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Integrable reductions of a soliton hierarchy associated with so(3, $\mathbb{R}$)

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ABSTRACT

We construct integrable reductions of a soliton hierarchy associated with the special orthogonal Lie algebra so$(3,\mathbb{R})$. The resulting reduced integrable equations include a nonlinear Schrödinger type equation and a modified Korteweg–de Vries type equation. There are two kinds of integrable reductions in our analysis, but they lead to essentially the same scalar integrable equations. This is a particular phenomenon for soliton equations associated with so$(3,\mathbb{R})$, which is different from the one for soliton equations associated with sl$(2,\mathbb{R})$.

1. Introduction

Soliton equations are a kind of nonlinear partial differential equations, which are generated from zero curvature equations associated with matrix Lie algebras [1]. The trace identity [2] and the variational identity [3] provide basic tools to show the Liouville integrability of soliton equations. Among the well-known hierarchies of soliton equations are the KdV hierarchy, the AKNS hierarchy and the Kaup–Newell hierarchy [1,4].

Let us consider the special orthogonal Lie algebra $\mathfrak{g} = \text{so}(3,\mathbb{R})$, and its basic representation presented by all 3×3 trace-free, skew-symmetric real matrices, whose basis could be taken as follows:

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

leading to the corresponding structure equations:

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2. \tag{2}$$

The derived algebra $[\mathfrak{g},\mathfrak{g}] = [\text{so}(3,\mathbb{R}), \text{so}(3,\mathbb{R})]$ is $\mathfrak{g} = \text{so}(3,\mathbb{R})$ itself. This is one of the only two three-dimensional real Lie algebras with a three-dimensional derived algebra. The other one is the special linear algebra sl$(2,\mathbb{R})$, which has been frequently used to construct soliton equations [1–4]. It is worth noting that their complexifications, i.e., the two complex Lie algebras, sl$(2,\mathbb{C})$ and so$(3,\mathbb{C})$, are isomorphic to each other.

We will use the matrix loop algebra in our construction:

$$\tilde{\mathfrak{g}} = \widetilde{\text{so}}(3,\mathbb{R}) = \{M \in \text{so}(3,\mathbb{R}) \mid \text{entries of } M \text{ - Laurent series in } \lambda\}, \tag{3}$$

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where λ is a spectral parameter. The loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ contains matrices of the form $\lambda^{m_1}e_1+\lambda^{m_2}e_2+\lambda^{m_3}e_3$ with arbitrary integers m_i, $1\le i\le 3$. This matrix loop algebra lays a foundation for constructing soliton equations [5–7]. Starting from the perturbation-type loop algebras of $\widetilde{\mathrm{so}}(3,\mathbb{R})$, we can also construct integrable couplings such as bi-integrable couplings and tri-integrable couplings [8].

In this paper, based on zero curvature equations, we would first like to reformulate a soliton hierarchy associated with $\mathrm{so}(3,\mathbb{R})$ [5]. We will then make integrable reductions for the adopted spectral matrix to generate a reduced hierarchy of soliton equations, which keeps the Liouville integrability of the original hierarchy and so possesses infinitely many commuting symmetries and conservation laws. Two reduced examples are two scalar integrable equations: a nonlinear Schrödinger type equation

$$ip_t=-p^*_{xx}+\frac{1}{2}|p|^2p+\frac{1}{2}(p^*)^3,$$

and a modified Korteweg–de Vries type equation

$$p_t=-p_{xxx}+\frac{3}{2}[p^2+(p^*)^2]p_x,$$

where p^* denotes the complex conjugate of p.

2. Reformulation of a soliton hierarchy

2.1. Soliton hierarchy

We would like to reformulate a soliton hierarchy associated with the matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ [5]. Let us take a spatial matrix spectral problem

$$-i\phi_x=U\phi=U(u,\lambda)\phi, \tag{4}$$

with

$$U=U(u,\lambda)=\lambda e_1+pe_2+qe_3=\begin{bmatrix}0&-q&-\lambda\\ q&0&-p\\ \lambda&p&0\end{bmatrix}, \tag{5}$$

where i is the unit imaginary number, λ is a spectral parameter, $u=(p,q)^T$ is a potential and $\phi=(\phi_1,\phi_2,\phi_3)^T$ is a column eigenfunction. The original spectral matrix for a soliton hierarchy in [5] is U, but here the adopted spectral matrix is iU with a factor difference i, which helps us determine integrable reductions successfully.

As normal, we begin with the stationary zero curvature equation

$$W_x=i[U,W]. \tag{6}$$

It becomes

$$\begin{cases}a_x=i(pc-qb),\\ b_x=i(-\lambda c+qa),\\ c_x=i(\lambda b-pa),\end{cases} \tag{7}$$

when W is taken as

$$W=ae_1+be_2+ce_3=\begin{bmatrix}0&-c&-a\\ c&0&-b\\ a&b&0\end{bmatrix}=\sum_{m\ge0}W_{0,i}\lambda^{-m}, \tag{8}$$

with

$$W_{0,m}=\begin{bmatrix}0&-c_m&-a_m\\ c_m&0&-b_m\\ a_m&b_m&0\end{bmatrix},\ m\ge0. \tag{9}$$

Upon taking the initial values

$$a_0=-1,\ b_0=c_0=0, \tag{10}$$

the system (7) equivalently gives rise to

$$\begin{cases}b_{m+1}=-ic_{m,x}+pa_m,\\ c_{m+1}=ib_{m,x}+qa_m,\\ a_{m+1,x}=i(pc_{m+1}-qb_{m+1}),\end{cases}\quad m\ge0. \tag{11}$$

We impose the integration conditions

$$a_m|_{u=0}=0,\ m\ge1;\ \text{and so},\ b_m|_{u=0}=c_m|_{u=0}=0,\ m\ge1.$$

This means that we take the constants of integration to be zero, in order to determine the sequence of $\{a_m, b_m, c_m | \, m \geq 1\}$ uniquely. In this way, the first few sets can be worked out as follows:

$$\begin{aligned}
&b_1 = -p,\ c_1 = -q,\ a_1 = 0;\\
&b_2 = iq_x,\ c_2 = -ip_x,\ a_2 = \tfrac{1}{2}(p^2+q^2);\\
&b_3 = -p_{xx} + \tfrac{1}{2}p^3 + \tfrac{1}{2}pq^2,\\
&c_3 = -q_{xx} + \tfrac{1}{2}p^2q + \tfrac{1}{2}q^3,\\
&a_3 = i(p_xq - pq_x);\\
&b_4 = i(q_{xxx} - \tfrac{3}{2}p^2q_x - \tfrac{3}{2}q^2q_x),\\
&c_4 = i(-p_{xxx} + \tfrac{3}{2}p^2p_x + \tfrac{3}{2}p_xq^2),\\
&a_4 = pp_{xx} + qq_{xx} - \tfrac{1}{2}p_x^2 - \tfrac{1}{2}q_x^2 - \tfrac{3}{8}(p^2+q^2)^2;\\
&b_5 = -p_{xxxx} + \tfrac{5}{2}p^2p_{xx} + \tfrac{5}{2}pp_x^2 + \tfrac{3}{2}p_{xx}q^2 + 3p_xqq_x\\
&\qquad + pqq_{xx} - \tfrac{1}{2}pq_x^2 - \tfrac{3}{8}p(p^2+q^2)^2,\\
&c_5 = -q_{xxxx} + \tfrac{5}{2}q^2q_{xx} + \tfrac{5}{2}qq_x^2 + \tfrac{3}{2}p^2q_{xx} + 3pp_xq_x\\
&\qquad + pp_{xx}q - \tfrac{1}{2}p_x^2q - \tfrac{3}{8}q(p^2+q^2)^2,\\
&a_5 = i(p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_xq_{xx}\\
&\qquad - \tfrac{3}{2}p^2p_xq + \tfrac{3}{2}pq^2q_x + \tfrac{3}{2}p^3q_x - \tfrac{3}{2}p_xq^3).
\end{aligned}$$

Now we take

$$V^{[m]} = (\lambda^m W)_+ = \sum_{i=0}^{m} W_{0,i}\lambda^{m-i},\ m \geq 0, \tag{12}$$

to introduce the temporal matrix spectral problems:

$$-i\phi_t = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi,\ m \geq 0. \tag{13}$$

Then, the zero curvature equations

$$U_t - V_x^{[m]} + i[U, V^{[m]}] = 0,\ m \geq 0, \tag{14}$$

generate a hierarchy of soliton equations:

$$u_t = K_m = i\begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} iq \\ -ip \end{bmatrix},\ m \geq 0, \tag{15}$$

where the operator Φ can be determined by the recursion relation (11):

$$\Phi = i\begin{bmatrix} q\partial^{-1}p & -\partial + q\partial^{-1}q \\ \partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix},\ \partial = \frac{\partial}{\partial x}. \tag{16}$$

2.2. Hamiltonian structure and the Liouville integrability

We apply the trace identity [2] for our spectral matrix iU:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})\,dx = \lambda^{-\gamma}\frac{\lambda}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}(W\frac{\partial U}{\partial u}), \tag{17}$$

where the constant γ is determined by

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|. \tag{18}$$

Obviously, we see

$$\frac{\partial U}{\partial \lambda} = e_1,\ \frac{\partial U}{\partial p} = e_2,\ \frac{\partial U}{\partial q} = e_3,$$

and so we have

$$\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = -2a,\ \mathrm{tr}(W\frac{\partial U}{\partial p}) = -2b,\ \mathrm{tr}(W\frac{\partial U}{\partial q}) = -2c.$$

Then the corresponding trace identity (17) reads

$$\frac{\delta}{\delta u}\int a\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} b \\ c \end{bmatrix}.$$

A balance of the coefficients of each power of λ in this equality yields

$$\frac{\delta}{\delta u}\int a_{m+1}\,dx=(\gamma-m)\begin{bmatrix}b_m\\ c_m\end{bmatrix},\ m\geq 0.$$

The case of $m=1$ leads to $\gamma=0$, and thus, we obtain

$$\frac{\delta}{\delta u}\int(-\frac{a_{m+2}}{m+1})\,dx=\begin{bmatrix}b_{m+1}\\ c_{m+1}\end{bmatrix},\ m\geq 0. \tag{19}$$

Consequently, we obtain a Hamiltonian structure for the soliton hierarchy (15):

$$u_t=K_m=i\begin{bmatrix}-c_{m+1}\\ b_{m+1}\end{bmatrix}=J\frac{\delta\mathcal{H}_m}{\delta u},\ m\geq 0, \tag{20}$$

with the Hamiltonian operator

$$J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{21}$$

and the Hamiltonian functionals

$$\mathcal{H}_m=\int(-\frac{ia_{m+2}}{m+1})\,dx,\ m\geq 0. \tag{22}$$

These lead to infinitely many conservation laws of each system in the soliton hierarchy (15), which can often be generated through symbolic computation by computer algebra systems (see, e.g., [9]).

To exhibit the Liouville integrability, let us prove that the operator Φ defined by (16) is a hereditary recursion operator for the soliton hierarchy (15).

First, a direct but lengthy computation can show that the operator Φ is hereditary (see [10] for definition), i.e., it satisfies

$$\Phi'(u)[\Phi K]S-\Phi\Phi'(u)[K]S=\Phi'(u)[\Phi S]K-\Phi\Phi'(u)[S]K \tag{23}$$

for all vector fields K and S; and that J and

$$M=\Phi J=i\begin{bmatrix}-\partial+q\partial^{-1}q & -q\partial^{-1}p\\ -p\partial^{-1}q & -\partial+p\partial^{-1}p\end{bmatrix}, \tag{24}$$

where J is defined by (21), constitute a Hamiltonian pair (see [11] for details), i.e., any linear combination N of J and M satisfies the Jacobi identity

$$\int K^T N'(u)[NS]T\,dx+\text{cycle}(K,S,T)=0 \tag{25}$$

for all vector fields K, S and T.

We point out that the hereditary property (23) is equivalent to

$$L_{\Phi K}\Phi=\Phi L_K\Phi, \tag{26}$$

where K is an arbitrary vector field and $L_K\Phi$ is the Lie derivative $L_K\Phi$:

$$(L_K\Phi)S=\Phi[K,S]-[K,\Phi S],$$

with $[\cdot,\cdot]$ being the Lie bracket of vector fields.

Second, note that an autonomous operator $\Phi=\Phi(u,u_x,\ldots)$ is a recursion operator of an evolution equation $u_t=K$ if and only if the operator Φ needs to satisfy

$$L_K\Phi=0. \tag{27}$$

Obviously, for the operator Φ defined by (16), we have

$$L_{K_0}\Phi=0,\ K_0=i\begin{bmatrix}q\\ -p\end{bmatrix},$$

and thus

$$L_{K_m}\Phi=L_{\Phi K_{m-1}}\Phi=\Phi L_{K_{m-1}}\Phi=0,\ m\geq 1, \tag{28}$$

where the K_m's are given by (15). This implies that the operator Φ defined by (16) is a common hereditary recursion operator for the soliton hierarchy (15).

Now, the soliton hierarchy (15) is bi-Hamiltonian (see, e.g., [11,12]):

$$u_t=K_m=J\frac{\delta\mathcal{H}_m}{\delta u}=M\frac{\delta\mathcal{H}_{m-1}}{\delta u},\ m\geq 1, \tag{29}$$

where J, M and $\mathcal{H}_m$ are defined by (21), (24) and (22) respectively, and so, every member in the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra:

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0,\ \ k, l \geq 0, \tag{30}$$

and the Abelian algebras of conserved functionals:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int (\frac{\delta \mathcal{H}_k}{\delta u})^T J \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0,\ \ k, l \geq 0, \tag{31}$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int (\frac{\delta \mathcal{H}_k}{\delta u})^T M \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0,\ \ k, l \geq 0. \tag{32}$$

2.3. Two particular examples

The first two nonlinear integrable systems in the soliton hierarchy (15) are a nonlinear Schrödinger type system

$$u_t = K_2, \tag{33}$$

i.e.,

$$\begin{cases} p_t = i(q_{xx} - \frac{1}{2}p^2 q - \frac{1}{2}q^3), \\ q_t = i(-p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2), \end{cases} \tag{34}$$

and a modified Korteweg–de Vries type system

$$u_t = K_3, \tag{35}$$

i.e.,

$$\begin{cases} p_t = -p_{xxx} + \frac{3}{2}p^2 p_x + \frac{3}{2}p_x q^2, \\ q_t = -q_{xxx} + \frac{3}{2}p^2 q_x + \frac{3}{2}q^2 q_x. \end{cases} \tag{36}$$

They possess the following bi-Hamiltonian structures

$$u_t = K_2 = J\frac{\delta \mathcal{H}_2}{\delta u} = M\frac{\delta \mathcal{H}_1}{\delta u}, \tag{37}$$

and

$$u_t = K_3 = J\frac{\delta \mathcal{H}_3}{\delta u} = M\frac{\delta \mathcal{H}_2}{\delta u}, \tag{38}$$

where the Hamiltonian pair $\{J, M\}$ is given by (21) and (24), and the Hamiltonian functionals, $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, are determined by

$$\mathcal{H}_1 = -\frac{1}{2}\int (pq_x - p_x q)\, dx, \tag{39}$$

$$\mathcal{H}_2 = -\frac{i}{3}\int [pp_{xx} + qq_{xx} - \frac{1}{2}p_x^2 - \frac{1}{2}q_x^2 - \frac{3}{8}(p^2+q^2)^2]\, dx, \tag{40}$$

$$\begin{aligned}\mathcal{H}_3 = \frac{1}{4}\int (&p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_x q_{xx} \\ &- \frac{3}{2}p^2 p_x q + \frac{3}{2}pq^2 q_x + \frac{3}{2}p^3 q_x - \frac{3}{2}p_x q^3)\, dx.\end{aligned} \tag{41}$$

It is worth pointing out that the transformation

$$\tilde{p}(x,t) = \frac{1}{2}(-p + iq)(ix, it),\ \ \tilde{q}(x,t) = \frac{1}{2}(-p - iq)(ix, it), \tag{42}$$

puts the system (34) of NLS equations associated with so(3) into th system of NLS equations associated with sl(2):

$$\tilde{p}_t = -i(\tilde{p}_{xx} + 2\tilde{p}^2\tilde{q}),\ \ \tilde{q}_t = i(\tilde{q}_{xx} + 2\tilde{p}\tilde{q}^2), \tag{43}$$

and the system (36) of mKdV equations associated with so(3) into the system of mKdV equations associated with sl(2):

$$\tilde{p}_t = \tilde{p}_{xxx} + 6\tilde{p}\tilde{q}\tilde{p}_x,\ \ \tilde{q}_t = \tilde{q}_{xxx} + 6\tilde{p}\tilde{q}\tilde{q}_x. \tag{44}$$

But there is no nontrivial transformation if we consider only real potentials $p, q, \tilde{p}, \tilde{q}$. This subtle difference is a reflection of the fact that the two complex Lie algebras, sl(2, $\mathbb{C}$) and so(3, $\mathbb{C}$), are isomorphic, but not so are the two real Lie algebras, sl(2, $\mathbb{R}$) and so(3, $\mathbb{R}$).

3. Integrable reductions

Let us introduce two specific reductions for the spectral matrix:

$$(U(\lambda^*))^\dagger = CU(\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{45}$$

where † denotes the Hermitian transpose. They lead to the potential reductions

$$p = \delta q^*,\ \delta = \pm 1, \tag{46}$$

and thus, the reduced spectral matrices read

$$U = \begin{bmatrix} 0 & -\delta p^* & -\lambda \\ \delta p^* & 0 & -p \\ \lambda & p & 0 \end{bmatrix},\ \delta = \pm 1.$$

Under the above potential reductions, one has

$$a_m^* = a_m,\ b_m^* = \delta c_m,\ m \geq 1. \tag{47}$$

This statement can be proved by the mathematical induction, since under the induction assumption for $m = l$ and the recursion relation (11), one can compute

$$b_{l+1}^* = ic_{l,x}^* + p^* a_l^* = i\delta b_{l,x} + \delta q a_l = \delta c_{l+1},$$
$$a_{l+1,x}^* = -i(p^* c_{l+1}^* - q^* b_{l+1}^*) = i(pc_{l+1} - qb_{l+1}) = a_{l+1,x}.$$

Therefore, one obtains

$$(V^{[m]}(\lambda^*))^\dagger = CV^{[m]}(\lambda)C^{-1},\ m \geq 1, \tag{48}$$

and further

$$((U_t - V_x^{[m]} + i[U, V^{[m]}])(\lambda^*))^\dagger = C(U_t - V_x^{[m]} + i[U, V^{[m]}])(\lambda)C^{-1},\ m \geq 1. \tag{49}$$

This tells that the potential reductions in (46) are compatible with the zero curvature equations of the soliton hierarchy (15).

Therefore, we have got two reduced soliton hierarchies associated with so(3, $\mathbb{R}$):

$$p_t = K_{m,1}|_{q=\delta p^*},\ m \geq 1, \tag{50}$$

where $K_m = (K_{m,1}, K_{m,2})^T$, $m \geq 1$, are defined by (15). The infinitely many symmetries and conservation laws for the soliton hierarchy (15) are reduced to infinitely many ones for the above scalar soliton hierarchies in (50).

With $\delta = 1$, the first two reduced integrable equations read

$$ip_t = -p_{xx}^* + \frac{1}{2}|p|^2 p + \frac{1}{2}(p^*)^3, \tag{51}$$

and

$$p_t = -p_{xxx} + \frac{3}{2}[p^2 + (p^*)^2]p_x. \tag{52}$$

The first one is a nonlinear Schrödinger type equation and the second one is a modified Korteweg–de Vries type equation, associated with Lax pairs from the Lie algebra so(3, $\mathbb{R}$).

Note that there exist even and odd properties with respect to p and q in the two components of K_m, $m \geq 1$. Actually, $K_{2l,1}$, $l \geq 1$, are odd with respect to q and even with respect to p, and $K_{2l+1,1}$, $l \geq 1$, are even with respect to q and odd with respect to p. Similarly, $K_{2l,2}$, $l \geq 1$, are odd with respect to p and even with respect to q, and $K_{2l+1,2}$, $l \geq 1$, are even with respect to p and odd with respect to q. Therefore, the case of $\delta = -1$ does not lead to essentially new scalar integrable equations. For example, the reduced second-order equation with $\delta = -1$ has just a different sign from the nonlinear Schrödinger type equation (51), and the reduced third-order equation with $\delta = -1$ is exactly the same as the modified Korteweg–de Vries type equation (52).

This is a new phenomenon for soliton equations associated with so(3, $\mathbb{R}$), which is totally different from the one for soliton equations associated with sl(2, $\mathbb{R}$).

4. Conclusions and remarks

We have reformulated a hierarchy of soliton equations based on zero curvature equations associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$) and presented two integrable reductions for the soliton hierarchy successfully. Two particular examples of the reduced scalar integrable equations are a nonlinear Schrödinger type equation and a modified Korteweg–de Vries type equation.

There are interesting questions for soliton equations associated with the special orthogonal Lie algebras. First, what kind of general soliton hierarchies could exist? Some novel structures of soliton equations associated to so(4, $\mathbb{R}$) have been explored [13].

Second, how can we formulate Riemann–Hilbert problems, based on matrix spectral problems? The above spectral matrix iU with zero potential has three eigenvalues, which brings difficulties in establishing relevant theories. The existing examples belong to the class with two eigenvalues.

It is known that soliton equations can be generated from zero curvature equations associated with non-semisimple Lie algebras. Bi-integrable couplings and tri-integrable couplings are such examples and exhibit insightful thoughts about general structures of multi-component soliton equations [14]. Multi-integrable couplings provide abundant examples of recursion operators in block matrix form, indeed. There are rich mathematical structures related to integrable couplings [8,14]. Hamiltonian structures could be furnished for the perturbation equations [15,16] by the variational identity, but not for all integrable couplings. Non-semisimple matrix Lie algebras may not possess any non-degenerate and ad-invariant bilinear forms required in the variational identities [17,18]. It still remains an open problem how to guarantee the existence of Hamiltonian structures for bi- or tri-integrable couplings, based on zero curvature equations. Moreover, we do not even know if there exists any Hamiltonian structure for a perturbation type coupling:

$$u_t = K(u),\ v_t = K'(u)[v],\ w_t = K'(u)[w].$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] M.A. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991, http://dx.doi.org/10.1017/CBO9780511623998.
[2] G.Z. Tu, T on Liouville integrability of zero-curvature equations and the Yang hierarchy, J. Phys. A: Math. Gen. 22 (1989) 2375–2392, http://dx.doi.org/10.1088/0305-4470/22/13/031.
[3] W.X. Ma, M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras, J. Phys. A: Math. Gen. 39 (2006) 10787–10801, http://dx.doi.org/10.1088/0305-4470/39/34/013.
[4] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315, http://dx.doi.org/10.1002/sapm1974534249.
[5] W.X. Ma, A soliton hierarchy associated with so(3, ℝ), Appl. Math. Comput. 220 (2013) 117–122, http://dx.doi.org/10.1016/j.amc.2013.04.062.
[6] W.X. Ma, A spectral problem based on so(3, ℝ) and its associated commuting soliton equations, J. Math. Phys. 54 (2013) 103509, http://dx.doi.org/10.1063/1.4826104.
[7] S.F. Shen, L.Y. Jiang, Y.Y. Jin, W.X. Ma, New soliton hierarchies associated with the Lie algebra so(3, R) and their bi-Hamiltonian structures, Rep. Math. Phys. 75 (2015) 113–133, http://dx.doi.org/10.1016/S0034-4877(15)60028-3.
[8] W.X. Ma, in: W.X. Ma, D. Kaup (Eds.), Nonlinear and Modern Mathematical Physics, in: AIP Conference Proceedings, vol. 1562, American Institute of Physics, Melville, NY, 2013, pp. 105–122, http://dx.doi.org/10.1063/1.4828687.
[9] W. Hereman, P.J. Adams, H.L. Eklund, M.S. Hickman, B.M. Herbst, Direct methods and symbolic software for conservation laws of nonlinear equations, in: Advances in Nonlinear Waves and Symbolic Computation, Nova Sci. Publ., New York, 2009, pp. 19–78, http://dx.doi.org/10.48550/arXiv.0803.0083.
[10] B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations, Nonlinear Anal. 3 (1979) 849–862, http://dx.doi.org/10.1016/0362-546X(79)90052-X.
[11] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162, http://dx.doi.org/10.1063/1.523777.
[12] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986, http://dx.doi.org/10.1007/978-1-4684-0274-2.
[13] S.D. Zhu, S.F. Shen, Y.Y. Jin, C.X. Li, W.X. Ma, New soliton hierarchies associated with the real Lie algebra so(4, ℝ), Math. Methods Appl. Sci. 40 (2017) 680–698, http://dx.doi.org/10.1002/mma.4001.
[14] W.X. Ma, J.H. Meng, H.Q. Zhang, Integrable. couplings, Variational identities and Hamiltonian formulations, Global J. Math. Sci. 1 (2012) 1–17, Corpus ID: 37738154.
[15] W.X. Ma, B. Fuchssteiner, Integrable theory of the perturbation equations, Chaos Solitons Fractals 7 (1996) 1227–1250, http://dx.doi.org/10.1016/0960-0779(95)00104-2.
[16] S. Yu Sakovich, On integrability of a $(2+1)$-dimensional perturbed KdV equation, J. Nonlinear Math. Phys. 5 (1998) 230–233, http://dx.doi.org/10.2991/jnmp.1998.5.3.1.
[17] W.X. Ma, L. Gao, Coupling integrable couplings, 1 Mod. Phys. Lett. B 23 (2009) 1847–1860, http://dx.doi.org/10.1142/S0217984909020011.
[18] W.X. Ma, Enlarging spectral problems to construct integrable couplings of soliton equations, Phys. Lett. A 316 (2003) 72–76, http://dx.doi.org/10.1016/S0375-9601(03)01137-X.