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Riemann–Hilbert problems of a six-component fourth-order AKNS system and its soliton solutions

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Abstract
Associated with a 4×4 matrix spectral problem, a six-component AKNS soliton hierarchy is presented, together with the first three nonlinear soliton systems. From an equivalent spectral problem, a kind of Riemann–Hilbert problems is formulated for a six-component system of fourth-order AKNS equations in the resulting AKNS hierarchy. Soliton solutions to the considered system of coupled fourth-order AKNS equations are worked out from a reduced Riemann–Hilbert problem where an identity jump matrix is taken.

Keywords Soliton hierarchy · Riemann–Hilbert problem · Soliton solution

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1 Introduction

In soliton theory, the Riemann–Hilbert approach is one of the most powerful techniques to generate integrable equations and their soliton solutions (Novikov et al. 1984). The approach

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is based on a kind of matrix spectral problems, which possess bounded eigenfunctions analytically extendable to the upper or lower half-plane. It is closely connected with the inverse scattering method, also known as a nonlinear Fourier method (Ablowitz and Clarkson 1991). The normalization conditions at infinity on the real line in constructing the scattering coefficients are used in solving the associated Riemann–Hilbert problems (Novikov et al. 1984). Upon taking the jump matrix to be the identity matrix, reduced Riemann–Hilbert problems lead, under imposed evolution rules, to soliton solutions, whose specific limits can engender rational solutions and periodic solutions. Applications have been developed for a few integrable equations such as the multiple wave interaction equations (Novikov et al. 1984), the Harry Dym equation (Xiao and Fan 2016), the generalized Sasa–Satsuma equation (Geng and Wu 2016) and the general coupled nonlinear Schrödinger equations (Wang et al. 2010).

We shall follow the standard procedure suited for Riemann–Hilbert problems, in which the unit imaginary number i is consistently used. We, therefore, begin with a pair of matrix spectral problems as follows:

$$-i\phi_x = U\phi, \quad -i\phi_t = V\phi, \quad U = A(\lambda) + P(u,\lambda), \quad V = B(\lambda) + Q(u,\lambda),$$

where λ is a spectral parameter, u is a vector potential, ϕ is an $n \times n$ matrix eigenfunction, A, B are constant commuting $n \times n$ matrices, and P, Q are trace-less $n \times n$ matrices. Their compatibility condition is presented by the zero curvature equation

$$U_t - V_x + i[U, V] = 0,$$

where $[\cdot,\cdot]$ is the matrix commutator. To formulate a Riemann–Hilbert problem on the real line for this zero curvature equation, we adopt a pair of equivalent matrix spectral problems as follows:

$$\psi_x = i[A(\lambda), \psi] + \check{P}(u,\lambda)\psi, \quad \psi_t = i[B(\lambda), \psi] + \check{Q}(u,\lambda)\psi,$$

where ψ is an $n \times n$ matrix eigenfunction, $\check{P} = iP$ and $\check{Q} = iQ$, and assume that $\mathbb{C}^{\pm}$ and $\mathbb{C}_0^{\pm}$ denote the upper and lower half-planes and the closed upper and lower half-planes, respectively:

$$\mathbb{C}^{\pm} = \{z \in \mathbb{C} \mid \pm\,\mathrm{Im}(z) > 0\}, \quad \mathbb{C}_0^{\pm} = \{z \in \mathbb{C} \mid \pm\,\mathrm{Im}(z) \geq 0\}.$$

The relation between ϕ and ψ is

$$\phi = \psi E_g, \quad E_g = \mathrm{e}^{iA(\lambda)x + iB(\lambda)t}.$$

This offers a possibility for us to have two analytical matrix eigenfunctions with the asymptotic conditions

$$\psi^{\pm} \to I_n, \quad \text{when } x, t \to \pm\infty,$$

where I_n stands for the identity matrix of size n. Then, from those two matrix eigenfunctions $\psi^{\pm}$ and the associated adjoint matrix eigenfunctions, we try to determine two analytical matrix functions $P^{\pm}(x,t,\lambda)$, which are analytical in the upper and lower half-planes and continuous in the closed upper and lower half-planes respectively, to build a Riemann–Hilbert problem on the real line:

$$G^{+}(x,t,\lambda) = G^{-}(x,t,\lambda)G(x,t,\lambda), \quad \lambda \in \mathbb{R},$$

where $G^{+} = P^{+}$ and $G^{-} = (P^{-})^{-1}$ are analytical in the upper and lower half-planes and continuous in the closed upper and lower half-planes, respectively. Upon taking the jump matrix G to be the identity matrix I_n, the reduced corresponding Riemann–Hilbert problem can be normally solved to generate soliton solutions, by observing asymptotic behaviors of the matrix functions $P^{\pm}$ at infinity of λ. In this paper, we shall present an application example by considering a six-component system of fourth-order AKNS equations and compute its soliton solutions by a special Riemann–Hilbert problem.

The rest of the paper is structured as follows. In Sect. 2, within the zero-curvature formulation, we rederive the six-component AKNS soliton hierarchy and furnish its bi-Hamiltonian structure, associated with a new matrix spectral problem suited for the Riemann–Hilbert theory. In Sect. 3, taking a system of coupled fourth-order AKNS equations as an example, we analyze analytical properties of matrix eigenfunctions for an equivalent spectral problem, and build a kind of Riemann–Hilbert problems associated with the newly introduced spectral problem. In Sect. 4, we compute soliton solutions to the considered six-component system of coupled fourth-order AKNS equations from a specific Riemann–Hilbert problem on the real line, in which the jump matrix is taken as the identity matrix. In Sect. 5, we present a summary of the results and some concluding remarks.

2 Six-component AKNS soliton hierarchy

2.1 Zero curvature formulation

Let us first recall the zero curvature formulation to construct soliton hierarchies (Tu 1989). Assume that u denotes a vector potential and λ, a spectral parameter. Choose a square spectral matrix $U = U(u, \lambda)$ from a given matrix loop algebra. Try a Laurent series

$$W = W(u, \lambda) = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} W_k(u) \lambda^{-k} \tag{2.1}$$

to solve the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.2}$$

Based on this solution W, we introduce a series of Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (\lambda^r W)_+ + \Delta_r, \quad r \geq 0, \tag{2.3}$$

where the subscript $+$ means to take a polynomial part in λ and Δ_r, $r \geq 0$, are appropriate modification terms, and then generate a soliton hierarchy

$$u_t = K_r(u) = K_r(x, t, u, u_x, \ldots), \quad r \geq 0, \tag{2.4}$$

from a series of zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r \geq 0. \tag{2.5}$$

The two matrices U and $V^{[r]}$ are called a Lax pair (Lax 1968) of the r-th soliton equation in the hierarchy (2.4). Note that the zero curvature equations in (2.5) are the compatibility conditions of the spatial and temporal matrix spectral problems

$$-i\phi_x = U\phi = U(u, \lambda)\phi, \qquad -i\phi_t = V^{[r]}\phi = V^{[r]}(u, \lambda)\phi, \quad r \geq 0, \tag{2.6}$$

where ϕ is the matrix eigenfunction.

To explore the Liouville integrability of the soliton hierarchy (2.4), we normally furnish a bi-Hamiltonian structure (Magri 1978):

$$u_t = K_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u}, \quad r \geq 1, \tag{2.7}$$

where J and M form a Hamiltonian pair and $\frac{\delta}{\delta u}$ denotes the variational derivative (see, e.g., Ma and Fuchssteiner 1996). The Hamiltonian structures can be often achieved through applying the trace identity (Tu 1989; Ma 1992a, b):

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right)\right], \quad \gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\mathrm{tr}(W^2)|, \tag{2.8}$$

or more generally, the variational identity (Ma and Chen 2006):

$$\frac{\delta}{\delta u}\int \left\langle W, \frac{\partial U}{\partial \lambda}\right\rangle \mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\left\langle W, \frac{\partial U}{\partial u}\right\rangle\right], \quad \gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle W, W\rangle|, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra (Ma 2009). The bi-Hamiltonian structure guarantees (Magri 1978) that there exist infinitely many commuting Lie symmetries $\{K_n\}_{n=0}^{\infty}$ and conserved quantities $\{\tilde{H}_n\}_{n=0}^{\infty}$:

$$[K_{n_1}, K_{n_2}] = K'_{n_1}[K_{n_2}] - K'_{n_2}[K_{n_1}] = 0,$$
$$\{\tilde{\mathcal{H}}_{n_1}, \tilde{\mathcal{H}}_{n_2}\}_N = \int \left(\frac{\delta \tilde{\mathcal{H}}_{n_1}}{\delta u}\right)^{\mathrm{T}} N \frac{\delta \tilde{\mathcal{H}}_{n_2}}{\delta u}\,\mathrm{d}x = 0,$$

where $n_1, n_2 \geq 0$, $N = J$ or M, and K' stands for the Gateaux derivative of K with respect to u:

$$K'(u)[S] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u+\varepsilon S, u_x + \varepsilon S_x, \ldots).$$

It is well recognized that for an evolution equation with a vector potential u, $\tilde{H} = \int H\,\mathrm{d}x$ is a conserved functional iff $\frac{\delta \tilde{H}}{\delta u}$ is an adjoint symmetry (Ma and Zhou 2002) and, thus, the Hamiltonian structures links conserved functionals to adjoint symmetries and further symmetries. Moreover, adjoint symmetries play a crucial role in formulating conservation laws (Ma 2018a).

When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations produce classical soliton hierarchies (Drinfeld and Sokolov 1982); when semisimple, the associated zero curvature equations lead to a collection of different soliton hierarchies; and when non-semisimple, we obtain hierarchies of integrable couplings (Ma et al. 2006), which require extra care in presenting soliton solutions.

2.2 Six-component AKNS hierarchy

Let us begin with a 4×4 matrix spectral problem

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = (U_{kl})_{4\times 4} = \begin{bmatrix} \alpha_1\lambda & p_1 & p_2 & p_3 \\ q_1 & \alpha_2\lambda & 0 & 0 \\ q_2 & 0 & \alpha_2\lambda & 0 \\ q_3 & 0 & 0 & \alpha_2\lambda \end{bmatrix}, \tag{2.10}$$

where α_1 and α_2 are two given real constants, λ is a spectral parameter and u is a six-dimensional potential

$$u = (p, q^T)^T, \quad p = (p_1, p_2, p_3), \quad q = (q_1, q_2, q_3)^T. \tag{2.11}$$

A special case of $p_2 = p_3 = q_2 = q_3 = 0$ transforms (2.10) into the AKNS spectral problem (Ablowitz et al. 1974) and, therefore, this spectral problem is called a six-component AKNS spectral problem. Since the matrix $\Lambda = \text{diag}(\alpha_1, \alpha_2, \alpha_2, \alpha_2)$ has a multiple eigenvalue, the spectral problem (2.10) is degenerate.

To generate the associated AKNS soliton hierarchy, we first solve the stationary zero curvature equation (2.2) corresponding to (2.10). We write a solution W as a compact form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.12}$$

where a is a scalar, b^T and c are three-dimensional columns, and d is a 3×3 matrix. Obviously, the stationary zero curvature equation (2.2) becomes

$$\begin{aligned} &a_x = i(pc - bq), \quad b_x = i(\alpha\lambda b + pd - ap), \quad c_x = i(-\alpha\lambda c + qa - dq), \\ &d_x = i(qb - cp), \end{aligned} \tag{2.13}$$

where $\alpha = \alpha_1 - \alpha_2$. As normal, we look for a Laurent series solution as:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m}, \quad W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix}, \qquad m \geq 0, \tag{2.14}$$

with $b^{[m]}$, $c^{[m]}$ and $d^{[m]}$ being denoted by

$$b^{[m]} = \left(b_1^{[m]}, b_2^{[m]}, b_3^{[m]}\right), \quad c^{[m]} = \left(c_1^{[m]}, c_2^{[m]}, c_3^{[m]}\right)^T, \quad d^{[m]} = \left(d_{kl}^{[m]}\right)_{3\times 3}, \qquad m \geq 0. \tag{2.15}$$

Then, the system (2.13) is equivalent to the following recursion relations:

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a_x^{[0]} = 0, \quad d_x^{[0]} = 0, \tag{2.16a}$$

$$b^{[m+1]} = \frac{1}{\alpha}\left(-ib_x^{[m]} - pd^{[m]} + a^{[m]}p\right), \quad m \geq 0, \tag{2.16b}$$

$$c^{[m+1]} = \frac{1}{\alpha}\left(ic_x^{[m]} + qa^{[m]} - d^{[m]}q\right), \quad m \geq 0, \tag{2.16c}$$

$$a_x^{[m]} = i\left(pc^{[m]} - b^{[m]}q\right), \quad d_x^{[m]} = i\left(qb^{[m]} - c^{[m]}p\right), \qquad m \geq 1. \tag{2.16d}$$

Let us now fix the initial values as follows:

$$a^{[0]} = \beta_1, \quad d^{[0]} = \beta_2 I_3, \tag{2.17}$$

where β_1, β_2 are arbitrary real constants and I_3 is the identity matrix of size 3, and take constants of integration in (2.16d) to be zero, that is, require

$$W_m|_{u=0} = 0, \quad m \geq 1. \tag{2.18}$$

Thus, with $a^{[0]}$ and $d^{[0]}$ given by (2.17), all matrices W_m, $m \geq 1$, will be uniquely determined. For instance, a direct computation, using (2.16), generates that

$$b_k^{[1]} = \frac{\beta}{\alpha} p_k, \quad c_k^{[1]} = \frac{\beta}{\alpha} q_k, \quad a^{[1]} = 0, \quad d_{kl}^{[1]} = 0; \tag{2.19a}$$

$$b_k^{[2]} = -\frac{\beta}{\alpha^2} i p_{k,x}, \quad c_k^{[2]} = \frac{\beta}{\alpha^2} i q_{k,x}, \quad a^{[2]} = -\frac{\beta}{\alpha^2} \sum_{j=1}^{3} p_j q_j, \quad d_{kl}^{[2]} = \frac{\beta}{\alpha^2} p_l q_k; \tag{2.19b}$$

$$b_k^{[3]} = -\frac{\beta}{\alpha^3} \left[p_{k,xx} + 2 \left(\sum_{j=1}^{3} p_j q_j \right) p_k \right], c_k^{[3]} = -\frac{\beta}{\alpha^3} \left[q_{k,xx} + 2 \left(\sum_{j=1}^{3} p_j q_j \right) q_k \right], \tag{2.19c}$$

$$a^{[3]} = -\frac{\beta}{\alpha^3} i \sum_{j=1}^{3} \left(p_j q_{j,x} - p_{j,x} q_j \right), \quad d_{kl}^{[3]} = -\frac{\beta}{\alpha^3} i \left(p_{l,x} q_k - p_l q_{k,x} \right); \tag{2.19d}$$

$$b_k^{[4]} = \frac{\beta}{\alpha^4} i \left[p_{k,xxx} + 3 \left(\sum_{j=1}^{3} p_j q_j \right) p_{k,x} + 3 \left(\sum_{j=1}^{3} p_{j,x} q_j \right) p_k \right], \tag{2.19e}$$

$$c_k^{[4]} = -\frac{\beta}{\alpha^4} i \left[q_{k,xxx} + 3 \left(\sum_{j=1}^{3} p_j q_j \right) q_{k,x} + 3 \left(\sum_{j=1}^{3} p_j q_{j,x} \right) q_k \right], \tag{2.19f}$$

$$a^{[4]} = \frac{\beta}{\alpha^4} \left[3 \left(\sum_{j=1}^{3} p_j q_j \right)^2 + \sum_{j=1}^{3} \left(p_j q_{j,xx} - p_{j,x} q_{j,x} + p_{j,xx} q_j \right) \right], \tag{2.19g}$$

$$d_{kl}^{[4]} = -\frac{\beta}{\alpha^4} \left[3 p_l \left(\sum_{j=1}^{3} p_j q_j \right) q_k + p_{l,xx} q_k - p_{l,x} q_{k,x} + p_l q_{k,xx} \right]; \tag{2.19h}$$

$$b_k^{[5]} = \frac{\beta}{\alpha^5} \left\{ p_{k,xxxx} + 4 \left(\sum_{j=1}^{3} p_j q_j \right) p_{k,xx} + \left(6 \sum_{j=1}^{3} p_{j,x} q_j + 2 \sum_{j=1}^{3} p_j q_{j,x} \right) p_{k,x} \right.$$
$$\left. + \left[4 \sum_{j=1}^{3} p_{j,xx} q_j + 2 \sum_{j=1}^{3} p_{j,x} q_{j,x} + 2 \sum_{j=1}^{3} p_j q_{j,xx} + 6 \left(\sum_{j=1}^{3} p_j q_j \right)^2 \right] p_k \right\}, \tag{2.19i}$$

$$c_k^{[5]} = \frac{\beta}{\alpha^5} \left\{ q_{k,xxxx} + 4 \left(\sum_{j=1}^{3} p_j q_j \right) q_{k,xx} + \left(6 \sum_{j=1}^{3} p_j q_{j,x} + 2 \sum_{j=1}^{3} p_{j,x} q_j \right) q_{k,x} \right.$$
$$\left. + \left[4 \sum_{j=1}^{3} p_j q_{j,xx} + 2 \sum_{j=1}^{3} p_{j,x} q_{j,x} + 2 \sum_{j=1}^{3} p_{j,xx} q_j + 6 \left(\sum_{j=1}^{3} p_j q_j \right)^2 \right] q_k \right\}, \tag{2.19j}$$

$$a^{[5]} = \frac{\beta}{\alpha^5} i \left[6 \left(\sum_{j=1}^{3} p_j q_j \right) \sum_{j=1}^{3} \left(p_j q_{j,x} - p_{j,x} q_j \right) \right.$$
$$\left. + \sum_{j=1}^{3} \left(p_j q_{j,xxx} - p_{j,xxx} q_j + p_{j,xx} q_{j,x} - p_{j,x} q_{j,xx} \right) \right], \tag{2.19k}$$

$$d_{kl}^{[5]} = \frac{\beta}{\alpha^5} i \left[2p_l \left(\sum_{j=1}^{3} p_{j,x} q_j - p_j q_{j,x} \right) q_k + 4p_{l,x} \left(\sum_{j=1}^{3} p_j q_j \right) q_k \right.$$
$$\left. - p_l \left(\sum_{j=1}^{3} p_j q_j \right) q_{k,x} + p_{l,xxx} q_k - p_l q_{k,xxx} + p_{l,x} q_{k,xx} - p_{l,xx} q_{k,x} \right]; \quad (2.19l)$$

where $\beta = \beta_1 - \beta_2$ and $1 \leq k, l \leq 3$. Based on (2.16d), we can have, from (2.16b) and (2.16c), a recursion relation for $b^{[m]}$ and $c^{[m]}$:

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix}, \quad m \geq 1, \quad (2.20)$$

where Ψ is a 6×6 matrix operator

$$\Psi = \frac{i}{\alpha} \begin{bmatrix} \left(\partial + \sum_{k=1}^{3} q_k \partial^{-1} p_k \right) I_3 + q\partial^{-1} p & -q\partial^{-1} q^{\mathrm{T}} - (q\partial^{-1} q^{\mathrm{T}})^{\mathrm{T}} \\ p^{\mathrm{T}} \partial^{-1} p + (p^{\mathrm{T}} \partial^{-1} p)^{\mathrm{T}} & -\left(\partial + \sum_{k=1}^{3} p_k \partial^{-1} q_k \right) I_3 - p^{\mathrm{T}} \partial^{-1} q^{\mathrm{T}} \end{bmatrix}. \quad (2.21)$$

To generate the six-component AKNS soliton hierarchy, we introduce, for all integers $r \geq 0$, the following Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (V_{kl}^{[r]})_{4\times 4} = (\lambda^r W)_+ = \sum_{k=0}^{r} W_k \lambda^{r-k}, \quad r \geq 0, \quad (2.22)$$

where the modification terms are chosen as zero. The compatibility conditions of (2.6), i.e., the zero curvature equations (2.5), engender the six-component AKNS soliton hierarchy:

$$u_t = \begin{bmatrix} p^{\mathrm{T}} \\ q \end{bmatrix}_t = K_r = i \begin{bmatrix} \alpha b^{[r+1]\mathrm{T}} \\ -\alpha c^{[r+1]} \end{bmatrix}, \quad r \geq 0. \quad (2.23)$$

The first two nonlinear systems in the above soliton hierarchy (2.23) read

$$p_{k,t} = -\frac{\beta}{\alpha^2} i \left[p_{k,xx} + 2 \left(\sum_{j=1}^{3} p_j q_j \right) p_k \right], \quad 1 \leq k \leq 3, \quad (2.24a)$$

$$q_{k,t} = \frac{\beta}{\alpha^2} i \left[q_{k,xx} + 2 \left(\sum_{j=1}^{3} p_j q_j \right) q_k \right], \quad 1 \leq k \leq 3, \quad (2.24b)$$

and

$$p_{k,t} = -\frac{\beta}{\alpha^3} \left[p_{k,xxx} + 3 \left(\sum_{j=1}^{3} p_j q_j \right) p_{k,x} + 3 \left(\sum_{j=1}^{3} p_{j,x} q_j \right) p_k \right], \ 1 \leq k \leq 3, \quad (2.25a)$$

$$q_{k,t} = -\frac{\beta}{\alpha^3} \left[q_{k,xxx} + 3 \left(\sum_{j=1}^{3} p_j q_j \right) q_{k,x} + 3 \left(\sum_{j=1}^{3} p_j q_{j,x} \right) q_k \right], \ 1 \leq k \leq 3, \quad (2.25b)$$

which are the six-component versions of the AKNS systems of coupled nonlinear Schrödinger equations and coupled mKdV equations, respectively. Under a symmetric reduction, the six-component AKNS systems (2.24) can be reduced to the Manakov system (Manakov 1974), for which a decomposition into finite-dimensional integrable Hamiltonian systems was made in (Chen and Zhou 2012), while as the six-component AKNS systems (2.25) contain various systems of mKdV equations, for which different kinds of integrable decompositions under symmetry constraints are made (see, e.g., Ma 1995; Yu and Zhou 2006).

We shall consider the third nonlinear system, i.e., the system of coupled fourth-order AKNS equations:

$$p_{k,t} = \frac{\beta}{\alpha^4} i \left\{ p_{k,xxxx} + 4\left(\sum_{j=1}^{3} p_j q_j\right) p_{k,xx} + \left(6\sum_{j=1}^{3} p_{j,x} q_j + 2\sum_{j=1}^{3} p_j q_{j,x}\right) p_{k,x} \right.$$
$$\left. + \left[4\sum_{j=1}^{3} p_{j,xx} q_j + 2\sum_{j=1}^{3} p_{j,x} q_{j,x} + 2\sum_{j=1}^{3} p_j q_{j,xx} + 6\left(\sum_{j=1}^{3} p_j q_j\right)^2\right] p_k \right\},$$
$$1 \le k \le 3, \tag{2.26a}$$
$$q_{k,t} = -\frac{\beta}{\alpha^4} i \left\{ q_{k,xxxx} + 4\left(\sum_{j=1}^{3} p_j q_j\right) q_{k,xx} + \left(6\sum_{j=1}^{3} p_j q_{j,x} + 2\sum_{j=1}^{3} p_{j,x} q_j\right) q_{k,x} \right.$$
$$\left. + \left[4\sum_{j=1}^{3} p_j q_{j,xx} + 2\sum_{j=1}^{3} p_{j,x} q_{j,x} + 2\sum_{j=1}^{3} p_{j,xx} q_j + 6\left(\sum_{j=1}^{3} p_j q_j\right)^2\right] q_k \right\},$$
$$1 \le k \le 3. \tag{2.26b}$$

There is no much study on this system of coupled fourth-order AKNS equations. We are going to build a kind of Riemann–Hilbert problems for this system and compute its soliton solutions by solving reduced Riemann–Hilbert problems.

There exists a Hamiltonian structure (Ma and Zhou 2002) for the six-component AKNS soliton hierarchy (2.23), which can be furnished through applying the trace identity (Tu 1989), or more generally, the variational identity (Ma and Chen 2006). Actually, we have

$$-i\,\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = \alpha_1 a + \alpha_2 \mathrm{tr}(d) = \sum_{m=0}^{\infty}\left(\alpha_1 a^{[m]} + \alpha_2 d_{11}^{[m]} + \alpha_2 d_{22}^{[m]}\right)\lambda^{-m},$$

and

$$-i\,\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right) = \begin{bmatrix} c \\ b^T \end{bmatrix} = \sum_{m\ge 0} G_{m-1}\lambda^{-m}.$$

Inserting these into the trace identity and considering the case of $m = 2$ tell $\gamma = 0$, and, thus

$$\frac{\delta \tilde{H}_m}{\delta u} = iG_{m-1}, \quad m \ge 1, \tag{2.27}$$

where

$$\tilde{H}_m = \frac{i}{m}\int\left(-\alpha_1 a^{[m+1]} - \alpha_2 d_{11}^{[m+1]} - \alpha_2 d_{22}^{[m+1]}\right)\mathrm{d}x,\ G_{m-1} = \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix}, \quad m \ge 1.$$

It then follows that a bi-Hamiltonian structure of the six-component AKNS systems (2.23):

$$u_t = K_r = JG_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u}, \quad r \geq 1, \tag{2.28}$$

where the Hamiltonian pair $(J, M = J\Psi)$ is given by

$$J = \begin{bmatrix} 0 & \alpha I_3 \\ -\alpha I_3 & 0 \end{bmatrix}, \tag{2.29a}$$

$$M = i\begin{bmatrix} p^{\mathrm{T}}\partial^{-1}p + (p^{\mathrm{T}}\partial^{-1}p)^{\mathrm{T}} & -\left(\partial + \sum_{k=1}^{3} p_k\partial^{-1}q_k\right) I_3 - p^{\mathrm{T}}\partial^{-1}q^{\mathrm{T}} \\ -\left(\partial + \sum_{k=1}^{3} p_k\partial^{-1}q_k\right) I_3 - q\partial^{-1}p & q\partial^{-1}q^{\mathrm{T}} + (q\partial^{-1}q^{\mathrm{T}})^{\mathrm{T}} \end{bmatrix}. \tag{2.29b}$$

Adjoint symmetry constraints (or equivalently symmetry constraints) decompose the six-component AKNS systems into two commuting finite-dimensional Liouville integrable Hamiltonian systems (Ma and Zhou 2002). In the next section, we shall concentrate on the six-component system of coupled fourth-order AKNS equations (2.26).

3 Riemann–Hilbert problems on the real line

The spectral problems of the six-component system of fourth-order AKNS equations (2.26) are

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad -i\phi_t = V^{[4]}\phi = V^{[4]}(u,\lambda)\phi, \tag{3.1}$$

with

$$U = \lambda\Lambda + P, \quad V^{[4]} = \lambda^4\Omega + Q, \tag{3.2}$$

where $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2, \alpha_2)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2, \beta_2, \beta_2)$, and

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} a^{[1]}\lambda^3 + a^{[2]}\lambda^2 + a^{[3]}\lambda + a^{[4]} & b^{[1]}\lambda^3 + b^{[2]}\lambda^2 + b^{[3]}\lambda + b^{[4]} \\ c^{[1]}\lambda^3 + c^{[2]}\lambda^2 + c^{[3]}\lambda + c^{[4]} & d^{[1]}\lambda^3 + d^{[2]}\lambda^2 + d^{[3]}\lambda + d^{[4]} \end{bmatrix}, \tag{3.3}$$

in which u, p, q are defined by (2.11), and $a^{[m]}, b^{[m]}, c^{[m]}, d^{[m]}$, $1 \leq m \leq 4$, are determined in (2.19).

In this section, we discuss the scattering and inverse scattering for the six-component fourth-order AKNS system (2.26), through using the Riemann–Hilbert formulation (Novikov et al. 1984) (see also Gerdjikov 2005; Doktorov and Leble 2007). The resulting results will lay the groundwork for a successful construction of soliton solutions in the next section. Assume that all the six potentials rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$ and satisfy the integrable conditions:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^{m_1}|t|^{m_2}\sum_{k=1}^{3}(|p_k| + |q_k|)\,\mathrm{d}x\,\mathrm{d}t < \infty, \quad m_1, m_2 = 0, 1. \tag{3.4}$$

For the sake of presentation, we also assume that

$$\alpha = \alpha_1 - \alpha_2 < 0, \quad \beta = \beta_1 - \beta_2 < 0, \tag{3.5}$$

and the other cases can be done similarly.

From the spectral problems in (3.1), we note, under (3.4), that when $x, t \to \pm\infty$, we have the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda \Lambda x + i\lambda^4 \Omega t}$. Therefore, upon making the variable transformation

$$\phi = \psi E_g, \quad E_g = \mathrm{e}^{i\lambda \Lambda x + i\lambda^4 \Omega t}, \tag{3.6}$$

we have the canonical normalization:

$$\psi \to I_4, \quad \text{when } x, t \to \pm\infty, \tag{3.7}$$

where I_4 is the identity matrix of size 4. Obviously, the equivalent pair of spectral problems to (3.1) is given by

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.8}$$

$$\psi_t = i\lambda^4[\Omega, \psi] + \check{Q}\psi, \tag{3.9}$$

where $\check{P} = iP$ and $\check{Q} = iQ$. Due to $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$, one obtains

$$\det \psi = 1, \tag{3.10}$$

by a generalized Liouville's formula (Ma et al. 2016a).

Let us now formulate an associated Riemann–Hilbert problem with the variable x. In the scattering problem, we first introduce the matrix solutions $\psi^{\pm}(x, \lambda)$ of (3.8) with the asymptotic conditions

$$\psi^{\pm} \to I_4, \quad \text{when } x \to \pm\infty, \tag{3.11}$$

respectively. The superscripts indicated above refer to which end of the x-axis the boundary conditions are required for. Then, based on (3.10), one sees $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Because $\phi^{\pm} = \psi^{\pm}E,\ E = \mathrm{e}^{i\lambda \Lambda x}$, are both solutions of (3.1), they are linearly dependent and, therefore, one can have

$$\psi^{-}E = \psi^{+}ES(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.12}$$

where

$$S(\lambda) = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}, \quad \lambda \in \mathbb{R}, \tag{3.13}$$

is the scattering matrix. Note that one has $\det S(\lambda) = 1$ because of $\det \psi^{\pm} = 1$.

Applying the method of variation in parameters and using the boundary condition (3.11), we can turn the x-part of (3.1) into the following Volterra integral equations for $\psi^{\pm}$ (Novikov et al. 1984):

$$\psi^{-}(\lambda, x) = I_4 + \int_{-\infty}^{x} \mathrm{e}^{i\lambda \Lambda (x-y)}\, \check{P}(y)\psi^{-}(\lambda, y)\mathrm{e}^{i\lambda \Lambda (y-x)}\, \mathrm{d}y, \tag{3.14}$$

$$\psi^{+}(\lambda, x) = I_4 - \int_{x}^{\infty} \mathrm{e}^{i\lambda \Lambda (x-y)}\, \check{P}(y)\psi^{+}(\lambda, y)\mathrm{e}^{i\lambda \Lambda (y-x)}\, \mathrm{d}y. \tag{3.15}$$

Thus, the two solutions $\psi^{\pm}$ allows analytical continuations off the real line $\lambda \in \mathbb{R}$ provided that the integrals on their right hand sides converge. Based on the diagonal form of Λ, we can directly see that the integral equation for the first column of ψ^{-} contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(y-x)}$, which decays because of $y < x$ in the integral, when λ is in the closed upper half-plane, and the integral equation for the last three columns of ψ^+ contains only the exponential factor $e^{-i\alpha\lambda(y-x)}$, which also decays because of $y > x$ in the integral, when λ is in the closed upper half-plane $\mathbb{C}^+$. Hence, these four columns can be analytically continued to the closed upper half-plane. In a similar manner, we can find that the last three columns of ψ^- and the first column of ψ^+ can be analytically continued to the closed lower half-plane. Upon expressing

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \psi_3^{\pm}, \psi_4^{\pm}), \tag{3.16}$$

that is, $\psi_k^{\pm}$ stands for the kth column of $\phi^{\pm}$ $(1 \le k \le 4)$, the matrix solution

$$P^+ = P^+(x, \lambda) = (\psi_1^-, \psi_2^+, \psi_3^+, \psi_4^+) = \psi^- H_1 + \psi^+ H_2 \tag{3.17}$$

is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}_0^+$, and the matrix solution

$$(\psi_1^+, \psi_2^-, \psi_3^-, \psi_4^-) = \psi^+ H_1 + \psi^- H_2 \tag{3.18}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}_0^-$, where the two basic matrices are

$$H_1 = \mathrm{diag}(1, 0, 0, 0), \quad H_2 = \mathrm{diag}(0, 1, 1, 1). \tag{3.19}$$

In addition, from the Volterra integral equation (3.14), we see that

$$P^+(x, \lambda) \to I_4, \quad \text{when } \lambda \in \mathbb{C}_0^+ \to \infty, \tag{3.20}$$

and

$$(\psi_1^+, \psi_2^-, \psi_3^-, \psi_4^-) \to I_4, \quad \text{when } \lambda \in \mathbb{C}_0^- \to \infty. \tag{3.21}$$

Next, we construct the analytic counterpart of P^+ in the lower half-plane. Note that the adjoint equation of the x-part of (3.1) and the adjoint equation of (3.8) are as follows:

$$i\tilde{\phi}_x = \tilde{\phi} U, \tag{3.22}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi} P. \tag{3.23}$$

Just a direct computation shows that the inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ solve these adjoint equations, respectively. Let we express $\tilde{\psi}^{\pm}$ as follows:

$$\tilde{\psi}^{\pm} = \begin{bmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \tilde{\psi}^{\pm,3} \\ \tilde{\psi}^{\pm,4} \end{bmatrix}, \tag{3.24}$$

that is, $\tilde{\psi}^{\pm,k}$ stands for the kth row of $\tilde{\psi}^{\pm}$ $(1 \le k \le 4)$, and then we can explore by similar arguments that the adjoint matrix solution

$$P^- = \begin{bmatrix} \tilde{\psi}^{-,1} \\ \tilde{\psi}^{+,2} \\ \tilde{\psi}^{+,3} \\ \tilde{\psi}^{+,4} \end{bmatrix} = H_1 \tilde{\psi}^- + H_2 \tilde{\psi}^+ = H_1 (\psi^-)^{-1} + H_2 (\psi^+)^{-1} \tag{3.25}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}_0^-$, and the other matrix solution

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{-,3} \\ \tilde{\psi}^{-,4} \end{bmatrix} = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1} \tag{3.26}$$

is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}_0^+$. In the same way, one sees that

$$P^-(x,\lambda) \to I_4, \quad \text{when } \lambda \in \mathbb{C}_0^- \to \infty, \tag{3.27}$$

and

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{-,3} \\ \tilde{\psi}^{-,4} \end{bmatrix} \to I_4, \quad \text{when } \lambda \in \mathbb{C}_0^+ \to \infty. \tag{3.28}$$

Now, we have built the two matrix functions P^+ and P^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$, respectively. We can directly find that on the real line, the two matrix functions P^+ and P^- are related by

$$P^-(x,\lambda)P^+(x,\lambda) = G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{3.29}$$

where

$$\begin{aligned} G(x,\lambda) &= E(H_1 + H_2 S)(H_1 + S^{-1}H_2)E^{-1} \\ &= E \begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \hat{s}_{14} \\ s_{21} & 1 & 0 & 0 \\ s_{31} & 0 & 1 & 0 \\ s_{41} & 0 & 0 & 1 \end{bmatrix} E^{-1} \end{aligned} \tag{3.30}$$

with $S^{-1} = (\hat{s}_{ij})_{4\times4}$. Then, the corresponding Riemann–Hilbert problems are determined by

$$G^+(x,\lambda) = G^-(x,\lambda)G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{3.31}$$

where $G^+ = P^+$ and $G^- = (P^-)^{-1}$ are analytical in the upper and lower half-planes and continuous in the closed upper and lower half-planes, respectively. The Eq. (3.31) with (3.30) is exactly the associated matrix Riemann–Hilbert problem we wanted to build. The asymptotic properties

$$G^\pm(x,\lambda) \to I_4, \quad \text{when } \lambda \in \mathbb{C}_0^\pm \to \infty, \tag{3.32}$$

provide the canonical normalization conditions for the presented Riemann–Hilbert problem.

To complete the direct scattering transform, we take the derivative of (3.12) with time t and use the vanishing conditions of the potentials. This way, we can show that S satisfies

$$S_t = i\lambda^4[\Omega, S], \tag{3.33}$$

which gives rise to

$$\begin{cases} s_{11,t} = s_{22,t} = s_{33,t} = s_{44,t} = s_{23,t} = s_{24,t} = s_{32,t} = s_{34,t} = s_{42,t} = s_{43,t} = 0, \\ s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^4 t}, \quad s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^4 t}, \quad s_{14} = s_{14}(0,\lambda)\mathrm{e}^{i\beta\lambda^4 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^4 t}, \quad s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^4 t}, \quad s_{41} = s_{41}(0,\lambda)\mathrm{e}^{-i\beta\lambda^4 t}. \end{cases} \tag{3.34}$$

Those are the time evolution of the scattering coefficients.

4 Soliton solutions

The Riemann–Hilbert problems with zeros lead to soliton solutions, which can be solved by transforming into the ones without zeros (Novikov et al. 1984). The uniqueness of the associated Riemann–Hilbert problem (3.31) does not hold unless the zeros of $\det P^+$ and $\det P^-$ in the upper and lower half-planes are specified and the kernel structures of $P^\pm$ at these zeros are well determined (Shchesnovich 2002; Shchesnovich and Yang 2003). Based on the definitions of $P^\pm$ and the scattering relation between the two matrix eigenfunctions ψ^+ and ψ^-, one finds, noting that $\det \psi^\pm = 1$, that

$$\det P^+(x,\lambda) = s_{11}(\lambda), \quad \det P^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{4.1}$$

where, due to $\det S = 1$, one has

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} \\ s_{32} & s_{33} & s_{34} \\ s_{42} & s_{43} & s_{44} \end{vmatrix}. \tag{4.2}$$

As usual, assume that s_{11} has zeros $\{\lambda_k \in \mathbb{C}^+,\ 1 \le k \le N\}$, and $\hat{s}_{11}$ has zeros $\{\hat{\lambda}_k \in \mathbb{C}^-,\ 1 \le k \le N\}$. To get soliton solutions, we also assume that these zeros, λ_k and $\hat{\lambda}_k,\ 1 \le k \le N$, are all simple. Thus, each of $\ker P^+(x,\lambda_k)$, $1 \le k \le N$, contains only a single column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker P^-(x,\hat{\lambda}_k)$, $1 \le k \le N$, a row vector, denoted by $\hat{v}_k$, $1 \le k \le N$:

$$P^+(x,\lambda_k)v_k = 0, \quad \hat{v}_k P^-(x,\hat{\lambda}_k) = 0, \qquad 1 \le k \le N. \tag{4.3}$$

The Riemann–Hilbert problem (3.31) with the canonical normalization conditions in (3.32) and the zero structures in (4.3) can be solved precisely (Novikov et al. 1984; Kawata 1984) and, therefore, one can readily compute the matrix P determining the potentials as follows. Note that P^+ is a solution to the spectral problem (3.8). Thus, as long as we expand P^+ at large λ as

$$P^+(x,\lambda) = I_4 + \frac{1}{\lambda}P_1^+(x) + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \infty, \tag{4.4}$$

by inserting this series expansion into (3.8) and balancing O(1) terms, we obtain

$$\check{P} = -i[\Lambda, P_1^+], \tag{4.5}$$

which leads to that

$$P = -[\Lambda, P_1^+] = \begin{bmatrix} 0 & -\alpha(P_1^+)_{12} & -\alpha(P_1^+)_{13} & -\alpha(P_1^+)_{14} \\ \alpha(P_1^+)_{21} & 0 & 0 & 0 \\ \alpha(P_1^+)_{31} & 0 & 0 & 0 \\ \alpha(P_1^+)_{41} & 0 & 0 & 0 \end{bmatrix}, \tag{4.6}$$

where we denote $P_1^+ = ((P_1^+)_{kl})_{1\le k,l\le 4}$. Now, the six potentials p_i and q_i, $1 \le i \le 3$, can be presented as follows:

$$\begin{cases} p_1 = -\alpha(P_1^+)_{12}, \quad p_2 = -\alpha(P_1^+)_{13}, \quad p_3 = -\alpha(P_1^+)_{14}, \\ q_1 = \alpha(P_1^+)_{21}, \quad q_2 = \alpha(P_1^+)_{31}, \quad q_3 = \alpha(P_1^+)_{41}. \end{cases} \tag{4.7}$$

To compute soliton solutions, we take $G = I_4$ in the above Riemann–Hilbert problem (3.31). This can be achieved exactly if we assume $s_{12} = s_{13} = s_{14} = s_{21} = s_{31} = s_{41} = 0$, which means that there is no reflection in the scattering problem. The solutions to this specific

Riemann–Hilbert problem can be determined by (see, e.g., Novikov et al. 1984; Kawata 1984):

$$P^{+}(x,\lambda)=I_4-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\quad P^{-}(x,\lambda)=I_4+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_l},\tag{4.8}$$

where $M=(m_{kl})_{N\times N}$ is a square matrix whose entries are defined by

$$m_{kl}=\frac{\hat{v}_k v_l}{\lambda_l-\hat{\lambda}_k},\quad 1\le k,l\le N.\tag{4.9}$$

Note that the zeros λ_k and $\hat{\lambda}_k$ are constants, i.e., space and time independent and, thus, we can easily find the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1\le k\le N$. For example, let us take the x-derivative of both sides of the equations

$$P^{+}(x,\lambda_k)v_k=0,\quad 1\le k\le N.\tag{4.10}$$

Further using (3.8) first and then (4.10), we obtain

$$P^{+}(x,\lambda_k)\left(\frac{\mathrm{d}v_k}{\mathrm{d}x}-i\lambda_k\Lambda v_k\right)=0,\quad 1\le k\le N.$$

Without loss of generality, we can take

$$\frac{\mathrm{d}v_k}{\mathrm{d}x}=i\lambda_k\Lambda v_k,\quad 1\le k\le N.\tag{4.11}$$

The time dependence of v_k:

$$\frac{\mathrm{d}v_k}{\mathrm{d}t}=i\lambda_k^4\Omega v_k,\quad 1\le k\le N,\tag{4.12}$$

can be obtained similarly through the t-part of the matrix spectral problem (3.9). To conclude, one can have

$$v_k(x,t)=\mathrm{e}^{i\lambda_k\Lambda x+i\lambda_k^4\Omega t}v_{k,0},\quad 1\le k\le N,\tag{4.13}$$

$$\hat{v}_k(x,t)=\hat{v}_{k,0}\mathrm{e}^{-i\hat{\lambda}_k\Lambda x-i\hat{\lambda}_k^4\Omega t},\quad 1\le k\le N,\tag{4.14}$$

where $v_{k,0}$ and $\hat{v}_{k,0}$, $1\le k\le N$, are arbitrary constant column and row vectors, respectively.

Lastly, from the solutions in (4.8), we get

$$P_1^{+}=-\sum_{k,l=1}^{N}v_k(M^{-1})_{kl}\hat{v}_l,\tag{4.15}$$

and thus, further through the presentations in (4.7), the N-soliton solution to the six-component system of coupled fourth-order AKNS equations (2.26):

$$\begin{cases} p_1=\alpha\sum_{k,l=1}^{N}v_{k,1}(M^{-1})_{kl}\hat{v}_{l,2}, & p_2=\alpha\sum_{k,l=1}^{N}v_{k,1}(M^{-1})_{kl}\hat{v}_{l,3},\\ p_3=\alpha\sum_{k,l=1}^{N}v_{k,1}(M^{-1})_{kl}\hat{v}_{l,4}, & q_1=-\alpha\sum_{k,l=1}^{N}v_{k,2}(M^{-1})_{kl}\hat{v}_{l,1},\\ q_2=-\alpha\sum_{k,l=1}^{N}v_{k,3}(M^{-1})_{kl}\hat{v}_{l,1}, & q_3=-\alpha\sum_{k,l=1}^{N}v_{k,4}(M^{-1})_{kl}\hat{v}_{l,1}, \end{cases}\tag{4.16}$$

where $v_k=(v_{k,1},v_{k,2},v_{k,3},v_{k,4})^{\mathrm{T}}$ and $\hat{v}_k=(\hat{v}_{k,1},\hat{v}_{k,2},\hat{v}_{k,3},\hat{v}_{k,4})$, $1\le k\le N$, are defined by (4.13) and (4.14) involving $2N$ arbitrary constant vectors $v_{k,0}$'s and $\hat{v}_{k,0}$'s, respectively.

5 Concluding remarks

The paper is dedicated to a general formulation of Riemann–Hilbert problems and associated N-soliton solutions for integrable equations. An important step is to introduce a kind of equivalent matrix spectral problems, which guarantee the existence of bounded analytical eigenfunctions in the upper or lower half-plane. We considered a 4×4 degenerate AKNS matrix spatial spectral problem and generated the corresponding soliton hierarchy which has a bi-Hamiltonian structure. Taking the system of coupled fourth-order AKNS equations as an illustrative example, we built its associated Riemann–Hilbert problems and computed an explicit formula for the jump matrix. Upon taking the jump matrix to be the identity matrix in the presented Riemann–Hilbert problems, we worked out N-soliton solutions to the considered six-component system of coupled fourth-order AKNS equations.

The Riemann–Hilbert approach is quite effective in computing soliton solutions (see also, e.g., Xiao and Fan 2016; Geng and Wu 2016; Wang et al. 2010; Ma 2018b). Moreover, the approach has been successfully generalized to attempt initial-boundary value problems of integrable equations on the half-line (see, e.g., Fokas and Lenells 2012; Hu et al. 2018). There are many other approaches to soliton solutions in the field of integrable equations, which include the Hirota direct method (Hirota 2004), the generalized bilinear technique (Ma 2011), the Wronskian technique (Freeman and Nimmo 1983; Ma and You 2005) and the Darboux transformation (Matveev and Salle 1991). All kinds of connections among different approaches would be interesting and important. Moreover, about coupled mKdV equations, there exist many studies such as integrable couplings (Xu 2010; Wang et al. 2014), super hierarchies (Dong et al. 2015) and fractional analogous equations (Dong et al. 2016; Guo et al. 2018), and an important topic for further study is a Riemann–Hilbert formulation for solving those generalized integrable counterparts.

It is always interesting to look for other kinds of exact solutions to integrable equations, including position and complexiton solutions (Matveev 1992; Ma 2002), lump solutions (Satsuma and Ablowitz 1979; Ma et al. 2016b; Zhang et al. 2017; Ma and Zhou 2018), and algebro-geometric solutions (Belokolos et al. 1994; Gesztesy and Holden 2003), through applying Riemann–Hilbert techniques. It is hoped that our results could be helpful in recognizing those exact solutions from the perspective of Riemann–Hilbert problems. Particularly, interaction solutions (see, e.g., Ma et al. 2018) would deserve our further investigation by Riemann–Hilbert techniques.

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