directly observe that the corresponding stationary zero curvature equation leads to the initial conditions:

$$a_x^{[0]} = 0,\ b_1^{[0]} = b_2^{[0]} = b_3^{[0]} = c_1^{[0]} = c_2^{[0]} = c_3^{[0]} = 0,\ d_{1,x}^{[0]} = d_{2,x}^{[0]} = d_{3,x}^{[0]} = 0, \tag{2.7}$$

and the recursion relations:

$$\begin{cases} b_1^{[s+1]} = -i b_{1,x}^{[s]} + p_1 a^{[s]} + \delta_1\delta_2 n_2 p_2 d_1^{[s]} + \delta_1\delta_3 n_3 p_3 d_2^{[s]}, \\ b_2^{[s+1]} = -i b_{2,x}^{[s]} + p_2 a^{[s]} - n_1 p_1 d_1^{[s]} + \delta_2\delta_3 n_3 p_3 d_3^{[s]}, \\ b_3^{[s+1]} = -i b_{3,x}^{[s]} + p_3 a^{[s]} - n_1 p_1 d_2^{[s]} - n_2 p_2 d_3^{[s]}, \end{cases} \tag{2.8}$$

$$\begin{cases} c_1^{[s+1]} = i c_{1,x}^{[s]} + q_1 a^{[s]} - n_2 q_2 d_1^{[s]} - n_3 q_3 d_2^{[s]}, \\ c_2^{[s+1]} = i c_{2,x}^{[s]} + q_2 a^{[s]} + \delta_1\delta_2 n_1 q_1 d_1^{[s]} - n_3 q_3 d_3^{[s]}, \\ c_3^{[s+1]} = i c_{3,x}^{[s]} + q_3 a^{[s]} + \delta_1\delta_3 n_1 q_1 d_2^{[s]} + \delta_2\delta_3 n_2 q_2 d_3^{[s]}, \end{cases} \tag{2.9}$$

$$\begin{cases} d_{1,x}^{[s+1]} = i(q_1 b_2^{[s+1]} - \delta_1\delta_2 q_2 b_1^{[s+1]} + \delta_1\delta_2 p_1 c_2^{[s+1]} - p_2 c_1^{[s+1]}), \\ d_{2,x}^{[s+1]} = i(q_1 b_3^{[s+1]} - \delta_1\delta_3 q_3 b_1^{[s+1]} + \delta_1\delta_3 p_1 c_3^{[s+1]} - p_3 c_1^{[s+1]}), \\ d_{3,x}^{[s+1]} = i(q_2 b_3^{[s+1]} - \delta_2\delta_3 q_3 b_2^{[s+1]} + \delta_2\delta_3 p_2 c_3^{[s+1]} - p_3 c_2^{[s+1]}), \end{cases} \tag{2.10}$$

and

$$\begin{aligned} a_x^{[s+1]} &= i(-n_1 q_1 b_1^{[s+1]} - n_2 q_2 b_2^{[s+1]} - n_3 q_3 b_3^{[s+1]} + n_1 p_1 c_1^{[s+1]} + n_2 p_2 c_2^{[s+1]} + n_3 p_3 c_3^{[s+1]}) \\ &= -(n_1 q_1 b_{1,x}^{[s]} + n_2 q_2 b_{2,x}^{[s]} + n_3 q_3 b_{3,x}^{[s]} + n_1 p_1 c_{1,x}^{[s]} + n_2 p_2 c_{2,x}^{[s]} + n_3 p_3 c_{3,x}^{[s]}), \end{aligned} \tag{2.11}$$

where $s \ge 0$. To uniquely determine a Laurent series solution, we fix the initial values,

$$a^{[0]} = 1,\ d_1^{[0]} = d_2^{[0]} = d_3^{[0]} = 0, \tag{2.12}$$

and take the constants of integration as zero,

$$a^{[s]}|_{u=0} = 0,\ d_1^{[s]}|_{u=0} = d_2^{[s]}|_{u=0} = d_3^{[s]}|_{u=0} = 0,\ s \ge 1. \tag{2.13}$$

Consequently, one can work out the first four sets of non-constant coefficients as follows:

$$\begin{cases} b_1^{[1]} = p_1,\ b_2^{[1]} = p_2,\ b_3^{[1]} = p_3, \\ c_1^{[1]} = q_1,\ c_2^{[1]} = q_2,\ c_3^{[1]} = q_3, \\ d_1^{[1]} = d_2^{[1]} = d_3^{[1]} = 0,\ a^{[1]} = 0; \end{cases}$$

$$\begin{cases} b_1^{[2]} = -ip_{1,x}, \ b_2^{[2]} = -ip_{2,x}, \ b_3^{[2]} = -ip_{3,x}, \\ c_1^{[2]} = iq_{1,x}, \ c_2^{[2]} = iq_{2,x}, \ c_3^{[2]} = iq_{3,x}, \\ d_1^{[2]} = -\delta_1\delta_2 p_1 q_2 + p_2 q_1, \ d_2^{[2]} = -\delta_1\delta_3 p_1 q_3 + p_3 q_1, \ d_3^{[2]} = -\delta_2\delta_3 p_2 q_3 + p_3 q_2, \\ a^{[2]} = -n_1 p_1 q_1 - n_2 p_2 q_2 - n_3 p_3 q_3; \end{cases}$$

$$\begin{cases} b_1^{[3]} = -p_{1,xx} + (-n_1 p_1^2 + \delta_1\delta_2 n_2 p_2^2 + \delta_1\delta_3 n_3 p_3^2) q_1 - 2(n_2 p_2 q_2 + n_3 p_3 q_3) p_1, \\ b_2^{[3]} = -p_{2,xx} + (\delta_1\delta_2 n_1 p_1^2 - n_2 p_2^2 + \delta_2\delta_3 n_3 p_3^2) q_2 - 2(n_1 p_1 q_1 + n_3 p_3 q_3) p_2, \\ b_3^{[3]} = -p_{3,xx} + (\delta_1\delta_3 n_1 p_1^2 + \delta_2\delta_3 n_2 p_2^2 - n_3 p_3^2) q_3 - 2(n_1 p_1 q_1 + n_2 p_2 q_2) p_3, \end{cases}$$

$$\begin{cases} c_1^{[3]} = -q_{1,xx} + (-n_1 q_1^2 + \delta_1\delta_2 n_2 q_2^2 + \delta_1\delta_3 n_3 q_3^2) p_1 - 2(n_2 p_2 q_2 + n_3 p_3 q_3) q_1, \\ c_2^{[3]} = -q_{2,xx} + (\delta_1\delta_2 n_1 q_1^2 - n_2 q_2^2 + \delta_2\delta_3 n_3 q_3^2) p_2 - 2(n_1 p_1 q_1 + n_3 p_3 q_3) q_2, \\ c_3^{[3]} = -q_{3,xx} + (\delta_1\delta_3 n_1 q_1^2 + \delta_2\delta_3 n_2 q_2^2 - n_3 q_3^2) p_3 - 2(n_1 p_1 q_1 + n_2 p_2 q_2) q_3, \end{cases}$$

$$\begin{cases} d_1^{[3]} = -i(\delta_1\delta_2 p_1 q_{2,x} - p_2 q_{1,x} - \delta_1\delta_2 p_{1,x} q_2 + p_{2,x} q_1), \\ d_2^{[3]} = -i(\delta_1\delta_3 p_1 q_{3,x} - p_3 q_{1,x} - \delta_1\delta_3 p_{1,x} q_3 + p_{3,x} q_1), \\ d_3^{[3]} = -i(\delta_2\delta_3 p_2 q_{3,x} - p_3 q_{2,x} - \delta_2\delta_3 p_{2,x} q_3 + p_{3,x} q_2), \end{cases}$$

$$a^{[3]} = -i(n_1 p_1 q_{1,x} - n_1 p_{1,x} q_1 + n_2 p_2 q_{2,x} - n_2 p_{2,x} q_2 + n_3 p_3 q_{3,x} - n_3 p_{3,x} q_3);$$

and

$$\begin{cases} b_1^{[4]} = i(p_{1,xxx} + 3n_1 p_1 p_{1,x} q_1 + 3n_2 p_1 p_{2,x} q_2 + 3n_3 p_1 p_{3,x} q_3 \\ \qquad -3\delta_1\delta_2 n_2 p_2 p_{2,x} q_1 - 3\delta_1\delta_3 n_3 p_3 p_{3,x} q_1 + 3n_2 p_{1,x} p_2 q_2 + 3n_3 p_{1,x} p_3 q_3), \\ b_2^{[4]} = i(p_{2,xxx} + 3n_1 p_1 p_{2,x} q_1 - 3\delta_1\delta_2 n_1 p_1 p_{1,x} q_2 + 3n_1 p_{1,x} p_2 q_1 \\ \qquad +3n_2 p_2 p_{2,x} q_2 + 3n_3 p_2 p_{3,x} q_3 + 3n_3 p_{2,x} p_3 q_3 - 3\delta_2\delta_3 n_3 p_3 p_{3,x} q_2), \\ b_3^{[4]} = i(p_{3,xxx} + 3n_1 p_1 p_{3,x} q_1 - 3\delta_1\delta_3 n_1 p_1 p_{1,x} q_3 + 3n_1 p_{1,x} p_3 q_1 \\ \qquad -3\delta_2\delta_3 n_2 p_2 p_{2,x} q_3 + 3n_2 p_2 p_{3,x} q_2 + 3n_2 p_{2,x} p_3 q_2 + 3n_3 p_3 p_{3,x} q_3), \end{cases}$$

$$\begin{cases} c_1^{[4]} = -i(q_{1,xxx} + 3n_1 p_1 q_1 q_{1,x} - 3\delta_1\delta_2 n_2 p_1 q_2 q_{2,x} - 3\delta_1\delta_3 n_3 p_1 q_3 q_{3,x} \\ \qquad +3n_2 p_2 q_{1,x} q_2 + 3n_2 p_2 q_1 q_{2,x} + 3n_3 p_3 q_{1,x} q_3 + 3n_3 p_3 q_1 q_{3,x}), \\ c_2^{[4]} = -i(q_{2,xxx} - 3\delta_1\delta_2 n_1 p_2 q_1 q_{1,x} + 3n_1 p_1 q_1 q_{2,x} + 3n_1 p_1 q_{1,x} q_2 \\ \qquad +3n_2 p_2 q_2 q_{2,x} - 3\delta_2\delta_3 n_3 p_2 q_3 q_{3,x} + 3n_3 p_3 q_2 q_{3,x} + 3n_3 p_3 q_{2,x} q_3), \\ c_3^{[4]} = -i(q_{3,xxx} + 3n_1 p_1 q_1 q_{3,x} + 3n_1 p_1 q_{1,x} q_3 - 3\delta_1\delta_3 n_1 p_3 q_1 q_{1,x} \\ \qquad +3n_2 p_2 q_2 q_{3,x} + 3n_2 p_2 q_{2,x} q_3 - 3\delta_2\delta_3 n_2 p_3 q_2 q_{2,x} + 3n_3 p_3 q_3 q_{3,x}), \end{cases}$$

$$\begin{cases} d_1^{[4]} = 3(n_1 p_1 q_1 + n_2 p_2 q_2 + n_3 p_3 q_3)(\delta_1\delta_2 p_1 q_2 - p_2 q_1) + \delta_1\delta_2 p_{1,xx} q_2 - p_{2,xx} q_1 \\ \qquad -p_2 q_{1,xx} + \delta_1\delta_2 p_1 q_{2,xx} - \delta_1\delta_2 p_{1,x} q_{2,x} + p_{2,x} q_{1,x}, \\ d_2^{[4]} = 3(n_1 p_1 q_1 + n_2 p_2 q_2 + n_3 p_3 q_3)(\delta_1\delta_3 p_1 q_3 - p_3 q_1) + \delta_1\delta_3 p_{1,xx} q_3 - p_{3,xx} q_1 \\ \qquad -p_3 q_{1,xx} + \delta_1\delta_3 p_1 q_{3,xx} - \delta_1\delta_3 p_{1,x} q_{3,x} + p_{3,x} q_{1,x}, \\ d_3^{[4]} = 3(n_1 p_1 q_1 + n_2 p_2 q_2 + n_3 p_3 q_3)(\delta_2\delta_3 p_2 q_3 - p_3 q_2) + \delta_2\delta_3 p_{2,xx} q_3 - p_{3,xx} q_2 \\ \qquad -p_3 q_{2,xx} + \delta_2\delta_3 p_2 q_{3,xx} - \delta_2\delta_3 p_{2,x} q_{3,x} + p_{3,x} q_{2,x}, \end{cases}$$

$$\begin{aligned} a^{[4]} = {} & \tfrac{3}{2} n_1 (n_1 q_1^2 - \delta_1\delta_2 n_2 q_2^2 - \delta_1\delta_3 n_3 q_3^2) p_1^2 - \tfrac{3}{2} n_2 (\delta_1\delta_2 n_1 q_1^2 - n_2 q_2^2 + \delta_2\delta_3 n_3 q_3^2) p_2^2 \\ & - \tfrac{3}{2} n_3 (\delta_1\delta_3 n_1 q_1^2 + \delta_2\delta_3 n_2 q_2^2 - n_3 q_3^2) p_3^2 + 6 n_1 p_1 (n_2 p_2 q_2 + n_3 p_3 q_3) q_1 \\ & + 6 n_2 n_3 p_2 p_3 q_2 q_3 + n_1 p_1 q_{1,xx} + n_1 p_{1,x,x} q_1 + n_2 p_2 q_{2,xx} + n_2 p_{2,xx} q_2 \\ & + n_3 p_3 q_{3,xx} + n_3 p_{3,xx} q_3 - n_1 p_{1,x} q_{1,x} - n_2 p_{2,x} q_{2,x} - n_3 p_{3,x} q_{3,x}. \end{aligned}$$

All these computations suggest that one can take $\Delta_r = 0,\ r \geq 0$, to introduce the temporal matrix spectral problems:

$$-i\phi_{t_r} = \mathcal{N}^{[r]}\phi = \mathcal{N}^{[r]}(u,\lambda)\phi,\ \mathcal{N}^{[r]} = (\lambda^r Z)_+ = \sum_{s=0}^{r} \lambda^s Z^{[r-s]},\ r \geq 0. \tag{2.14}$$

Being key objects in the zero curvature formulation, these temporal spectral problems pair up with the spatial spectral problem (2.1). The corresponding compatibility conditions, namely, the zero curvature equations in (1.4), generate a six-component integrable hierarchy:

$$u_{t_r} = X^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, ib_3^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]}, -ic_3^{[r+1]})^T,\ r \geq 0, \tag{2.15}$$

or more concretely,

$$\begin{cases} p_{1,t_r} = ib_1^{[r+1]},\ p_{2,t_r} = ib_2^{[r+1]},\ p_{3,t_r} = ib_3^{[r+1]}, \\ q_{1,t_r} = -ic_1^{[r+1]},\ q_{2,t_r} = -ic_2^{[r+1]},\ q_{3,t_r} = -ic_3^{[r+1]}, \end{cases} \quad r \geq 0. \tag{2.16}$$

Base on the previous computations, one can present the first two nonlinear examples in this integrable hierarchy, one of which is the integrable system of coupled nonlinear Schrödinger equations:

$$\begin{cases} ip_{1,t_2} = p_{1,xx} + (n_1 p_1^2 - \delta_1\delta_2 n_2 p_2^2 - \delta_1\delta_3 n_3 p_3^2)q_1 + 2(n_2 p_2 q_2 + n_3 p_3 q_3)p_1, \\ ip_{2,t_2} = p_{2,xx} - (\delta_1\delta_2 n_1 p_1^2 - n_2 p_2^2 + \delta_2\delta_3 n_3 p_3^2)q_2 + 2(n_1 p_1 q_1 + n_3 p_3 q_3)p_2, \\ ip_{3,t_2} = p_{3,xx} - (\delta_1\delta_3 n_1 p_1^2 + \delta_2\delta_3 n_2 p_2^2 - n_3 p_3^2)q_3 + 2(n_1 p_1 q_1 + n_2 p_2 q_2)p_3, \end{cases} \tag{2.17}$$

and

$$\begin{cases} iq_{1,t_2} = -q_{1,xx} + (-n_1 q_1^2 + \delta_1\delta_2 n_2 q_2^2 + \delta_1\delta_3 n_3 q_3^2)p_1 - 2(n_2 p_2 q_2 + n_3 p_3 q_3)q_1, \\ iq_{2,t_2} = -q_{2,xx} + (\delta_1\delta_2 n_1 q_1^2 - n_2 q_2^2 + \delta_2\delta_3 n_3 q_3^2)p_2 - 2(n_1 p_1 q_1 + n_3 p_3 q_3)q_2, \\ iq_{3,t_2} = -q_{3,xx} + (\delta_1\delta_3 n_1 q_1^2 + \delta_2\delta_3 n_2 q_2^2 - n_3 q_3^2)p_3 - 2(n_1 p_1 q_1 + n_2 p_2 q_2)q_3; \end{cases} \tag{2.18}$$

and the other, the integrable system of coupled modified Korteweg–de Vries equations:

$$\begin{cases} p_{1,t_3} = -p_{1,xxx} - 3n_1 p_1 p_{1,x} q_1 - 3n_2 p_1 p_{2,x} q_2 - 3n_3 p_1 p_{3,x} q_3 \\ \qquad +3\delta_1\delta_2 n_2 p_2 p_{2,x} q_1 + 3\delta_1\delta_3 n_3 p_3 p_{3,x} q_1 - 3n_2 p_{1,x} p_2 q_2 - 3n_3 p_{1,x} p_3 q_3, \\ p_{2,t_3} = -p_{2,xxx} - 3n_1 p_1 p_{2,x} q_1 + 3\delta_1\delta_2 n_1 p_1 p_{1,x} q_2 - 3n_1 p_{1,x} p_2 q_1 \\ \qquad -3n_2 p_2 p_{2,x} q_2 - 3n_3 p_2 p_{3,x} q_3 - 3n_3 p_{2,x} p_3 q_3 + 3\delta_2\delta_3 n_3 p_3 p_{3,x} q_2, \\ p_{3,t_3} = -p_{3,xxx} - 3n_1 p_1 p_{3,x} q_1 + 3\delta_1\delta_3 n_1 p_1 p_{1,x} q_3 - 3n_1 p_{1,x} p_3 q_1 \\ \qquad +3\delta_2\delta_3 n_2 p_2 p_{2,x} q_3 - 3n_2 p_2 p_{3,x} q_2 - 3n_2 p_{2,x} p_3 q_2 - 3n_3 p_3 p_{3,x} q_3, \end{cases} \tag{2.19}$$

and

$$\begin{cases} q_{1,t_3} = -q_{1,xxx} - 3n_1 p_1 q_1 q_{1,x} + 3\delta_1\delta_2 n_2 p_1 q_2 q_{2,x} + 3\delta_1\delta_3 n_3 p_1 q_3 q_{3,x} \\ \qquad -3n_2 p_2 q_{1,x} q_2 - 3n_2 p_2 q_1 q_{2,x} - 3n_3 p_3 q_{1,x} q_3 - 3n_3 p_3 q_1 q_{3,x}, \\ q_{2,t_3} = -q_{2,xxx} + 3\delta_1\delta_2 n_1 p_2 q_1 q_{1,x} - 3n_1 p_1 q_1 q_{2,x} - 3n_1 p_1 q_{1,x} q_2 \\ \qquad -3n_2 p_2 q_2 q_{2,x} + 3\delta_2\delta_3 n_3 p_2 q_3 q_{3,x} - 3n_3 p_3 q_2 q_{3,x} - 3n_3 p_3 q_{2,x} q_3, \\ q_{3,t_3} = -q_{3,xxx} - 3n_1 p_1 q_1 q_{3,x} - 3n_1 p_1 q_{1,x} q_3 + 3\delta_1\delta_3 n_1 p_3 q_1 q_{1,x} \\ \qquad -3n_2 p_2 q_2 q_{3,x} - 3n_2 p_2 q_{2,x} q_3 + 3\delta_2\delta_3 n_2 p_3 q_2 q_{2,x} - 3n_3 p_3 q_3 q_{3,x}. \end{cases} \tag{2.20}$$

The coefficients in these two integrable models depend on three integer numbers: n_1, n_2, n_3, and involve three signs: $\delta_1, \delta_2, \delta_3$. Taking different values of n_1, n_2, n_3 and positive or negative signs of $\delta_1, \delta_2, \delta_3$ leads to abundant integrable systems of coupled focusing or defocusing type nonlinear Schrödinger equations and modified Korteweg–de Vries equations with six potentials (see also, [29,30]).

3. Hamiltonian structures

To establish Hamiltonian structures [31] for the integrable hierarchy (2.16), we apply the trace identity (1.6) to the matrix spatial spectral problem (2.1). Following the solution Z given by (2.4), one can work out

$$\mathrm{tr}\big(Z\frac{\partial \mathcal{M}}{\partial \lambda}\big) = 2a,\ \mathrm{tr}\big(Z\frac{\partial \mathcal{M}}{\partial u}\big) = 2(n_1c_1, n_2c_2, n_3c_3, n_1b_1, n_2b_2, n_3b_3)^T, \tag{3.1}$$

and subsequently, it follows from the trace identity that

$$\frac{\delta}{\delta u}\int a\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma-s}(n_1c_1, n_2c_2, n_3c_3, n_1b_1, n_2b_2, n_3b_3)^T, \tag{3.2}$$

or precisely,

$$\frac{\delta}{\delta u}\int \lambda^{-s-1}a^{[s+1]}\,dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma-s}(n_1c_1^{[s]},n_2c_2^{[s]},n_3c_3^{[s]},n_1b_1^{[s]},n_2b_2^{[s]},n_3b_3^{[s]})^T,\ \ s\geq 0. \tag{3.3}$$

A check with $s=2$ leads to $\gamma=0$, and consequently, one arrives at

$$\frac{\delta}{\delta u}\mathcal{H}^{[s]}=(n_1c_1^{[s+1]},n_2c_2^{[s+1]},n_3c_3^{[s+1]},n_1b_1^{[s+1]},n_2b_2^{[s+1]},n_3b_3^{[s+1]})^T,\ \ s\geq 0, \tag{3.4}$$

where based on (3.2), the Hamiltonian functionals are given by

$$\mathcal{H}^{[s]}=-\int\frac{a^{[s+2]}}{s+1}\,dx,\ \ s\geq 0. \tag{3.5}$$

This allows us to present Hamiltonian structures for the hierarchy (2.16):

$$u_{t_r}=X^{[r]}=J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\ \ J=\left[\begin{array}{ccc|ccc} & & & \frac{1}{n_1}i & 0 & 0\\ & 0 & & 0 & \frac{1}{n_2}i & 0\\ & & & 0 & 0 & \frac{1}{n_3}i\\ \hline -\frac{1}{n_1}i & 0 & 0 & & & \\ 0 & -\frac{1}{n_2}i & 0 & & 0 & \\ 0 & 0 & -\frac{1}{n_3}i & & & \end{array}\right],\ \ r\geq 0, \tag{3.6}$$

where J is Hamiltonian and the functionals $\mathcal{H}^{[r]}$ are determined by (3.5). These Hamiltonian structures exhibit a relation $S=J\frac{\delta\mathcal{H}}{\delta u}$ from a conserved functional $\mathcal{H}$ to a symmetry S. The vector fields satisfy a characteristic property

$$[\![X^{[s_1]},X^{[s_2]}]\!]=X^{[s_1]\prime}(u)[X^{[s_2]}]-X^{[s_2]\prime}(u)[X^{[s_1]}]=0,\ \ s_1,s_2\geq 0, \tag{3.7}$$

which comes from a Lax operator algebra:

$$[\![\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}]\!]=\mathcal{N}^{[s_1]\prime}(u)[X^{[s_2]}]-\mathcal{N}^{[s_2]\prime}(u)[X^{[s_1]}]+[\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}]=0,\ \ s_1,s_2\geq 0, \tag{3.8}$$

or an algebraic structure associated with the isospectral zero curvature equations (see [32] for details). Further, following a recursion structure of the hierarchy, one can show that the conserved functionals also commute under the corresponding Poisson bracket:

$$\{\mathcal{H}^{[s_1]},\mathcal{H}^{[s_2]}\}_J=\int\Big(\frac{\delta\mathcal{H}^{[s_1]}}{\delta u}\Big)^TJ\frac{\delta\mathcal{H}^{[s_2]}}{\delta u}\,dx=0,\ \ s_1,s_2\geq 0. \tag{3.9}$$

Finally, by combining J with a recursion operator Φ [26], determined by $X^{[s+1]}=\Phi X^{[s]}$, bi-Hamiltonian structures [31] with a Hamiltonian pair of J and $M=\Phi J$ can be presented for the integrable hierarchy (2.16). This ensures the Liouville integrability of every model in the hierarchy (2.16).

4. Concluding remarks

A Liouville integrable hierarchy of Hamiltonian equations with six components and three signs has been generated from a specific matrix spectral problem of arbitrary order within the zero curvature formulation. A crucial step is to determine an appropriate Laurent series solution to the corresponding stationary zero curvature equation. The resulting integrable models possess Hamiltonian structures furnished by the trace identity, which show the Liouville integrability of the hierarchy, together with its recursion structure.

It would be of great importance to explore structures of solitons to the resulting integrable systems. The Riemann–Hilbert technique [33], the Zakharov–Shabat dressing method [34], the Darboux transformation [35,36] and the determinant approach [37,38] could be particularly helpful. Taking wave number reductions of solitons can yield other types of important solutions, such as kink, lump, rogue and breather wave solutions (see, e.g., [39–44]).

We also remark that our matrix spectral problem could be generalized further by taking conjugate copies of potentials and/or involving more potentials. Generalized matrix spectral problems could lead to more general coupled integrable models (see, e.g., [45–48]), including integrable couplings (see, e.g., [17,49,50]). Local and nonlocal reduced integrable systems can be generated by conducting group reductions of matrix spectral problems (see, e.g., [28,51–53], respectively).

The study of integrable models is an exciting and rapidly evolving field of research, and has the potential to shed light on a wide range of physical phenomena, particularly in nonlinear optics and shallow water waves. There is still much to be learned about these fascinating mathematical systems. The techniques used to study their mathematical structures and solitons continue to be an active area of research for many years to come.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21 (1968) 467–490, http://dx.doi.org/10.1002/cpa.3160210503.
[2] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981, http://dx.doi.org/10.1137/1.9781611970883.
[3] G.Z. Tu, On Liouville integrability of zero-curvature equations and the yang hierarchy, J. Phys. A: Math. Gen. 22 (1989) 2375–2392, http://dx.doi.org/10.1088/0305-4470/22/13/031.
[4] W.X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, Chin. Ann. Math. Ser. A 13 (1992) 115–123, (Chinese); Chin. J. Contemp. Math. 13 (1992) 79–89 (English).
[5] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315, http://dx.doi.org/10.1002/sapm1974534249.
[6] M. Antonowicz, A.P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, Physica D 28 (1987) 345–357, http://dx.doi.org/10.1016/0167-2789(87)90023-6.
[7] Y.T. Wu, X.G. Geng, X.B. Hu, S.M. Zhu, A generalized Hirota-Satsuma coupled Korteweg–de Vries equation and Miura transformations, Phys. Lett. A 255 (1999) 259–264, http://dx.doi.org/10.1016/S0375-9601(99)00163-2.
[8] T.C. Xia, F.J. Yu, Y. Zhang, The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions, Physica A 343 (2004) 238–246, http://dx.doi.org/10.1016/j.physa.2004.06.070.
[9] S. Manukure, Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints, Commun. Nonlinear Sci. Numer. Simul. 57 (2018) 125–135, http://dx.doi.org/10.1016/j.cnsns.2017.09.016.
[10] W.X. Ma, Matrix integrable fourth-order nonlinear Schrödinger equations and their exact soliton solutions, Chin. Phys. Lett. 39 (2022) 100201, http://dx.doi.org/10.1007/s11071-023-08275-3.
[11] T.S. Liu, T.C. Xia, Multi-component generalized Gerdjikov-Ivanov integrable hierarchy and its Riemann–Hilbert problem, Nonlinear Anal. RWA 68 (2022) 103667, http://dx.doi.org/10.1016/j.nonrwa.2022.103667.
[12] W.X. Ma, Matrix integrable fifth-order mKdV equations and their soliton solutions, Chin. Phys. B 32 (2023) 020201, http://dx.doi.org/10.1088/1674-1056/ac7dc1.
[13] H.F. Wang, Y.F. Zhang, Application of Riemann–Hilbert method to an extended coupled nonlinear Schrödinger equations, J. Comput. Appl. Math. 420 (2023) 114812, http://dx.doi.org/10.1016/j.cam.2022.114812.
[14] W.X. Ma, A Hamiltonian structure associated with a matrix spectral problem of arbitrary-order, Phys. Lett. A 367 (2007) 473–477, http://dx.doi.org/10.1016/j.physleta.2007.03.047.
[15] Y.Q. Yao, S.F. Shen, W.X. Ma, A soliton hierarchy associated with a spectral problem of 2nd degree in a spectral parameter and its bi-Hamiltonian structure, Adv. Math. Phys. 2016 (2016) 3589704, http://dx.doi.org/10.1155/2016/3589704.
[16] W.X. Ma, Integrable nonlocal nonlinear Schrödinger equations associated with so(3, ℝ), Proc. Amer. Math. Soc. Ser. B 9 (2022) 1–11, http://dx.doi.org/10.1090/bproc/116.
[17] Z.B. Wang, H.F. Wang, Integrable couplings of two expanded non-isospectral soliton hierarchies and their bi-Hamiltonian structures, Int. J. Geom. Methods Mod. Phys. 19 (2022) 2250160, http://dx.doi.org/10.1142/S0219887822501602.
[18] W.X. Ma, A multi-component integrable hierarchy and its integrable reductions, Phys. Lett. A 457 (2023) 128575, http://dx.doi.org/10.1016/j.physleta.2022.128575.
[19] D.J. Kaup, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19 (1978) 798–801, http://dx.doi.org/10.1063/1.523737.
[20] M. Wadati, K. Konno, Y.H. Ichikawa, New integrable nonlinear evolution equations, J. Phys. Soc. Japan 47 (1979) 1698–1700, http://dx.doi.org/10.1143/JPSJ.47.1698.
[21] L.A. Takhtajan, Integration of the continuous heisenberg spin chain through the inverses scattering method, Phys. Lett. A 64 (1977) 235–237, http://dx.doi.org/10.1016/0375-9601(77)90727-7.
[22] W.X. Ma, A soliton hierarchy associated with so(3, ℝ), Appl. Math. Comput. 220 (2013) 117–122, http://dx.doi.org/10.1016/j.amc.2013.04.062.
[23] W.X. Ma, A spectral problem based on so(3, ℝ) and its associated commuting soliton equations, J. Math. Phys. 54 (2013) 103509, http://dx.doi.org/10.1063/1.4826104.
[24] W.X. Ma, S. Manukure, H.C. Zheng, A counterpart of the Wadati–Konno–Ichikawa soliton hierarchy associated with so(3, ℝ), Z. Naturf. A 69 (2014) 411–419, http://dx.doi.org/10.5560/zna.2014-0035.
[25] W.X. Ma, S.F. Shen, S.M. Yu, H.Q. Zhang, W.Y. Zhang, An integrable so(3, ℝ)-counterpart of the heisenberg soliton hierarchy, Rep. Math. Phys. 74 (2014) 283–299, http://dx.doi.org/10.1016/S0034-4877(15)60002-7.
[26] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D 4 (1981) 47–66, http://dx.doi.org/10.1016/0167-2789(81)90004-X.
[27] W.X. Ma, Reduced non-local integrable NLS hierarchies by pairs of local and non-local constraints, Int. J. Appl. Comput. Math. 8 (2022) 206, http://dx.doi.org/10.1007/s40819-022-01422-1.
[28] W.X. Ma, Sasa–Satsuma type matrix integrable hierarchies and their Riemann–Hilbert problems and soliton solutions, Physica D 446 (2023) 133672, http://dx.doi.org/10.1016/j.physd.2023.133672.
[29] W.X. Ma, A Liouville integrable hierarchy with four potentials and its bi-Hamiltonian structure, Romanian Rep. Phys. 75 (2023) 115, http://dx.doi.org/10.59277/RomRepPhys.2023.75.115.
[30] W.X. Ma, Four-component integrable hierarchies of Hamiltonian equations with $(m+n+2)$th-order Lax pairs, Theoret. Math. Phys. 216 (2023) 1180–1188, http://dx.doi.org/10.1134/S0040577923080093.
[31] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162, http://dx.doi.org/10.1063/1.523777.
[32] W.X. Ma, The algebraic structure of zero curvature representations and application to coupled KdV systems, J. Phys. A: Math. Gen. 26 (1993) 2573–2582, http://dx.doi.org/10.1088/0305-4470/26/11/009.
[33] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Consultants Bureau, New York, 1984.
[34] E.V. Doktorov, S.B. Leble, A Dressing Method in Mathematical Physics, Springer, Dordrecht, 2007.
[35] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin, 1991.

[36] X.G. Geng, R.M. Li, B. Xue, A vector general nonlinear Schrödinger equation with $(m+n)$ components, J. Nonlinear Sci. 30 (2020) 991–1013, http://dx.doi.org/10.1007/s00332-019-09599-4.

[37] W.X. Ma, Y. You, Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions, Trans. Amer. Math. Soc. 357 (2005) 1753–1778, http://dx.doi.org/10.1090/S0002-9947-04-03726-2.

[38] T. Aktosun, T. Busse, F. Demontis, C. van der Mee, Symmetries for exact solutions to the nonlinear Schrödinger equation, J. Phys. A: Math. Theor. 43 (2010) 025202, http://dx.doi.org/10.1088/1751-8113/43/2/025202.

[39] A.M. Wazwaz, Burgers hierarchy: Multiple kink solutions and multiple singular kink solutions, J. Franklin Inst. 347 (2010) 618–626, http://dx.doi.org/10.1016/j.jfranklin.2010.01.003.

[40] J.P. Yu, Y.L. Sun, Study of lump solutions to dimensionally reduced generalized KP equations, Nonlinear Dynam. 87 (2017) 2755–2763, http://dx.doi.org/10.1007/s11071-016-3225-z.

[41] S. Manukure, A. Chowdhury, Y. Zhou, Complexiton solutions to the asymmetric Nizhnik–Novikov–Veselov equation, Internat. J. Modern Phys. B 33 (2019) 1950098, http://dx.doi.org/10.1142/S021797921950098X.

[42] T.A. Sulaiman, A. Yusuf, A. Abdeljabbar, M. Alquran, Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation, J. Geom. Phys. 169 (2021) 104347, http://dx.doi.org/10.1016/j.geomphys.2021.104347.

[43] H.C. Ma, S.P. Yue, A.P. Deng, Lump and interaction solutions for a (2+1)-dimensional combined pKP-BKP equation in fluids, Modern Phys. Lett. B 36 (2022) 2250069, http://dx.doi.org/10.1142/S0217984922500695.

[44] C. Hang, H.Q. Zhang, Rogue waves of the coupled modified nonlinear Schrödinger equations, Appl. Math. Lett. 137 (2023) 108478, http://dx.doi.org/10.1016/j.aml.2022.108478.

[45] R.S. Ye, Y. Zhang, Two-component complex modified Korteweg–de Vries equations: New soliton solutions from novel binary Darboux transformation, Theoret. Math. Phys. 214 (2023) 183–193, http://dx.doi.org/10.1134/S0040577923020034.

[46] W.X. Ma, AKNS type reduced integrable bi-Hamiltonian hierarchies with four potentials, Appl. Math. Lett. 145 (2023) 108775, http://dx.doi.org/10.1016/j.aml.2023.108775.

[47] W.X. Ma, Four-component integrable hierarchies and their Hamiltonian structures, Commun. Nonlinear Sci. Numer. Simul. 126 (2023) 107460, http://dx.doi.org/10.1016/j.cnsns.2023.107460.

[48] W.X. Ma, A six-component integrable hierarchy and its Hamiltonian formulation, Modern Phys. Lett. B 37 (2023) 2350143, http://dx.doi.org/10.1142/S0217984923501439.

[49] S.F. Shen, C.X. Li, Y.Y. Jin, S.M. Yu, Multi-component integrable couplings for the Ablowitz–Kaup–Newell–Segur and Volterra hierarchies, Math. Methods Appl. Sci. 38 (2015) 4345–4356, http://dx.doi.org/10.1002/mma.3372.

[50] H.F. Wang, Y.F. Zhang, A new multi-component integrable coupling and its application to isospectral and nonisospectral problems, Commun. Nonlinear Sci. Numer. Simul. 105 (2022) 106075, http://dx.doi.org/10.1016/j.cnsns.2021.106075.

[51] W.X. Ma, A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations, Modern Phys. Lett. B 36 (2022) 2250094, http://dx.doi.org/10.1142/S0217984922500944.

[52] W.X. Ma, Integrable non-local nonlinear Schrödinger hierarchies of type $(-\lambda^*, \lambda)$ and soliton solutions, Rep. Math. Phys. 92 (2023) 19–36, http://dx.doi.org/10.1016/S0034-4877(23)00052-6.

[53] W.X. Ma, Soliton solutions to reduced nonlocal integrable nonlinear Schrödinger hierarchies of type $(-\lambda, \lambda)$, Int. J. Geom. Methods Mod. Phys. 20 (2023) 2350098, http://dx.doi.org/10.1142/S0219887823500986.