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A New Hierarchy of Liouville Integrable Generalized Hamiltonian Equations and Its Reduction

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A first-degree isospectral problem with three potentials is considered following Tu's approach. A new corresponding hierarchy of Lax integrable evolution equations is generated and a reduction hierarchy is discussed in detail. It is shown that both these hierarchies of equations possess Hamiltonian structures and are Liouville integrable.

§1. INTRODUCTION

In the theory of finite dimensional Hamiltonian systems, Liouville's theorem[1] shows that an N-dimensional Hamiltonian system over some region $\Omega \sqsubseteq R^{2N}$

$$p_{it} = -\frac{\partial H}{\partial q_i} \quad , \quad q_{it} = \frac{\partial H}{\partial p_i} \quad , \quad i = 1, 2, \cdots, N \tag{1.1}$$

is completely integrable, i.e. integrable by quadratures, so long as there exist N independent integrals of motion in involution over the region Ω. However, in the case of infinite dimensional systems, i.e., generalized Hamiltonian equations, we do not have such elegant geometrical theory yet. What is more, we have not yet exposed satisfactorily the nature of complete integrability for infinite dimensional systems. In this paper we adopt two kinds of special definitions on integrability: Lax integrability and Liouville integrability. A nonlinear evolution equation is called Lax integrable if it admits a zero curvature representation and Liouville integrability will be introduced in the following after giving some basic notation.

Let $u = u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_q(x,t))^T$, $x, t \in R$, be a q-dimensional function vector belonging to the q-tuple Schwartz space over R for any fixed t. We denote by $\mathcal{A}$ the linear space of smooth functions $F(x, u^{(n)}) = F[u]$ and consider two functions F and G to be equivalent if $F - G = \partial H = dH/dx$ holds for some $H \in \mathcal{A}$. The equivalent class which F belongs to is denoted by $\tilde{F} = \int F dx$. Each equivalent class $\tilde{F} = \int F dx$ is called a functional and its variational derivative is defined by

$$\frac{\delta \tilde{F}}{\delta u} = \frac{\delta F}{\delta u} = (\frac{\delta F}{\delta u_1}, \frac{\delta F}{\delta u_2}, \cdots, \frac{\delta F}{\delta u_q})^T \tag{1.2a}$$

where

$$\frac{\delta}{\delta u_i} = \sum_{j\geq 0}(-\partial)^j \frac{\partial}{\partial u_i^{(j)}} \quad , \quad u_i^{(j)} = \partial^j u_i \quad , \quad i = 1, 2, \cdots, q\,. \tag{1.2b}$$

Because $\frac{\delta}{\delta u}\partial G = 0$ for all $G \in \mathcal{A}$, the above definition (1.2a) is clear and unambiguous.

Definition 1.1[2] A generalized Hamiltonian equation with the Hamiltonian operator $J = J(x, u)$ and the Hamiltonian function $H = H(x, u)$

$$u_t = J\frac{\delta H}{\delta u} \quad \text{where} \quad u = u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_q(x,t))^T \tag{1.3}$$

is called Liouville integrable if there exists a sequence of conserved densities $\{F_n\}_{n=0}^{\infty}$ satisfying the following conditions

(1) $\{\tilde{F}_m, \tilde{F}_n\} = \int (\frac{\delta \tilde{F}_m}{\delta u})^T J \frac{\delta \tilde{F}_n}{\delta u} dx = o$ for $0 \leq m, n < \infty$;

(2) $\{dF_n\}_{n=0}^{\infty}$ constitute a set of linearly independent 1-forms, where

$$dF_n = \frac{\partial F_n}{\partial x}dx + \frac{\partial F_n}{\partial t}dt + \sum_{i,j}\frac{\partial F_n}{\partial u_i^{(j)}}du_i^{(j)} \quad \text{for} \quad 0 \leq n < \infty \quad .$$

Tu[3-5] proposed an algebraic scheme for generating hierarchies of Liouville integrable generalized Hamiltonian equations from a series of zero curvature representations of the following isospectral problem

$$\begin{cases} \phi_x = U\phi = U(u,\lambda)\phi \\ \phi_{t_n} = V^{(n)}\phi = V^{(n)}(u,\lambda)\phi \end{cases} \quad \text{for } n \geq -k \quad , \quad k \in Z \tag{1.4}$$

which is the form extended in Ref. [6]. Many hierarchies of soliton equations are indeed derived through this approach (for example, see References [7-13]).

In this paper, by means of Tu's method mentioned above, we consider the following isospectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi = \begin{bmatrix} \alpha_1\lambda + q & r \\ \alpha_3 & \alpha_2\lambda + s \end{bmatrix}\phi \quad , \quad \alpha_1 \neq \alpha_2 \ , \ \alpha_3 \neq 0 \ , \tag{1.5}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants and $u = (q, r, s)^T$. In Section 2 we regularly solve the coadjoint representation equation $V_x = [U, V]$ of (1.5) and tersely derive a sort of special hierarchies of Lax integrable evolution equations. In Section 3, we take out a hierarchy of generalized Hamiltonian equations from among those Lax integrable equations and show Liouville integrability of the hierarchy. Finally in Section 4. we carefully discuss a kind of reduction of the isospectral problem (1.5) and also show Liouville integrability of the corresponding reduction hierarchy.

§2. SPECIAL LAX INTEGRABLE EVOLUTION EQUATIONS

We rewrite the matrix U in (1.5) as

$$\begin{aligned} U =&(\alpha_1\lambda + q)h_1 + (\alpha_2\lambda + s)h_2 + re + \alpha_3 f \\ =&(\alpha_1\lambda h_1 + \alpha_2\lambda h_2 + \alpha_3 f) + qh_1 + sh_2 + re \end{aligned} \tag{2.1}$$

where

$$h_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, h_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for which we have

$$[h_1, e] = -[h_2, e] = e,\ [h_1, f] = -[h_2, f] = -f,\ [e, f] = h_1 - h_2\ , \tag{2.2}$$

$$< h_1, h_1 >=< h_2, h_2 >=< e, f >= 1\ . \tag{2.3}$$

Here and in the following we denote the Killing-Cartan form by $< y, z >= tr(yz)$.

We first solve the coadjoint representation equation $V_x = [U, V]$ of (2.1). We assume that V has the following form

$$V = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = ah + be + cf \quad , \quad h = h_1 - h_2\ . \tag{2.4}$$

It is easy to show that $V_x = [U, V]$ is equivalent to

$$\begin{cases} a_x = rc - \alpha_3 b, \\ b_x = \alpha\lambda b + (q - s)b - 2ra, \\ c_x = -\alpha\lambda c - (q - s)c + 2\alpha_3 a, \end{cases} \tag{2.5}$$

where $\alpha = \alpha_1 - \alpha_2$. Substituting

$$a = \sum_{n\geq 0} a_n\lambda^{-n}, \quad b = \sum_{n\geq 0} b_n\lambda^{-n}, \quad c = \sum_{n\geq 0} c_n\lambda^{-n}$$

into (2.5), we obtain

$$\begin{cases} b_0 = c_0 = 0\ ,\ a_0 = \eta = \text{const.} \neq 0\ , \\ a_{nx} = rc_n - \alpha_3 b_n\ , \\ b_{nx} = \alpha b_{n+1} + (q - s)b_n - 2ra_n\ , \\ c_{nx} = -\alpha c_{n+1} - (q - s)c_n + 2\alpha_3 a_n\ , \end{cases} \quad n \geq 0 \quad , \tag{2.6}$$

in which we have made that $\eta \neq 0$ in order to get nonzero vectors (b_n, c_n, a_n). In particular, one can recursively deduce the first three vectors (b_n, c_n, a_n)

$$b_1 = 2\alpha^{-1}\eta r\ ,\ c_1 = 2\alpha^{-1}\alpha_3\eta\ ,\ a_1 = 0\ ; \tag{2.7a}$$

$$b_2 = 2\alpha^{-2}\eta[r_x - (q-s)r]\,,\ c_2 = -2\alpha^{-2}\alpha_3\eta(q-s)\,,\ a_2 = -2\alpha^{-2}\alpha_3\eta r\,; \tag{2.7b}$$

$$b_3 = 2\alpha^{-3}\eta[r_{xx} + 2(q-s)r_x + (q_x - s_x)r + r(q^2+s^2) - 2(qs+\alpha_3 r)r]\,,$$
$$c_3 = 2\alpha^{-3}\alpha_3\eta[(q_x - s_x) + (q^2+s^2) - 2(qs+\alpha_3 r)]\,, \tag{2.7c}$$
$$a_3 = 2\alpha^{-3}\alpha_3\eta[-r_x + 2(q-s)r]\,.$$

Based on $(\det V)_x = 0$, we can determine recursively from (2.6) that the a_n , b_n , c_n are all local.

Now we turn to derive new Lax integrable evolution equations. For convenience sake, we will always write $g' = g(\mu)$ if $g = g(\lambda)$ and $g_+ = \sum_{n\geq 0} g_n\lambda^n$ if $g = \sum g_n\lambda^n$. By using (2.1) and (2.2), we have the following relation

$$[\mu(U(\lambda) - U(\mu))/(\lambda-\mu), V(\mu)] = [\mu(\alpha_1 h_1 + \alpha_2 h_2), a'h + b'e + c'f]$$
$$= [\mu(\alpha_1 h_1 + \alpha_2 h_2), b'e + c'f] = \alpha\mu b'e - \alpha\mu c'f\,.$$

Choosing $\Delta(\lambda) = \sum_{n\geq -1}\Delta_n\lambda^{-n} = \delta_1 h_1 + \delta_2 h_2 = \delta_1(\lambda)h_1 + \delta_2(\lambda)h_2\ (k=1)$, we obtain

$$\Delta_x(\mu) - [U(\lambda), \Delta(\mu)] = \delta'_{1x}h_1 + \delta'_{2x}h_2 + (\delta'_1 - \delta'_2)re - \alpha_3(\delta'_1 - \delta'_2)f\,.$$

To obtain Lax integrable equations, we need the condition

$$\alpha_3(\delta_1 - \delta_2) = -\alpha\lambda c \quad \text{or} \quad \delta_1 - \delta_2 = -\alpha\alpha_3^{-1}\lambda c\,. \tag{2.8}$$

At this moment, by (2.5),

$$\alpha\lambda b + (\delta_1 - \delta_2)r = \alpha\alpha_3^{-1}\lambda(rc - a_x) - \alpha\alpha_3^{-1}\lambda rc = -\alpha\alpha_3^{-1}\lambda a_x\,.$$

Therefore we get a determining equation for generating Lax integrable evolution equations

$$[\mu(U(\lambda) - U(\mu))/(\lambda-\mu), V(\mu)] + \Delta_x(\mu) - [U(\lambda), \Delta(\mu)]$$
$$= \delta'_{1x}h_1 + \delta'_{2x}h_2 - \alpha\alpha_3^{-1}\mu a'_x e := f'_1 h_1 + f'_3 h_2 + f'_2 e\,. \tag{2.9}$$

Set $V^{(n)} = (\lambda^n V)_+ + \Delta_n, n \geq -1$. According to Tu's scheme[4,6], we have

$$\sum_{n\geq -1}(V_x^{(n)}(\lambda) - [U(\lambda), V^{(n)}(\lambda)])\mu^{-n}$$
$$= [\mu(U(\lambda) - U(\mu))/(\lambda-\mu), V(\mu)] + \Delta_x(\mu) - [U(\lambda), \Delta(\mu)]\,.$$

Therefore we see by (2.9) that a sequence of zero curvature representation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$, $n \geq -1$, gives the following Lax integrable evolution equations

$$\begin{cases} q_{t_n} = \delta_{1nx}, \\ r_{t_n} = -\alpha\alpha_3^{-1}a_{n+1,x}, \qquad n \geq -1 \quad , \\ s_{t_n} = \delta_{2nx}, \end{cases} \tag{2.10}$$

where the a_n are defined by (2.6), and $\delta_i = \sum_{n\geq -1} \delta_{in}\lambda^{-n}$, $i = 1, 2$, satisfy the condition (2.8). Evidently here (2.10) includes an arbitrary function. In the next section we shall choose a sort of such functions so that (2.10) becomes a hierarchy of generalized Hamiltonian equations.

§3. A HAMILTONIAN STRUCTURE THROUGH THE TRACE IDENTITY

In order to derive a Hamiltonian structure by means of the trace identity, we need the following equalities which are easily worked out from (2.1) and (2.3)

$$\frac{\partial U}{\partial q} = h_1\,,\ \frac{\partial U}{\partial r} = e\,,\ \frac{\partial U}{\partial s} = h_2\,,\ \frac{\partial U}{\partial \lambda} = \alpha_1 h_1 + \alpha_2 h_2\,, \tag{3.1}$$

and

$$< V, \frac{\partial U}{\partial q} >= a\,, < V, \frac{\partial U}{\partial r} >= c\,, < V, \frac{\partial U}{\partial s} >= -a\,, < V, \frac{\partial U}{\partial \lambda} >= \alpha a\,. \tag{3.2}$$

We further choose that

$$\delta_1 = \lambda(\beta_1 a + \beta_2 c) \quad , \quad \delta_2 = \lambda(\beta_1 a + \beta_3 c) \quad , \tag{3.3}$$

where $\beta_1, \beta_2, \beta_3$ are constants and

$$\beta_2 - \beta_3 = -\alpha\alpha_3^{-1} \tag{3.4}$$

by which we have (2.8). Let

$$J = \begin{bmatrix} \beta_1\partial & \beta_2\partial & 0 \\ \beta_2\partial & 0 & \beta_3\partial \\ 0 & \beta_3\partial & -\beta_1\partial \end{bmatrix} \quad . \tag{3.5}$$

Then we have

$$\begin{aligned} &\lambda J(< V, \frac{\partial U}{\partial q} >, < V, \frac{\partial U}{\partial r} >, < V, \frac{\partial U}{\partial s} >)^T \\ &= \lambda J(a, c, -a)^T = (\delta_{1x}, -\alpha\alpha_3^{-1}\lambda a_x, \delta_{2x})^T = (f_1, f_2, f_3)^T \quad . \end{aligned} \tag{3.6}$$

Obviously J is a Hamiltonian operator and thus we can define a Poisson bracket for two functionals $\tilde{H} = \int H\,dx$, $\tilde{I} = \int I\,dx$:

$$\{\tilde{H}, \tilde{I}\} = \int (\frac{\delta \tilde{H}}{\delta u})^T J \frac{\delta \tilde{I}}{\delta u}\,dx \quad . \tag{3.7}$$

Now applying the trace identity[14,8]

$$\frac{\delta}{\delta u} < V, \frac{\partial U}{\partial \lambda} >= \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(< V, \frac{\partial U}{\partial q} >, < V, \frac{\partial U}{\partial r} >, < V, \frac{\partial U}{\partial s} >)^T \quad ,$$

we obtain at once

$$\frac{\delta}{\delta u}\alpha a = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(a, c, -a)^T \quad .$$

Equating the coefficients of λ^{-n-1} on the left and right sides, we have

$$\frac{\delta}{\delta u}\alpha a_{n+1} = (\gamma - n)(a_n, c_n, -a_n)^T \quad , \quad n \geq 0 \quad .$$

To fix the constant γ, we take simply $n = 0$ and obtain $\gamma(\eta, 0, -\eta)^T = 0$. Thus $\gamma = 0$. In this way we obtain an important equality

$$\frac{\delta H_n}{\delta u} = (a_n, c_n, -a_n)^T \quad , \quad n \geq 0 \tag{3.8}$$

where the H_n are defined as follows

$$H_0 = \eta(q - s) \quad , \quad H_n = -\frac{\alpha a_{n+1}}{n} \quad , \quad n \geq 1 \quad . \tag{3.9}$$

Therefore we see by (3.6) that Lax integrable evolution equations (2.10) read as the following hierarchy of generalized Hamiltonian equations

$$\begin{bmatrix} q \\ r \\ s \end{bmatrix}_{t_n} = J \begin{bmatrix} \frac{\delta H_{n+1}}{\delta q} \\ \frac{\delta H_{n+1}}{\delta r} \\ \frac{\delta H_{n+1}}{\delta s} \end{bmatrix} \quad , \quad n \geq -1 \quad . \tag{3.10}$$

Now let us to assume that $H = \sum_{n\geq 0} H_n \lambda^{-n}$. Then

$$\frac{\delta H}{\delta u} = (< V, \frac{\partial U}{\partial q} >, < V, \frac{\partial U}{\partial r} >, < V, \frac{\partial U}{\partial s} >)^T \quad .$$

It is shown by a direct computation (see [4,6]) that

$$\begin{aligned} &(< V(\lambda), \frac{\partial U(\lambda)}{\partial q} >, < V(\lambda), \frac{\partial U(\lambda)}{\partial r} >, < V(\lambda), \frac{\partial U(\lambda)}{\partial s} >)(f_1(\mu), f_2(\mu), f_3(\mu))^T \\ &=< V(\lambda), \mu V(\mu)/(\mu - \lambda) + \Delta(\mu) >_x \quad . \end{aligned}$$

Therefore we obtain by (3.6)

$$\mu \frac{\delta H(\lambda)}{\delta u} J \frac{\delta H(\mu)}{\delta u} =< V(\lambda), \mu V(\mu)/(\mu - \lambda) + \Delta(\mu) >_x \quad .$$

A further computation may yield that

$$< V(\lambda), \mu V(\mu)/(\mu - \lambda) + \Delta(\mu) >= \frac{\mu}{\mu - \lambda}(2aa' + bc' + cb') - \alpha\alpha_3^{-1}\mu ac' \ ,$$

by which we arrive at

$$(\mu - \lambda)\frac{\delta H(\lambda)}{\delta u} J \frac{\delta H(\mu)}{\delta u} = [(2aa' + bc' + cb') + (\lambda - \mu)\alpha\alpha_3^{-1} ac']_x \ , \tag{3.11}$$

which may also be proved directly. Based on (3.11), we can know that $\{H_n\}_{n=0}^{\infty}$ is a series of common conserved densities of the hierarchy (3.10) and is an involutive system with respect to the Poisson bracket (3.7). Noticing the relation (3.9) and the recursive formula (2.6), it is not difficult to see that $\{dH_n\}_{n=0}^{\infty}$ is linearly independent. Thus all generalized Hamiltonian equations in the hierarchy (3.10) are Liouville integrable. Moreover the hierarchy (3.10) possesses a series of common symmetries $\{X_{n-1} = J\frac{\delta H_n}{\delta u}\}_{n=0}^{\infty}$ since one has

$$[J\frac{\delta H_m}{\delta u}, J\frac{\delta H_n}{\delta u}] = J\frac{\delta}{\delta u}\{\tilde{H}_m, \tilde{H}_n\} = 0 \quad \text{for} \quad 0 \leq m, n < \infty .$$

In addition, we easily work out

$$\begin{cases} a_{n+1} = P_a a_n + P_c c_n, \\ c_{n+1} = Q_a a_n + Q_c c_n, \end{cases} \qquad n \geq 0 \quad , \tag{3.12}$$

where

$$\begin{cases} P_a = \alpha^{-1}[\partial - \partial^{-1}(q-s)\partial] \ , \ P_c = -\alpha^{-1}(\partial^{-1} r\partial + r) \ ; \\ Q_a = 2\alpha^{-1}\alpha_3 \ , \ Q_c = -\alpha^{-1}[\partial + (q-s)] \ . \end{cases} \tag{3.13}$$

Thus we obtain

$$\begin{bmatrix} a_{n+1} \\ c_{n+1} \\ -a_{n+1} \end{bmatrix} = \Psi \begin{bmatrix} a_n \\ c_n \\ -a_n \end{bmatrix} = \begin{bmatrix} P_a + R_1 & P_c & R_1 \\ Q_a + R_2 & Q_c & R_2 \\ -P_a - R_3 & -P_c - R_3 \end{bmatrix} \begin{bmatrix} a_n \\ c_n \\ -a_n \end{bmatrix} \ , \ n \geq 0 \ , \tag{3.14}$$

where R_1, R_2, R_3 are all arbitrary integro-differential operators. Therefore here the operator Ψ includes three arbitrary operators, which is a specific property of the integrable hierarchy (3.10). We have not yet know whether there exist three operators R_1, R_2, R_3 so that the operators $J, M = J\Psi$, being closely related with the bi-Hamiltonian structure of the hierarchy (3.10), constitute a Hamiltonian pair.

§4. A KIND OF REDUCTION

In this section we consider a kind of reduction of (1.5) with $q = \alpha_4 s$ where α_4 =const. and $\alpha_4 \neq 1$. Right now U reads as

$$U = \begin{bmatrix} \alpha_1\lambda + \alpha_4 s & r \\ \alpha_3 & \alpha_2\lambda + s \end{bmatrix} \ , \ \alpha_1 \neq \alpha_2 \ , \ \alpha_3 \neq 0 \ , \ \alpha_4 \neq 1 \ . \tag{4.1}$$

We choose $\delta_1 = \alpha_4\delta_2$. To fulfil the condition (2.8), we need to set

$$\delta_2 = -\frac{\alpha}{\alpha_3(\alpha_4 - 1)}\lambda c \quad . \tag{4.2}$$

In this case, a hierarchy of Lax integrable evolution equations (2.10) becomes

$$\begin{cases} r_{t_n} = -\alpha\alpha_3^{-1} a_{n+1,x}, \\ s_{t_n} = \delta_{2nx} = -\frac{\alpha}{\alpha_3(\alpha_4 - 1)} c_{n+1,x} \end{cases} \quad , \quad n \geq -1 \quad . \tag{4.3}$$

In order to derive Hamiltonian structures of (4.3), we similarly need the following

$$\frac{\partial U}{\partial r} = e \, , \frac{\partial U}{\partial s} = \alpha_4 h_1 + h_2 \, , \frac{\partial U}{\partial \lambda} = \alpha_1 h_1 + \alpha_2 h_2 \, ;$$

$$< V, \frac{\partial U}{\partial r} >= c \, , < V, \frac{\partial U}{\partial s} >= (\alpha_4 - 1)a \, , < V, \frac{\partial U}{\partial \lambda} >= \alpha a \, ,$$

which are easily calculated. Supposing that

$$J = \begin{bmatrix} 0 & -\frac{\alpha}{\alpha_3(\alpha_4 - 1)}\partial \\ -\frac{\alpha}{\alpha_3(\alpha_4 - 1)}\partial & 0 \end{bmatrix} \tag{4.4}$$

which is obviously a Hamiltonian operator, then we get

$$\lambda J \begin{bmatrix} c \\ (\alpha_4 - 1)a \end{bmatrix} = \begin{bmatrix} -\alpha\alpha_3^{-1}\lambda a_x \\ \delta_{2x} \end{bmatrix} \quad . \tag{4.5}$$

The trace identity of this case engenders

$$\frac{\delta}{\delta u}\alpha a = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} c \\ (\alpha_4 - 1)a \end{bmatrix} \quad , \quad u = \begin{bmatrix} r \\ s \end{bmatrix} \quad ,$$

from which we have

$$\frac{\delta}{\delta u}\alpha a_{n+1} = (\gamma - n)\begin{bmatrix} c_n \\ (\alpha_4 - 1)a_n \end{bmatrix} \quad , \quad n \geq 0 \quad .$$

After setting $n = 0$, one obtains $\gamma = 0$ immediately. Therefore

$$\frac{\delta H_n}{\delta u} = \begin{bmatrix} c_n \\ (\alpha_4 - 1)a_n \end{bmatrix} \quad , \quad n \geq 0 \quad ,$$

where $\{H_n\}_{n=0}^{\infty}$ is given by

$$H_0 = (\alpha_4 - 1)\eta s \quad , \quad H_n = -\frac{\alpha a_{n+1}}{n} \quad , \quad n \geq 1 \quad . \tag{4.6}$$

Now the hierarchy (4.3) becomes the following forms

$$\begin{bmatrix} r \\ s \end{bmatrix}_{t_n} = J\begin{bmatrix} c_{n+1} \\ (\alpha_4 - 1)a_{n+1} \end{bmatrix} = J\begin{bmatrix} \frac{\delta H_{n+1}}{\delta r} \\ \frac{\delta H_{n+1}}{\delta s} \end{bmatrix} \quad , \quad n \geq -1 \quad . \tag{4.7}$$

This shows that (4.3) possesses Hamiltonian structures.

It is easy to calculate that

$$\begin{bmatrix} c_{n+1} \\ (\alpha_4 - 1)a_{n+1} \end{bmatrix} = \Psi\begin{bmatrix} c_n \\ (\alpha_4 - 1)a_n \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha^{-1}[\partial + (\alpha_4 - 1)s] & \frac{2\alpha_3}{\alpha(\alpha_4 - 1)} \\ -\alpha^{-1}(\alpha_4 - 1)(\partial^{-1}r\partial + r) & \alpha^{-1}[\partial - (\alpha_4 - 1)\partial^{-1}s\partial] \end{bmatrix}\begin{bmatrix} c_n \\ (\alpha_4 - 1)a_n \end{bmatrix}, n \geq 0, \tag{4.8}$$

and

$$
\begin{aligned}
M = J\Psi &= \begin{bmatrix} \alpha_3^{-1}(r\partial + \partial r) & -\dfrac{1}{\alpha_3(\alpha_4-1)}\partial^2 + \alpha_3^{-1}s\partial \\ \dfrac{1}{\alpha_3(\alpha_4-1)}\partial^2 + \alpha_3^{-1}\partial s & -\dfrac{2}{(\alpha_4-1)^2}\partial \end{bmatrix} \\
&= \Psi^* J = -M^* \ .
\end{aligned} \tag{4.9}
$$

It appears to us that the conjugate operator Ψ^* of Ψ is a hereditary symmetry and that the operator M is a Hamiltonian operator. Also we conjecture that J, M constitute a Hamiltonian pair. By using (4.7) and (4.8), the hierarchy of equations (4.3) reads as

$$
\begin{bmatrix} r \\ s \end{bmatrix}_{t_n} = \begin{bmatrix} -\alpha\alpha_3^{-1}a_{n+1,x} \\ -\dfrac{\alpha}{\alpha_3(\alpha_4-1)}c_{n+1,x} \end{bmatrix} = J\Psi^{n+1}\begin{bmatrix} 0 \\ (\alpha_4-1)\eta \end{bmatrix} = J\begin{bmatrix} \dfrac{\delta H_{n+1}}{\delta r} \\ \dfrac{\delta H_{n+1}}{\delta s} \end{bmatrix}, \quad n \geq -1. \tag{4.10}
$$

The first two nonzero systems are as follows

$$
r_{t_1} = 2\alpha^{-3}\eta r_x \ , \ s_{t_1} = 2\alpha^{-3}\eta s_x \ , \tag{4.11}
$$

$$
\begin{aligned}
r_{t_2} &= -2\alpha^{-4}\eta[-r_{xx} + 2(\alpha_4-1)(rs)_x] \ , \\
s_{t_2} &= -2\alpha^{-4}\eta[-\frac{2\alpha_3}{\alpha_4-1}r_x + s_{xx} + 2(\alpha_4-1)ss_x] \ .
\end{aligned} \tag{4.12}
$$

On the other hand, one can also find an analogous equality for the Poisson bracket

$$
(\mu-\lambda)\frac{\delta H(\lambda)}{\delta u}J\frac{\delta H(\mu)}{\delta u} = [(2aa' + bc' + cb') + (\lambda-\mu)\alpha\alpha_3^{-1}ac']_x \ .
$$

This is exactly the same as (3.11) in the third section. But here J is given by (4.4) and $H(\lambda) = \sum_{n\geq 0} H_n\lambda^{-n}$ in which $\{H_n\}_{n=0}^{\infty}$ is defined by (4.6). From this fact, using similar derivation **as in Section 3, one can show that the hierarchy (4.10) is Liouville integrable and that $\{H_n\}_{n=0}^{\infty}$ is just the desired conserved densities for Liouville integrability. Moreover $\{X_{n-1} = J\frac{\delta H_n}{\delta u}\}_{n=0}^{\infty}$ is a series of common symmetries of the hierarchy (4.10).**

In the following we give two special cases of the reduction.

Case 1: **Let $\alpha_1 = -\alpha_2 = -1/2$, $\alpha_3 = 1$, $\alpha_4 = -1$, $s = -(1/2)u_1$, $r = -u_2$, then the matrix U of (4.1) reads as**

$$
U = \begin{bmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u_1 & -u_2 \\ 1 & \frac{1}{2}\lambda - \frac{1}{2}u_1 \end{bmatrix} .
$$

Cao[15] investigated the nonlinearization of the spectral problem that the U corresponds to. Xu and Zhao[16] established the Hamiltonian structures of the corresponding hierarchy of integrable systems by the trace identity. The first nonlinear system in the hierarchy is the following dispersive long wave equation[17]

$$
\begin{cases} u_{1t} = u_{1xx} + 2u_1u_{1x} + 2u_{2x}, \\ u_{2t} = -u_{2xx} + 2(u_1u_2)_x, \end{cases} \qquad \eta = -\frac{1}{2} \ .
$$

Case 2: Let $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = -1$, $\alpha_4 = 0$, $r = -u_1$, $s = -u_2$, then one has

$$U = \begin{bmatrix} 0 & -u_1 \\ -1 & \lambda - u_2 \end{bmatrix}$$

which has been introduced by Levi[18,19]. The first two typical integrable systems in the corresponding Levi's hierarchy are as follows

$$\begin{cases} u_{1t} = -u_{1xx} + 2(u_1u_2)_x, \\ u_{2t} = -2u_{1x} + u_{2xx} + 2u_2u_{2x}, \end{cases} \quad \eta = -\frac{1}{2},$$

$$\begin{cases} u_{1t} = u_{1xxx} - 6u_1u_{1x} - 3(u_{1x}u_2)_x + 3(u_1u_2^2)_x, \\ u_{2t} = u_{2xxx} + \frac{3}{2}(u_2^2)_{xx} + (u_2^3)_x - 6(u_1u_2)_x, \end{cases} \quad \eta = -\frac{1}{2}. \qquad (4.13)$$

The system (4.13) is called a coupled Korteweg-de Vries(CKdV) system since it reduces to the KdV equation by setting $u_2 = 0$.

Finally we point out that the spectral problem (1.5) itself is a reduction of the spectral problem which is discussed in Ref.[20]. Thus we also give a definite answer to the first question posed in Ref.[20].

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