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Loop Algebras and Bi-integrable Couplings*

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Abstract A class of non-semisimple matrix loop algebras consisting of triangular block matrices is introduced and used to generate bi-integrable couplings of soliton equations from zero curvature equations. The variational identities under non-degenerate, symmetric and ad-invariant bilinear forms are used to furnish Hamiltonian structures of the resulting bi-integrable couplings. A special case of the suggested loop algebras yields nonlinear bi-integrable Hamiltonian couplings for the AKNS soliton hierarchy.

Keywords Loop algebra, Bi-integrable coupling, Zero curvature equation, Symmetry, Hamiltonian structure
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1 Introduction

It is known that zero curvature equations on non-semisimple loop algebras generate integrable couplings (see [25–26]), and a kind of variational identities over loop algebras (see [20, 14]) is used to furnish Hamiltonian structures of the resulting integrable couplings (see [34, 38, 40]). A key step in generating Hamiltonian structures by the variational identity is to confirm the existence of non-degenerate, symmetric and ad-invariant bilinear forms on the underlying loop algebras (see [18]). Based on special non-semisimple loop algebras, Lax pairs of block matrix form and Lax pairs with several spectral parameters bring various interesting integrable couplings, including higher dimensional local bi-Hamiltonian integrable couplings (see [9–10, 31–32]).

Let us consider an integrable evolution equation

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \cdots), \tag{1.1}$$

where u is a column vector of dependent variables. We recall that the Gateaux derivative of a

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function or an operator P along a direction X is defined by

$$P'(u)[X] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} P(u + \varepsilon X). \tag{1.2}$$

A vector field

$$\sigma = \sigma(x, t, u)$$

is called a symmetry of the equation (1.1), if it satisfies

$$\frac{\partial \sigma}{\partial t} = K'(u)[\sigma] - \sigma'(u)[K], \tag{1.3}$$

where $\frac{\partial}{\partial t}$ denotes the partial derivative with respect to the second variable t. If σ is time-independent, i.e.,

$$\frac{\partial \sigma}{\partial t} = 0,$$

then the above condition (1.3) can be reduced to a commutativity condition between K and σ:

$$[K, \sigma] := K'(u)[\sigma] - \sigma'(u)[K] = 0. \tag{1.4}$$

We know that symmetries are crucial in exploring integrability (see [12]) and generate one-parameter solution transformation groups (see [2]). If σ is a symmetry of the equation (1.1), then the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\widetilde{u} = \sigma(x, t, \widetilde{u}), \quad \widetilde{u}|_{\varepsilon=0} = u \tag{1.5}$$

defines a solution transformation group

$$g_\varepsilon : u \mapsto \widetilde{u}(\varepsilon, x, t, u)$$

with one parameter ε in the interval of existence of the above Cauchy problem. That is to say, when u is a solution to the equation (1.1), for every ε in the interval of existence, $\widetilde{u}$ is again a solution and satisfies the group axioms because of $\widetilde{u}(\varepsilon, \widetilde{u}(\eta, u)) = \widetilde{u}(\varepsilon + \eta, u)$. An important question in soliton theory is to determine and classify integrable equations. It is helpful to collect examples of evolution equations which possess infinite dimensional symmetry algebras, to work towards complete classification of integrable equations.

One way to search for integrable equations is to use zero curvature equations (see [33, 8]). A zero curvature representation of the equation (1.1) means that there exists a Lax pair (see [6]), $U = U(u, \lambda)$ and $V = V(u, \lambda)$, belonging to a matrix loop algebra, such that

$$U_t - V_x + [U, V] = 0 \tag{1.6}$$

generates the equation (1.1) (see [33]). An integrable coupling of the equation (1.1) (see [9–10] for definition)

$$\overline{u}_t = \overline{K}_1(\overline{u}) = \begin{bmatrix} K(u) \\ S(u, v) \end{bmatrix}, \quad \overline{u} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{1.7}$$

is called nonlinear, if $S(u,v)$ is nonlinear with respect to the sub-vector v of dependent variables (see [28, 19]). An integrable system of the form

$$\overline{u}_t = \overline{K}(\overline{u}) = \begin{bmatrix} K(u) \\ S_1(u,v) \\ S_2(u,v,w) \end{bmatrix}, \quad \overline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{1.8}$$

is called a bi-integrable coupling of the equation (1.1). Note that in (1.8), S_2 depends on w but S_1 does not. In what follows, we would like to use zero curvature equations to explore possibilities of generating bi-integrable couplings and construct Hamiltonian structures for the resulting integrable couplings through the variational identity associated with the enlarged Lax pairs.

The rest of the paper is structured as follows. In Section 2, a kind of matrix loop algebras consisting of triangular block matrices is introduced. Zero curvature equations on the suggested loop algebras generate bi-integrable couplings. In Section 3, an application to the AKNS spectral problem is made to construct nonlinear bi-integrable couplings for the AKNS equations, and the corresponding variational identity furnishes Hamiltonian structures for the resulting integrable couplings. An important step in generating Hamiltonian structures is to look for non-degenerate, symmetric and ad-invariant bilinear forms on the underlying loop algebras. In the final section, a few conclusive remarks on other possibilities and problems on integrable couplings are given.

2 Constructing Bi-integrable Couplings Through Loop Algebras

2.1 Loop algebras

To construct bi-integrable couplings, let us fix an arbitrary constant α and introduce a class of triangular block matrices

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & A_2 + \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{2.1}$$

where A_1, A_2 and A_3 are square matrices of the same order. It is easy to see that the matrix product of such two block matrices is given by

$$M(A_1, A_2, A_3)M(B_1, B_2, B_3) = M(C_1, C_2, C_3) \tag{2.2}$$

with C_1, C_2 and C_3 being defined by

$$\begin{cases} C_1 = A_1 B_1, \\ C_2 = A_1 B_2 + A_2 B_1 + \alpha A_2 B_2, \\ C_3 = A_1 B_3 + A_3 B_1 + A_2 B_2 + \alpha A_2 B_3 + \alpha A_3 B_2. \end{cases} \tag{2.3}$$

This closure property under matrix multiplication guarantees that all the above block matrices defined by (2.1) form a matrix Lie algebra under the matrix commutator

$$[M_1, M_2] = M_1M_2 - M_2M_1. \tag{2.4}$$

The corresponding loop matrix algebra consists of all block matrices of the form

$$M(A_1, A_2, A_3)f(\lambda)$$

with f being smooth, and its Lie product is defined by

$$[M_1f_1(\lambda), M_2f_2(\lambda)] = [M_1, M_2]f_1(\lambda)f_2(\lambda). \tag{2.5}$$

Let us denote this matrix loop algebra by $\overline{g}$. It is non-semisimple, due to the following semi-direct sum decomposition:

$$\overline{g} = g \Subset g_c, \quad g = \{M(A_1, A_1, A_1)f(\lambda)\}, \quad g_c = \{M(0, A_2, A_3)f(\lambda)\}, \tag{2.6}$$

where g and g_c are Lie subalgebras of $\overline{g}$, and g is a non-trivial ideal Lie subalgebra of $\overline{g}$.

This kind of matrix loop algebras establishes a basis for constructing nonlinear Hamiltonian bi-integrable couplings through zero curvature equations. The first block A_1 gives the original integrable equation as required, and the second and third blocks A_2 and A_3 yield the supplementary vector fields S_1 and S_2 in (1.8) that we are looking for. More interestingly, the commutators $[A_2, B_2]$, $[A_2, B_3]$ and $[A_3, B_2]$ often generate nonlinear terms in the resulting bi-integrable couplings.

2.2 Constructing bi-integrable couplings

Let us assume that an integrable equation

$$u_t = K(u) \tag{2.7}$$

possesses a zero curvature representation

$$U_t - V_x + [U, V] = 0, \tag{2.8}$$

where two square matrices

$$U = U(u, \lambda) \quad \text{and} \quad V = V(u, \lambda)$$

usually belong to a semisimple matrix loop algebra (see, e.g., [3]) and constitute a Lax pair (see [6]). Let us then take an enlarged spectral matrix $\overline{U}$ in $\overline{g}$ as

$$\overline{U} = \overline{U}(\overline{u}, \lambda) = \begin{bmatrix} U(u,\lambda) & U_1(u_1,\lambda) & U_2(u_2,\lambda) \\ 0 & U(u,\lambda) + \alpha U_1(u_1,\lambda) & U_1(u_1,\lambda) + \alpha U_2(u_1,u_2,\lambda) \\ 0 & 0 & U(u,\lambda) + \alpha U_1(u_1,\lambda) \end{bmatrix}, \tag{2.9}$$

where $\overline{u}$ consists of three dependent variables u, u_1 and u_2. Then an enlarged zero curvature equation

$$\overline{U}_t - \overline{V}_x + [\overline{U}, \overline{V}] = 0 \tag{2.10}$$

with an enlarged Lax matrix $\overline{V}$ in $\overline{g}$:

$$\overline{V} = \overline{V}(\overline{u}, \lambda) = \begin{bmatrix} V(u,\lambda) & V_1(u,u_1,\lambda) & V_2(u,u_1,u_2,\lambda) \\ 0 & V(u,\lambda) + \alpha V_1(u,u_1,\lambda) & V_1(u,u_1,\lambda) + \alpha V_2(u,u_1,u_2,\lambda) \\ 0 & 0 & V(u,\lambda) + \alpha V_1(u,u_1,\lambda) \end{bmatrix}, \tag{2.11}$$

yields the following triangular system:

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{1,t} - V_{1,x} + [U, V_1] + [U_1, V] + \alpha[U_1, V_1] = 0, \\ U_{2,t} - V_{2,x} + [U, V_2] + [U_2, V] + [U_1, V_1] + \alpha[U_1, V_2] + \alpha[U_2, V_1] = 0. \end{cases} \tag{2.12}$$

Note that the zero curvature representation (2.8) of the evolution equation (2.7) presents a bi-integrable coupling of the equation (2.7). It is, usually, nonlinear with respect to the two supplementary variables u_1 and u_2, thereby providing candidates for nonlinear bi-integrable couplings.

To generate infinitely many symmetries, we search for a solution $\overline{W}$ in $\overline{g}$:

$$\overline{W} = \overline{W}(\overline{u}, \lambda) = \begin{bmatrix} W(u,\lambda) & W_1(u,u_1,\lambda) & W_2(u,u_1,u_2,\lambda) \\ 0 & W(u,\lambda) + \alpha W_1(u,u_1,\lambda) & W_1(u,u_1,\lambda) + \alpha W_2(u,u_1,\lambda) \\ 0 & 0 & W(u,\lambda) + \alpha W_1(u,u_1,\lambda) \end{bmatrix} \tag{2.13}$$

to the enlarged stationary zero curvature equation

$$\overline{W}_x = [\overline{U}, \overline{W}]. \tag{2.14}$$

This equation equivalently engenders

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W] + \alpha[U_1, W_1], \\ W_{2,x} = [U, W_2] + [U_2, W] + [U_1, W_1] + \alpha[U_1, W_2] + \alpha[U_2, W_1]. \end{cases} \tag{2.15}$$

We can often (see, e.g., [33, 8]) have a solution of the type

$$W = \sum_{i=0}^{\infty} W_i \lambda^{-i}, \quad W_1 = \sum_{i=0}^{\infty} W_{1,i} \lambda^{-i}, \quad W_2 = \sum_{i=0}^{\infty} W_{2,i} \lambda^{-i}. \tag{2.16}$$

Then, we introduce a set of enlarged matrix modifications $\overline{\Delta}_m$, $m \geq 0$, and define the enlarged Lax matrices to be

$$\overline{V}^{[m]} = (\lambda^m \overline{W})_+ + \overline{\Delta}_m, \quad m \geq 0, \tag{2.17}$$

where the subscript "+" denotes the polynomial part, such that the enlarged zero curvature equations

$$\overline{U}_{t_m} - \overline{V}^{[m]}_x + [\overline{U}, \overline{V}^{[m]}] = 0, \quad m \geq 0 \tag{2.18}$$

generate a soliton hierarchy of nonlinear bi-integrable couplings for the equation (2.7). In general, integrable couplings in this hierarchy commute with each other, and so, they provide infinitely many common symmetries for the hierarchy.

The next crucial step is to find a class of bilinear forms over the underlying loop algebra, which should satisfy the non-degenerate property, the symmetric property and the ad-invariant property. Then use the corresponding variational identity (see [20]):

$$\frac{\delta}{\delta \overline{u}} \int \langle \overline{W}, \overline{U}_\lambda \rangle \mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle \overline{W}, \overline{U}_{\overline{u}} \rangle, \quad \gamma = \text{const.} \tag{2.19}$$

to furnish Hamiltonian structures for the resulting bi-integrable couplings. In the above variational identity (2.19), $\langle \cdot, \cdot \rangle$ is the required non-degenerate, symmetric and ad-invariant bilinear form over the underlying loop algebra consisting of square matrices of the form (2.9) (see [20, 27, 17] for details). Hamiltonian structures link symmetries and conservation laws together.

In the next section, to show an illustrative example, we apply the above general computational paradigm to the AKNS spectral problem and generate nonlinear Hamiltonian bi-integrable couplings for the AKNS equations.

3 An Application to the AKNS Spectral Problem

3.1 AKNS hierarchy

The spectral matrix

$$U = U(u, \lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \lambda\text{-spectral parameter} \tag{3.1}$$

generates the AKNS hierarchy of soliton equations (see [1, 39]). There are other integrable equations associated with gl(2) (see, e.g., [41]). Upon setting

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i \geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{3.2}$$

the stationary zero curvature equation

$$W_x = [U, W]$$

yields

$$b_{i+1} = -\frac{1}{2} b_{i,x} - p a_i, \quad c_{i+1} = \frac{1}{2} c_{i,x} - q a_i, \quad a_{i+1,x} = p c_{i+1} - q b_{i+1}, \quad i \geq 0. \tag{3.3}$$

Taking the initial data as

$$a_0 = -1, \quad b_0 = c_0 = 0, \tag{3.4}$$

and assuming

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1$$

(equivalently selecting constants of integration to be zero), the recursion relation (3.3) uniquely defines all differential polynomial functions a_i, b_i and c_i, $i \geq 1$. The first few sets are listed as follows:

$$\begin{aligned}
&b_1 = p, \quad c_1 = q, \quad a_1 = 0;\\
&b_2 = -\frac{1}{2}p_x, \quad c_2 = \frac{1}{2}q_x, \quad a_2 = \frac{1}{2}pq;\\
&b_3 = \frac{1}{4}p_{xx} - \frac{1}{2}p^2 q, \quad c_3 = \frac{1}{4}q_{xx} - \frac{1}{2}pq^2, \quad a_3 = \frac{1}{4}(pq_x - p_x q);\\
&b_4 = -\frac{1}{8}p_{xxx} + \frac{3}{4}p_x pq, \quad c_4 = \frac{1}{8}q_{xxx} - \frac{3}{4}pq_x q;\\
&a_4 = \frac{1}{8}p_{xx}q - \frac{1}{8}p_x q_x + \frac{1}{8}pq_{xx} - \frac{3}{8}p^2 q^2.
\end{aligned}$$

Note that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad V^{[m]} = (\lambda^m W)_+, \quad m \geq 0 \tag{3.5}$$

generate the AKNS hierarchy of soliton equations

$$u_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ 2q \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0. \tag{3.6}$$

The Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functions in (3.6) are given by

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \quad \mathcal{H}_m = \int \frac{2a_{m+2}}{m+1}\,\mathrm{d}x, \tag{3.7}$$

where $\partial = \frac{\partial}{\partial x}$ and $m \geq 0$.

3.2 Bi-integrable couplings

We begin with an enlarged spectral matrix

$$\overline{U} = \overline{U}(\overline{u}, \lambda) = \begin{bmatrix} U & U_1 & U_2 \\ 0 & U + \alpha U_1 & U_1 + \alpha U_2 \\ 0 & 0 & U + \alpha U_1 \end{bmatrix}, \quad \overline{u} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}, \tag{3.8}$$

where U is defined as in (3.1) and the supplementary spectral matrices U_1 and U_2 read

$$U_1 = U_1(u_1) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} r \\ s \end{bmatrix},$$
$$U_2 = U_2(u_2) = \begin{bmatrix} 0 & v \\ w & 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} v \\ w \end{bmatrix}.$$

As before, to solve the enlarged stationary zero curvature equation (2.14), we take a solution of the following type:

$$\overline{W} = \overline{W}(\overline{u}, \lambda) = \begin{bmatrix} W & W_1 & W_2 \\ 0 & W + \alpha W_1 & W_1 + \alpha W_2 \\ 0 & 0 & W + \alpha W_1 \end{bmatrix}, \tag{3.9}$$

where W, defined by (3.2), solves $W_x = [U, W]$, and

$$W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix},$$
$$W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix}.$$

Then, the second and third equations in (2.15) equivalently generate

$$\begin{cases} e_x = pg - qf + \alpha rg - \alpha sf + rc - sb, \\ f_x = -2\lambda f - 2pe - 2\alpha re - 2ra, \\ g_x = 2qe + 2\lambda g + 2\alpha se + 2sa \end{cases}$$

and

$$\begin{cases} e'_x = (p + \alpha r)g' - (q + \alpha s)f' + (r + \alpha v)g - (s + \alpha w)f + vc - wb, \\ f'_x = -2\lambda f' - 2(p + \alpha r)e' - 2(r + \alpha v)e - 2va, \\ g'_x = 2(q + \alpha s)e' + 2\lambda g' + 2(s + \alpha w)e + 2wa, \end{cases}$$

respectively. Trying a formal series solution $\overline{W}$ by assuming

$$e = \sum_{i=0}^{\infty} e_i \lambda^{-i}, \quad f = \sum_{i=0}^{\infty} f_i \lambda^{-i}, \quad g = \sum_{i=0}^{\infty} g_i \lambda^{-i},$$
$$e' = \sum_{i=0}^{\infty} e'_i \lambda^{-i}, \quad f' = \sum_{i=0}^{\infty} f'_i \lambda^{-i}, \quad g' = \sum_{i=0}^{\infty} g'_i \lambda^{-i},$$

we arrive at

$$\begin{cases} f_{i+1} = -\frac{1}{2}f_{i,x} - pe_i - \alpha re_i - ra_i, \\ g_{i+1} = \frac{1}{2}g_{i,x} - qe_i - \alpha se_i - wa_i, \\ e_{i+1,x} = pg_{i+1} - qf_{i+1} + \alpha rg_{i+1} - \alpha sf_{i+1} + rc_{i+1} - sb_{i+1}, \\ f'_{i+1} = -\frac{1}{2}f'_{i,x} - (p+\alpha r)e'_i - (r+\alpha v)e_i - va_i, \\ g'_{i+1} = \frac{1}{2}g'_{i,x} - (q+\alpha s)e'_i - (s+\alpha w)e_i - wa_i, \\ e'_{i+1,x} = (p+\alpha r)g'_i - (q+\alpha s)f'_i + (r+\alpha v)g_i - (s+\alpha w)f_i + vc_i - wb_i, \end{cases} \tag{3.10}$$

where $i \geq 0$. We select the initial data to be

$$e_0 = -1, \quad f_0 = g_0 = 0; \quad e'_0 = -1, \quad f'_0 = g'_0 = 0, \tag{3.11}$$

and assume that

$$e_i|_{\overline{u}=0} = f_i|_{\overline{u}=0} = g_i|_{\overline{u}=0} = 0, \quad e'_i|_{\overline{u}=0} = f'_i|_{\overline{u}=0} = g'_i|_{\overline{u}=0} = 0, \quad i \geq 1.$$

Then the recursion relation (3.10) uniquely determines the sequence of e_i, f_i, g_i and e'_i, f'_i, g'_i, $i \geq 1$, recursively. It is now direct to compute the first few sets of functions:

$$\begin{aligned}
f_1 &= p + r + \alpha r, \\
g_1 &= q + s + \alpha s, \\
e_1 &= 0; \\
f_2 &= -\frac{1}{2}p_x - \frac{1}{2}\alpha r_x - \frac{1}{2}r_x, \\
g_2 &= \frac{1}{2}q_x + \frac{1}{2}\alpha s_x + \frac{1}{2}s_x, \\
e_2 &= \frac{1}{2}pq + \frac{1}{2}ps + \frac{1}{2}qr + \frac{\alpha}{2}ps + \frac{\alpha}{2}qr + \frac{\alpha+\alpha^2}{2}rs; \\
f_3 &= \frac{1}{4}p_{xx} + \frac{\alpha+1}{4}r_{xx} - \frac{1}{2}p^2q - \frac{\alpha+1}{2}p^2s - (\alpha+1)p(q+\alpha s)r \\
&\quad - \frac{\alpha(\alpha+1)}{2}qr^2 - \frac{\alpha^2(\alpha+1)}{2}r^2s, \\
g_3 &= \frac{1}{4}q_{xx} + \frac{\alpha+1}{4}s_{xx} - \frac{1}{2}q^2p - \frac{\alpha+1}{2}q^2r - (\alpha+1)(p+\alpha r)qs \\
&\quad - \frac{\alpha(\alpha+1)}{2}ps^2 - \frac{\alpha^2(\alpha+1)}{2}rs^2, \\
e_3 &= -\frac{1}{4}p_xq + \frac{1}{4}pq_x - \frac{\alpha+1}{4}p_xs + \frac{\alpha+1}{4}ps_x + \frac{\alpha+1}{4}q_xr - \frac{\alpha+1}{4}qr_x \\
&\quad - \frac{\alpha(\alpha+1)}{4}r_xs + \frac{\alpha(\alpha+1)}{4}rs_x; \\
f'_1 &= p + (\alpha+1)r + (\alpha+1)v, \\
g'_1 &= q + (\alpha+1)s + (\alpha+1)w,
\end{aligned}$$

$$
\begin{aligned}
e_1' &= 0; \\
f_2' &= -\frac{1}{2}p_x - \frac{\alpha+1}{2}r_x - \frac{\alpha+1}{2}v_x, \\
g_2' &= \frac{1}{2}q_x + \frac{\alpha+1}{2}s_x + \frac{\alpha+1}{2}w_x, \\
e_2' &= \frac{1}{2}pq + \frac{\alpha+1}{2}ps + \frac{\alpha+1}{2}pw + \frac{\alpha+1}{2}qr + \frac{\alpha+1}{2}qv + \frac{(\alpha+1)^2}{2}rs \\
&\quad + \frac{\alpha(\alpha+1)}{2}rw + \frac{\alpha(\alpha+1)}{2}sv; \\
f_3' &= \frac{1}{4}p_{xx} + \frac{\alpha+1}{4}r_{xx} + \frac{\alpha+1}{4}v_{xx} - \frac{1}{2}p^2q - \frac{\alpha+1}{2}p^2s - \frac{\alpha+1}{2}p^2w \\
&\quad - (\alpha+1)pqr - (\alpha+1)pqv - (\alpha+1)^2prs - \frac{(\alpha+1)^2}{2}qr^2 \\
&\quad - \frac{\alpha(\alpha+1)(\alpha+2)}{2}r^2s - \alpha(\alpha+1)prw - \alpha(\alpha+1)psv \\
&\quad - \alpha(\alpha+1)qrv - \frac{\alpha^2(\alpha+1)}{2}r^2w - \alpha^2(\alpha+1)rsv, \\
g_3' &= \frac{1}{4}q_{xx} + \frac{\alpha+1}{4}s_{xx} + \frac{\alpha+1}{4}w_{xx} - \frac{\alpha+1}{2}pq^2 - \frac{\alpha+1}{2}q^2r - \frac{\alpha+1}{2}q^2v \\
&\quad - (\alpha+1)pqs - (\alpha+1)pqw - (\alpha+1)^2qrs - \frac{(\alpha+1)^2}{2}ps^2 \\
&\quad - \frac{\alpha(\alpha+1)(\alpha+2)}{2}rs^2 - \alpha(\alpha+1)psv - \alpha(\alpha+1)prw \\
&\quad - \alpha(\alpha+1)qsw - \frac{\alpha^2(\alpha+1)}{2}s^2v - \alpha^2(\alpha+1)rsw, \\
e_3' &= -\frac{1}{4}p_xq + \frac{1}{4}pq_x - \frac{\alpha+1}{4}p_xs + \frac{\alpha+1}{4}ps_x - \frac{\alpha+1}{4}p_xw + \frac{\alpha+1}{4}pw_x \\
&\quad + \frac{\alpha+1}{4}q_xr - \frac{\alpha+1}{4}qr_x + \frac{\alpha+1}{4}q_xv - \frac{\alpha+1}{4}qv_x - \frac{(\alpha+1)^2}{4}r_xs \\
&\quad + \frac{(\alpha+1)^2}{4}rs_x - \frac{\alpha(\alpha+1)}{4}r_xw + \frac{\alpha(\alpha+1)^2}{4}rw_x \\
&\quad + \frac{\alpha(\alpha+1)}{4}s_xv - \frac{\alpha(\alpha+1)^2}{4}sv_x.
\end{aligned}
$$

We point out that those functions are all differential polynomials in six variables p, q, r, s, v, w.

For each integer $m \geq 0$, let us further define an enlarged Lax matrix by

$$
\overline{V}^{[m]} = (\lambda^m \overline{W})_+ = \begin{bmatrix} V^{[m]} & V_1^{[m]} & V_2^{[m]} \\ 0 & V^{[m]} + \alpha V_1^{[m]} & V_1^{[m]} + \alpha V_2^{[m]} \\ 0 & 0 & V^{[m]} + \alpha V_1^{[m]} \end{bmatrix}, \tag{3.12}
$$

where

$$
V_i^{[m]} = (\lambda^m W_i)_+, \quad i = 1, 2.
$$

Then the enlarged zero curvature equation

$$
\overline{U}_{t_m} - (\overline{V}^{[m]})_x + [\overline{U}, \overline{V}^{[m]}] = 0 \tag{3.13}
$$

generates

$$\begin{cases} U_{1,t_m} - V_{1,x}^{[m]} + [U, V_1^{[m]}] + [U_1, V^{[m]}] + \alpha[U_1, V_1^{[m]}] = 0, \\ U_{2,t_m} - V_{2,x}^{[m]} + [U, V_2^{[m]}] + [U_2, V^{[m]}] + [U_1, V_1^{[m]}] + \alpha[U_1, V_2^{[m]}] + \alpha[U_2, V_1^{[m]}] = 0, \end{cases}$$

together with the m-th AKNS system in (3.6). This presents the supplementary systems

$$\overline{v}_{t_m} = S_m = S_m(\overline{u}) = \begin{bmatrix} S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix}, \quad \overline{v} = (r, s, v, w)^{\mathrm{T}}, \quad m \geq 0, \tag{3.14}$$

where

$$S_{1,m}(u, u_1) = \begin{bmatrix} -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \quad S_{2,m}(u, u_1, u_2) = \begin{bmatrix} -2f'_{m+1} \\ 2g'_{m+1} \end{bmatrix}.$$

In this way, the hierarchy of enlarged zero curvature equations gives a hierarchy of bi-integrable couplings

$$\overline{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}_{t_m} = \overline{K}_m(\overline{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \\ -2f'_{m+1} \\ 2g'_{m+1} \end{bmatrix}, \quad m \geq 0 \tag{3.15}$$

for the AKNS hierarchy (3.6).

Except the first two, all bi-integrable couplings presented above are nonlinear, since the supplementary systems (3.14) with $m \geq 2$ are nonlinear with respect to the four dependent variables r, s, v, w. This implies that (3.15) provides a hierarchy of nonlinear bi-integrable couplings for the AKNS hierarchy of soliton equations. The first nonlinear bi-integrable coupling system is given by

$$\begin{cases} p_{t_2} = -2b_3, \quad q_{t_2} = 2c_3, \\ r_{t_2} = -2f_3, \quad s_{t_2} = 2g_3, \\ v_{t_2} = -2f'_3, \quad w_{t_2} = 2g'_3, \end{cases} \tag{3.16}$$

where $b_3, c_3, f_3, g_3, f'_3, g'_3$ are defined before.

3.3 Hamiltonian structures

To furnish Hamiltonian structures of the obtained bi-integrable couplings, we need to compute non-degenerate, symmetric and ad-invariant bilinear forms on the adopted matrix loop algebra:

$$\overline{g} = \left\{ \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & A_2 + \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 \end{bmatrix} f(\lambda) \,\middle|\, A_i \in \mathrm{sl}(2),\ 1 \leq i \leq 3,\ f \in C^\infty(S^1) \right\}. \tag{3.17}$$

For convenience, the general algorithm (see [18]) suggests us to transform the Lie algebra $\overline{g}$ into a vector form through the mapping

$$\delta : \overline{g} \to \mathbb{R}^9, \quad A \mapsto (a_1, a_2, \cdots, a_9)^{\mathrm{T}}, \tag{3.18}$$

where

$$A = A(a_1, a_2, \cdots, a_9) = \begin{bmatrix} a_1 & a_2 & a_4 & a_5 & a_7 & a_8 \\ a_3 & -a_1 & a_6 & -a_4 & a_9 & -a_7 \\ 0 & 0 & a_1 + \alpha a_4 & a_2 + \alpha a_5 & a_4 + \alpha a_7 & a_5 + \alpha a_8 \\ 0 & 0 & a_3 + \alpha a_6 & -a_1 - \alpha a_4 & a_6 + \alpha a_9 & -a_4 - \alpha a_7 \\ 0 & 0 & 0 & 0 & a_1 + \alpha a_4 & a_2 + \alpha a_5 \\ 0 & 0 & 0 & 0 & a_3 + \alpha a_6 & -a_1 - \alpha a_4 \end{bmatrix}. \quad (3.19)$$

This mapping δ induces a Lie algebraic structure on $\mathbb{R}^9$, isomorphic to the above matrix Lie algebra $\overline{g}$. The corresponding commutator $[\,\cdot\,, \cdot\,]$ on the resulting Lie algebra $\mathbb{R}^9$ reads

$$[a, b]^{\mathrm{T}} = a^{\mathrm{T}} R(b), \quad a = (a_1, a_2, \cdots, a_9)^{\mathrm{T}}, \ \ b = (b_1, b_2, \cdots, b_9)^{\mathrm{T}} \in \mathbb{R}^9, \qquad (3.20)$$

where $R(b)$ is given by

$$R(b) = \begin{bmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 & 0 & 2b_8 & -2b_9 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 & b_9 & -2b_7 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 & -b_8 & 0 & 2b_7 \\ 0 & 0 & 0 & 0 & 2b_2 + 2\alpha b_5 & -2b_3 - 2\alpha b_6 & 0 & 2b_5 + 2\alpha b_8 & -2b_6 - 2\alpha b_9 \\ 0 & 0 & 0 & b_3 + \alpha b_6 & -2b_1 - 2\alpha b_4 & 0 & b_6 + \alpha b_9 & -2b_4 - 2\alpha b_7 & 0 \\ 0 & 0 & 0 & -b_2 - \alpha b_5 & 0 & 2b_1 + 2\alpha b_4 & -b_5 - \alpha b_8 & 0 & 2b_4 + 2\alpha b_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2b_2 + 2\alpha b_5 & -2b_3 - 2\alpha b_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_3 + \alpha b_6 & -2b_1 - 2\alpha b_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_2 - \alpha b_5 & 0 & 2b_1 + 2\alpha b_4 \end{bmatrix}.$$

An arbitrary bilinear form on $\mathbb{R}^9$ takes the form

$$\langle a, b \rangle = a^{\mathrm{T}} F b, \qquad (3.21)$$

where F is a constant matrix. Two of the three required properties, the symmetric property

$$\langle a, b \rangle = \langle b, a \rangle$$

and the ad-invariance property

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle, \qquad (3.22)$$

mean that

$$F^{\mathrm{T}} = F$$

and

$$(R(b)F)^{\mathrm{T}} = -R(b)F \quad \text{for all} \ \ b \in \mathbb{R}^9.$$

This matrix equation on F yields a system of linear equations on the elements of F. Solving the resulting linear system gives

$$F = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 & 0 & 0 & 2\eta_3 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\eta_1 & 0 & 0 & \frac{1}{2}\eta_2 & 0 & 0 & \eta_3 \\ 0 & \frac{1}{2}\eta_1 & 0 & 0 & \frac{1}{2}\eta_2 & 0 & 0 & \eta_3 & 0 \\ \eta_2 & 0 & 0 & \alpha\eta_2 + 2\eta_3 & 0 & 0 & 2\alpha\eta_3 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\eta_2 & 0 & 0 & \frac{1}{2}\alpha\eta_2 + \eta_3 & 0 & 0 & \alpha\eta_3 \\ 0 & \frac{1}{2}\eta_2 & 0 & 0 & \frac{1}{2}\alpha\eta_2 + \eta_3 & 0 & 0 & \alpha\eta_3 & 0 \\ 2\eta_3 & 0 & 0 & 2\alpha\eta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_3 & 0 & 0 & \alpha\eta_3 & 0 & 0 & 0 \\ 0 & \eta_3 & 0 & 0 & \alpha\eta_3 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (3.23)$$

where η_1, η_2 and η_3 are arbitrary constants.

Therefore, a required bilinear form on the underlying Lie algebra $\overline{g}$ is determined by

$$\begin{aligned}\langle A,B\rangle_{\overline{g}} &= \langle \delta^{-1}(A),\delta^{-1}(B)\rangle_{\mathbb{R}^9}\\ &= (a_1,a_2,\cdots,a_9)F(b_1,b_2,\cdots,b_9)^{\mathrm{T}}\\ &= \eta_1\Big(a_1b_1+\frac{1}{2}a_2b_3+\frac{1}{2}a_3b_2\Big)\\ &\quad+\eta_2\Big[a_1b_4+\frac{1}{2}a_2b_6+\frac{1}{2}a_3b_5+a_4(b_1+\alpha b_4)+\frac{1}{2}a_5(b_3+\alpha b_6)+\frac{1}{2}a_6(b_2+\alpha b_5)\Big]\\ &\quad+\eta_3[2a_1b_7+a_2b_9+a_3b_8+2a_4(b_4+\alpha b_7)+a_5(b_6+\alpha b_9)+a_6(b_5+\alpha b_8)\\ &\quad+2a_7(b_1+\alpha b_4)+a_8(b_3+\alpha b_6)+a_9(b_2+\alpha b_5)], \end{aligned}\tag{3.24}$$

where $A=A(a_1,a_2,\cdots,a_9)$ and $B=B(b_1,b_2,\cdots,b_9)$ are two block matrices of the form defined by (3.19). This bilinear form (3.24) is symmetric and ad-invariant:

$$\langle A,B\rangle=\langle B,A\rangle,\quad \langle A,[B,C]\rangle=\langle [A,B],C\rangle,\quad A,B,C\in\overline{g},$$

and it is non-degenerate if and only if

$$\det(F)=(\alpha^2\eta_1-\alpha\eta_2+2\eta_3)^3\eta_3^6\neq 0.\tag{3.25}$$

To apply the variational identity, let us further compute that

$$\begin{aligned}\langle \overline{W},\overline{U}_\lambda\rangle &= -\eta_1 a-\eta_2 e-2\eta_3 e',\\ \langle \overline{W},\overline{U}_{\overline{u}}\rangle &= \Big(\frac{1}{2}\eta_1 c+\frac{1}{2}\eta_2 g+\eta_3 g',\frac{1}{2}\eta_1 b+\frac{1}{2}\eta_2 f+\eta_3 f',\frac{1}{2}\eta_2(c+\alpha g)+\eta_3(g+\alpha g'),\\ &\quad \frac{1}{2}\eta_2(b+\alpha f)+\eta_3(f+\alpha f'),\eta_3(c+\alpha g),\eta_3(b+\alpha f)\Big)^{\mathrm{T}},\\ \gamma &= -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle \overline{W},\overline{W}\rangle|=0,\end{aligned}$$

where $\overline{W}$ is given by (3.9). Therefore, the corresponding variational identity (2.19) leads to

$$\begin{aligned}&\frac{\delta}{\delta\overline{u}}\int\frac{\eta_1 a_{m+1}+\eta_2 e_{m+1}+2\eta_3 e'_{m+1}}{m}\,\mathrm{d}x\\ =&\Big(\frac{1}{2}\eta_1 c_m+\frac{1}{2}\eta_2 g_m+\eta_3 g'_m,\frac{1}{2}\eta_1 b_m+\frac{1}{2}\eta_2 f_m+\eta_3 f'_m,\frac{1}{2}\eta_2(c_m+\alpha g_m)+\eta_3(g_m+\alpha g'_m),\\ &\frac{1}{2}\eta_2(b_m+\alpha f_m)+\eta_3(f_m+\alpha f'_m),\eta_3(c_m+\alpha g_m),\eta_3(b_m+\alpha f_m)\Big)^{\mathrm{T}},\quad m\geq 1.\end{aligned}$$

It follows thus that the AKNS bi-integrable couplings in (3.15) possess the following Hamiltonian structures:

$$\overline{u}_{t_m}=\overline{K}_m(\overline{u})=\overline{J}\frac{\delta\overline{\mathcal{H}}_m}{\delta\overline{u}},\quad m\geq 0,\tag{3.26}$$

where the Hamiltonian operator is

$$\overline{J}=\begin{bmatrix} 0 & -\frac{4\alpha^2}{\beta} & 0 & \frac{4\alpha}{\beta} & 0 & -\frac{4}{\beta} \\ \frac{4\alpha^2}{\beta} & 0 & -\frac{4\alpha}{\beta} & 0 & \frac{4}{\beta} & 0 \\ 0 & \frac{4\alpha}{\beta} & 0 & -\frac{4}{\beta} & 0 & \frac{2(\eta_2-\alpha\eta_1)}{\beta\eta_3} \\ -\frac{4\alpha}{\beta} & 0 & \frac{4}{\beta} & 0 & -\frac{2(\eta_2-\alpha\eta_1)}{\beta\eta_3} & 0 \\ 0 & -\frac{4}{\beta} & 0 & \frac{2(\eta_2-\alpha\eta_1)}{\beta\eta_3} & 0 & \frac{\alpha\eta_1\eta_2+2\eta_1\eta_3-\eta_2^2}{\beta\eta_3^2} \\ \frac{4}{\beta} & 0 & -\frac{2(\eta_2-\alpha\eta_1)}{\beta\eta_3} & 0 & -\frac{\alpha\eta_1\eta_2+2\eta_1\eta_3-\eta_2^2}{\beta\eta_3^2} & 0 \end{bmatrix} \quad (3.27)$$

with β being defined by $\beta=\alpha^2\eta_1-\alpha\eta_2+2\eta_3$, and the Hamiltonian functionals read

$$\overline{\mathcal{H}}_m=\int\frac{\eta_1 a_{m+2}+\eta_2 e_{m+2}+2\eta_3 e'_{m+2}}{m+1}\,\mathrm{d}x,\quad m\geq 0. \quad (3.28)$$

3.4 Commutativity of symmetries and conserved functionals

Based on (3.3) and (3.10), a direct computation shows a recursion relation

$$\overline{K}_{m+1}=\overline{\Phi}\,\overline{K}_m,\quad m\geq 1, \quad (3.29)$$

where the recursion operator $\overline{\Phi}$ (see [30] for definition of recursion operators) is given by

$$\overline{\Phi}=\begin{bmatrix}\Phi & 0 & 0\\ \Phi_2 & \Phi_1 & 0\\ \Phi_5 & \Phi_4 & \Phi_3\end{bmatrix} \quad (3.30)$$

with Φ being given by (3.7) and

$$\Phi_1=\begin{bmatrix}-\frac{1}{2}\partial+(p+\alpha r)\partial^{-1}(q+\alpha s) & (p+\alpha r)\partial^{-1}(p+\alpha r)\\ -(q+\alpha s)\partial^{-1}(q+\alpha s) & \frac{1}{2}\partial-(q+\alpha s)\partial^{-1}(p+\alpha r)\end{bmatrix},$$

$$\Phi_2=\begin{bmatrix} r\partial^{-1}q+(p+\alpha r)\partial^{-1}s & r\partial^{-1}p+(p+\alpha r)\partial^{-1}r\\ -s\partial^{-1}q-(q+\alpha s)\partial^{-1}s & -s\partial^{-1}p-(q+\alpha s)\partial^{-1}r\end{bmatrix},$$

$$\Phi_3=\begin{bmatrix}-\frac{1}{2}\partial+(p+\alpha r)\partial^{-1}(q+\alpha s) & (p+\alpha r)\partial^{-1}(p+\alpha r)\\ -(q+\alpha s)\partial^{-1}(q+\alpha s) & \frac{1}{2}\partial-(q+\alpha s)\partial^{-1}(p+\alpha r)\end{bmatrix},$$

$$\Phi_4=\begin{bmatrix}(r+\alpha v)\partial^{-1}(q+\alpha s)+(p+\alpha r)\partial^{-1}(s+\alpha w) & (r+\alpha v)\partial^{-1}(p+\alpha r)+(p+\alpha r)\partial^{-1}(r+\alpha v)\\ -(s+\alpha w)\partial^{-1}(q+\alpha s)-(q+\alpha s)\partial^{-1}(s+\alpha w) & -(s+\alpha w)\partial^{-1}(p+\alpha r)-(q+\alpha s)\partial^{-1}(r+\alpha v)\end{bmatrix},$$

$$\Phi_5=\begin{bmatrix} v\partial^{-1}q+(r+\alpha v)\partial^{-1}s+(p+\alpha r)\partial^{-1}w & v\partial^{-1}p+(r+\alpha v)\partial^{-1}r+(p+\alpha r)\partial^{-1}v\\ -w\partial^{-1}q-(s+\alpha w)\partial^{-1}s-(q+\alpha s)\partial^{-1}w & -w\partial^{-1}p-(s+\alpha w)\partial^{-1}r-(q+\alpha s)\partial^{-1}r\end{bmatrix}.$$

Furthermore, we can show

$$\overline{J}\,\overline{\Phi}^{\dagger}=\overline{\Phi}\,\overline{J},$$

where $\overline{\Phi}^{\dagger}$ denotes the adjoint operator of $\overline{\Phi}$. This tells that all bi-integrable couplings in (3.15) commute with each other and so do all conserved functionals in (3.28). It is also direct to verify that $\overline{J}$ and $\overline{M} = \overline{\Phi}\,\overline{J}$ constitute a Hamiltonian pair (see [29, 5]), and so, $\overline{\Phi}$ is a hereditary recursion operator (see [4]) for the hierarchy of Hamiltonian bi-integrable couplings (3.15). It now follows that there exist infinitely many commuting symmetries and conserved functionals:

$$[\overline{K}_m, \overline{K}_n] := \overline{K}'_m(\overline{u})[\overline{K}_n] - \overline{K}'_n(\overline{u})[\overline{K}_m] = 0, \quad m, n \geq 0, \tag{3.31}$$

$$\{\overline{\mathcal{H}}_m, \overline{\mathcal{H}}_n\}_{\overline{J}} := \int \left(\frac{\delta \overline{\mathcal{H}}_m}{\delta \overline{u}}\right)^{\mathrm{T}} \overline{J} \frac{\delta \overline{\mathcal{H}}_n}{\delta \overline{u}} \mathrm{d}x = 0, \qquad m, n \geq 0, \tag{3.32}$$

and that the resulting bi-integrable couplings possess a bi-Hamiltonian structure

$$\overline{u}_{t_m} = \overline{K}_m = \overline{J} \frac{\delta \overline{\mathcal{H}}_m}{\delta \overline{u}} = \overline{M} \frac{\delta \overline{\mathcal{H}}_{m-1}}{\delta \overline{u}}, \quad m \geq 1. \tag{3.33}$$

In particular, the bi-integrable coupling (3.16) has the following bi-Hamiltonian structure:

$$\overline{u}_{t_2} = \overline{K}_2 = \overline{J} \frac{\delta \overline{\mathcal{H}}_2}{\delta \overline{u}} = \overline{M} \frac{\delta \overline{\mathcal{H}}_1}{\delta \overline{u}}, \tag{3.34}$$

where

$$\overline{\mathcal{H}}_1 = \int \frac{1}{2}(\eta_1 a_3 + \eta_2 e_3 + 2\eta_3 e'_3)\mathrm{d}x, \quad \overline{\mathcal{H}}_2 = \int \frac{1}{3}(\eta_1 a_4 + \eta_2 e_4 + 2\eta_3 e'_4)\mathrm{d}x. \tag{3.35}$$

4 Concluding Remarks

We have introduced a kind of matrix loop algebras which provide an architecturally rich context for understanding and constructing integrable couplings. The variational identities on the adopted matrix loop algebras were used to furnish Hamiltonian structures of the resulting bi-integrable couplings. An application to the AKNS spectral problem presented a hierarchy of nonlinear bi-integrable Hamiltonian couplings for the AKNS soliton hierarchy. Our results complement well some of the previous ideas of generating linear and nonlinear integrable couplings (see [9, 11, 19–20]), and help us gain fresh insights into rich structures that integrable couplings possess.

We remark that matrix loop algebras of high order block type will allow us to generate multi-integrable couplings and more diverse integrable couplings, and provide supplements to the spectral matrices of the other forms in the literature (see, e.g., [13, 22]). Moreover, enlarged Lax pairs of direct-sum type always hold for degenerate coupled systems. For instance, we can specify an enlarged spectral matrix $\widehat{U}(\widehat{u})$ in either of the following forms:

$$\left[\begin{array}{cc|ccc} U(u) & U_1(u,v) & & 0 & \\ 0 & U(u) & & & \\ \hline & & U(u) & U_2(u,r) & U_3(u,r,s) \\ & 0 & 0 & U(u)+\alpha U_2(u,r) & U_2(u,r)+\alpha U_3(u,r,s) \\ & & 0 & 0 & U(u)+\alpha U_2(u,r) \end{array}\right]$$

and

$$\left[\begin{array}{cc|ccc} U(u) & U_1(u,v) & & 0 & \\ 0 & 0 & & & \\ \hline & & U(u) & U_2(u,r) & U_3(u,r,s) \\ & 0 & 0 & U(u)+\alpha U_2(u,r) & U_2(u,r)+\alpha U_3(u,r,s) \\ & & 0 & 0 & U(u)+\alpha U_2(u,r) \end{array}\right].$$

Those can also be generalized to enlarged spectral matrices which generate tri-integrable couplings:

$$u_t = K(u), \quad u_{1,t} = S_1(u,u_1), \quad u_{2,t} = S_2(u,u_1,u_2), \quad u_{3,t} = S_3(u,u_1,u_2,u_3).$$

Further, by an idea of using the Kronecker product (see [23, 37]), we can construct many other enlarged Lax pairs generating integrable couplings.

Moreover, various other integrable characteristics such as Hirota bilinear forms can be exhibited for integrable couplings (see, e.g., [24]). Integrable couplings may possess another interesting property: the linear superposition principle on subspaces of solutions, and the closure of such subspaces of exponential wave solutions should contain all soliton solutions (see [21]). In particular, it is interesting to see what kinds of subspaces of solutions the bi-integrable coupling

$$u_t = K(u), \quad v_t = K'(u)[v], \quad w_t = K'(u)[w]$$

can possess. To enrich multi-component integrable equations (see, e.g., [15–16, 35–36, 7, 42]), it has been an important task to explore more integrable properties for multi-integrable couplings including the above intriguing bi-integrable coupling.

References

[1] Ablowitz, M. J., Kaup, D. J., Newell, A. C., et al., The inverse scattering transform— Fourier analysis for nonlinear problems, *Stud. Appl. Math.*, **53**, 1974, 249–315.

[2] Bluman, G. W. and Cole, J. D., Similarity Methods for Differential Equations, Applied Mathematical Sciences, **13**, Springer-Verlag, New York, Heidelberg, 1974.

[3] Drinfel'd, V. G. and Sokolov, V. V., Equations of Korteweg-de Vries type and simple Lie algebras, *Soviet Math. Dokl.*, **23**, 1981, 457–462.

[4] Fuchssteiner, B., Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Analysis TMA*, **3**, 1979, 849–862.

[5] Fuchssteiner, B. and Fokas, A. S., Symplectic structures, their Bcklund transformations and hereditary symmetries, *Phys. D*, **4**, 1981/1982, 47–66.

[6] Lax, P., Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.*, **21**, 1968, 467–490.

[7] Li, Z. and Dong, H. H., Two integrable couplings of the Tu hierarchy and their Hamiltonian structures, *Comput. Math. Appl.*, **55**, 2008, 2643–2652.

[8] Ma, W. X., A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction (in Chinese), *Chin. Ann. Math.*, **13A**(1), 1992, 115–123.

[9] Ma, W. X. and Fuchssteiner, B., Integrable theory of the perturbation equations, *Chaos Solitons Fractals*, **7**, 1996, 1227–1250.

[10] Ma, W. X., Integrable couplings of soliton equations by perturbations I — A general theory and application to the KdV hierarchy, *Methods Appl. Anal.*, **7**, 2000, 21–55.

[11] Ma, W. X., Enlarging spectral problems to construct integrable couplings of soliton equations, *Phys. Lett. A*, **316**, 2003, 72–76.

[12] Ma, W. X., Integrability, Encyclopedia of Nonlinear Science, A. Scott (ed.), Taylor & Francis, New York, 2005, 250–253.

[13] Ma, W. X., Integrable couplings of vector AKNS soliton equations, *J. Math. Phys.*, **46**, 2005, 033507, 19 pages.

[14] Ma, W. X., A discrete variational identity on semi-direct sums of Lie algebras, *J. Phys. A, Math. Theoret.*, **40**, 2007, 15055–15069.

[15] Ma, W. X., A Hamiltonian structure associated with a matrix spectral problem of arbitrary-order, *Phys. Lett. A*, **367**, 2007, 473–477.

[16] Ma, W. X., Multi-component bi-Hamiltonian Dirac integrable equations, *Chaos Solitons Fractals*, **39**, 2009, 282–287.

[17] Ma, W. X., Variational identities and applications to Hamiltonian structures of soliton equations, *Nonlinear Anal. TMA*, **71**, 2009, e1716–e1726.

[18] Ma, W. X., Variational identities and Hamiltonian structures, Nonlinear and Modern Mathematical Physics, W. X. Ma, X. B. Hu and Q. P. Liu (eds.), AIP, New York, 2010, 1–27.

[19] Ma, W. X., Nonlinear continuous integrable Hamiltonian couplings, *Appl. Math. Comput.*, **217**, 2011, 7238–7244.

[20] Ma, W. X. and Chen, M., Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras, *J. Phys. A, Math. Gen.*, **39**, 2006, 10787–10801.

[21] Ma, W. X. and Fan, E. G., Linear superposition principle applying to Hirota bilinear equations, *Comput. Math. Appl.*, **61**, 2011, 950–959.

[22] Ma, W. X. and Gao, L., Coupling integrable couplings, *Modern Phys. Lett. B*, **23**, 2009, 1847–1860.

[23] Ma, W. X. and Guo, F. K., Lax representations and zero curvature representations by Kronecker product, *Int. J. Theor. Phys.*, **36**, 1997, 697–704.

[24] Ma, W. X. and Strampp, W., Bilinear forms and Bäcklund transformations of the perturbation systems, *Phys. Lett. A*, **341**, 2005, 441–449.

[25] Ma, W. X., Xu, X. X. and Zhang, Y. F., Semi-direct sums of Lie algebras and continuous integrable couplings, *Phys. Lett. A*, **351**, 2006, 125–130.

[26] Ma, W. X., Xu, X. X. and Zhang, Y. F., Semi-direct sums of Lie algebras and discrete integrable couplings, *J. Math. Phys.*, **47**, 2006, 053501, 16 pages.

[27] Ma, W. X. and Zhang, Y., Component-trace identities for Hamiltonian structures, *Appl. Anal.*, **89**, 2010, 457–472.

[28] Ma, W. X. and Zhu, Z. N., Constructing nonlinear discrete integrable Hamiltonian couplings, *Comput. Math. Appl.*, **60**, 2010, 2601–2608.

[29] Magri, F., A simple model of the integrable Hamiltonian equation, *J. Math. Phys.*, **19**, 1978, 1156–1162.

[30] Olver, P. J., Evolution equations possessing infinitely many symmetries, *J. Math. Phys.*, **18**, 1977, 1212–1215.

[31] Sakovich, S. Y., On integrability of a $(2+1)$-dimensional perturbed KdV equation, *J. Nonlinear Math. Phys.*, **5**, 1998, 230–233.

[32] Sakovich, S. Y., Coupled KdV equations of Hirota-Satsuma type, *J. Nonlinear Math. Phys.*, **6**, 1999, 255–262.

[33] Tu, G. Z., On Liouville integrability of zero-curvature equations and the Yang hierarchy, *J. Phys. A, Math. Gen.*, **22**, 1989, 2375–2392.

[34] Xia, T. C., Chen, X. H. and Chen, D. Y., A new Lax integrable hierarchy, N Hamiltonian structure and its integrable couplings system, *Chaos Solitons Fractals*, **23**, 2005, 451–458.

[35] Xia, T. C., Yu, F. J. and Zhang, Y., The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions, *Phys. A*, **343**, 2004, 238–246.

[36] Xu, X. X., Integrable couplings of relativistic Toda lattice systems in polynomial form and rational form, their hierarchies and bi-Hamiltonian structures, *J. Phys. A, Math. Theoret.*, **42**, 2009, 395201, 21 pages.

[37] Yu, F. J. and Li, L., A new method to construct the integrable coupling system for discrete soliton equation with the Kronecker product, *Phys. Lett. A*, **372**, 2008, 3548–3554.

[38] Yu, Y. J. and Zhang, H. Q., Hamiltonian structure of the integrable couplings for the multicomponent Dirac hierarchy, *Appl. Math. Comput.*, **197**, 2008, 828–835.

[39] Zakharov, V. E. and Shabat, A. B., A scheme for generating the nonlinear equations of mathematical physics by the method of the inverse scattering problem I, *Funct. Anal. Appl.*, **8**, 1974, 226–235.

[40] Zhang, Y. F. and Feng, B. L., A few Lie algebras and their applications for generating integrable hierarchies of evolution types, *Commun. Nonlinear Sci. Numer. Simul.*, **16**, 2011, 3045–3061.

[41] Zhang, Y. F. and Tam, H. M., Applications of the Lie algebra gl(2), *Modern Phys. Lett. B*, **23**, 2009, 1763–1770.

[42] Zhang, Y. F. and Tam, H. W., Coupling commutator pairs and integrable systems, *Chaos Solitons Fractals*, **39**, 2009, 1109–1120.