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An Integrated Integrable Hierarchy Arising from a Broadened Ablowitz–Kaup–Newell–Segur Scenario

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Abstract: This study introduces a 4×4 matrix eigenvalue problem and develops an integrable hierarchy with a bi-Hamiltonian structure. Integrability is ensured by the zero-curvature condition, while the Hamiltonian structure is supported by the trace identity. Explicit derivations yield second-order and third-order integrable equations, illustrating the integrable hierarchy.

Keywords: matrix eigenvalue problem; Lax pair; zero-curvature equation; integrable model; bi-Hamiltonian formulation

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1. Introduction

In soliton theory, Lax pairs play a crucial role in exploring integrable models. The concept of a Lax pair [1] involves formulating a linear eigenvalue problem associated with a given nonlinear partial differential equation. By constructing an appropriate Lax pair, we can generate a compatible set of model equations that possess remarkable integrable properties, such as infinitely countless symmetries and conserved quantities. These equations exhibit soliton solutions, making them amenable to analytical techniques and providing deep insights into their dynamics [2,3].

To construct integrable models using Lax pairs, we typically start with a column potential vector u, and denote an eigenvalue parameter by k. The formulation entails defining a set of linear differential equations, referred to as the Lax pair, that are related through the compatibility condition, ensuring the integrability of the associated nonlinear equations. The Lax pair consists of two eigenvalue equations:

$$\phi_x = \mathcal{P}(u,k)\phi,\ \phi_t = \mathcal{Q}(u,k)\phi, \tag{1}$$

where ϕ is the eigenfunction, $\mathcal{P}(u,k)$ is the spatial spectral matrix, and $\mathcal{Q}(u,k)$ is the temporal spectral matrix. These matrices depend on both the potential vector u and the eigenvalue parameter k. The zero-curvature condition, or the compatibility condition, is given by

$$\mathcal{P}_t - \mathcal{Q}_x + [\mathcal{P},\mathcal{Q}] = 0, \tag{2}$$

where $[\mathcal{P},\mathcal{Q}] = \mathcal{P}\mathcal{Q} - \mathcal{Q}\mathcal{P}$ is the commutator of $\mathcal{P}$ and $\mathcal{Q}$. This condition ensures that the eigenfunction ϕ evolves consistently in both the spatial and temporal directions, leading to an integrable system.

To illustrate, consider the AKNS (Ablowitz–Kaup–Newell–Segur) system, which is a well-known framework for generating integrable equations. The AKNS system [4] defines $\mathcal{P}$ and $\mathcal{Q}$ as follows:

$$\mathcal{P}(u,k) = \begin{bmatrix} -k & p \\ q & k \end{bmatrix}, \ \mathcal{Q}(u,k) = \begin{bmatrix} -A & B \\ C & A \end{bmatrix},$$

where p and q are components of the potential vector u, and A, B and C are functions of u and k. The specific forms of A, B and C depend on the particular integrable model under consideration. The zero-curvature condition, Equation (2), then leads to a set of nonlinear partial differential equations for p and q. For instance, in the case of the nonlinear Schrödinger (NLS) equation, the condition results in

$$\begin{cases} p_t = -p_{xx} + 2p^2q, \\ q_t = q_{xx} - 2pq^2. \end{cases}$$

Solving these equations reveals the integrable structure of the system, characterized by soliton solutions, infinite symmetries, and conserved quantities.

By appropriately choosing $\mathcal{P}$ and $\mathcal{Q}$, one can derive various integrable models such as the sine-Gordon equation, the Korteweg–de Vries (KdV) equation, and others, all exhibiting remarkable integrable properties and amenable to powerful analytical techniques like the inverse scattering transform.

Hamiltonian structures are fundamental in the study of integrable systems, as they provide a framework for exploring the integrability of the resultant models. One method to generate Hamiltonian structures is by utilizing the trace identity or the variational identity. The trace identity, in particular, is a robust technique in this context.

The trace identity reads as follows (see [5] for details):

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(\mathcal{R}\frac{\partial \mathcal{P}}{\partial k}\big)\,dx = k^{-\tau}\frac{\partial}{\partial k}k^{\tau}\mathrm{tr}\big(\mathcal{R}\frac{\partial \mathcal{P}}{\partial u}\big), \tag{3}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u and tr stands for the trace of a matrix; τ remains invariant with respect to the eigenvalue parameter k. Here, $\mathcal{R}$ solves

$$\mathcal{R}_x = [\mathcal{P}, \mathcal{R}], \tag{4}$$

where $\mathcal{P}$ is the spectral matrix. The trace identity connects the variational derivative of an integral involving the eigenvalue parameter k to the trace of a matrix expression, linking the eigenvalue problem with the system's Hamiltonian structure.

A plethora of Liouville integrable hierarchies of soliton Hamiltonian equations can be derived using the aforementioned Lax pair formulation, utilizing loop algebras derived from both special linear algebras (see, for instance, [4–14]), special orthogonal algebras (see, for example, [15,16]), and non-semisimple Lie algebras (see, e.g., [17–24]). These hierarchies are pivotal in the study of integrable models, providing a structured framework to explore the solutions and properties of soliton equations.

This paper proposes a novel spectral matrix and constructs a Liouville integrable hierarchy comprising four-component bi-Hamiltonian equations using the Lax pair formulation. The resulting soliton equations exhibit established bi-Hamiltonian structures, demonstrated through the application of the trace identity. Several demonstrative examples are provided, including four-component coupled integrable NLS equations and modified Korteweg–de Vries (mKdV) equations. The conclusion in the final section summarizes the findings and offers summary remarks.

2. Commuting Integrable Hamiltonian Models

Motivated by a study on non-perturbation-type integrable couplings within the AKNS hierarchy via the Lax pair formulation [25], we consider a newly proposed matrix eigenvalue problem of the following form:

$$\phi_x = \mathcal{P}\phi = \mathcal{P}(u,k)\phi,\ \mathcal{P} = \begin{bmatrix} \xi_1 k & u_1 & \eta_1 k & u_3 \\ u_2 & \xi_2 k & u_4 & \eta_2 k \\ \eta_1 k & u_3 & \xi_1 k & u_1 \\ u_4 & \eta_2 k & u_2 & \xi_2 k \end{bmatrix}, \tag{5}$$

where k is again the eigenvalue parameter and u stands for the dependent variable consisting of four components:

$$u = u(x,t) = (u_1, u_2, u_3, u_4)^T. \tag{6}$$

If the bottom-left 2×2 block is taken to be zero, then this spectral problem with $\xi_1 = -\xi_2 = -1$ and $\eta_1 = -\eta_2 = -1$ becomes the one discussed in the above reference. To guarantee that an integrable hierarchy can be generated via the Lax pair formation from this new spectral problem, we need to impose a necessary and sufficient condition:

$$\xi^2 - \eta^2 \neq 0,\ \xi = \xi_1 - \xi_2,\ \eta = \eta_1 - \eta_2. \tag{7}$$

When $\eta_1 = \eta_2 = 0$ and $u_3 = u_4 = 0$, the spectral problem reduces to two identical copies of the standard AKNS eigenvalue problem [4], and thus, it provides a broadened version of the AKNS eigenvalue problem.

To establish a corresponding four-component Liouville integrable hierarchy, we initially solve the associated stationary zero-curvature Equation (4) by seeking a specific Laurent series solution:

$$\mathcal{R} = \begin{bmatrix} a & b & e & f \\ c & -a & g & -e \\ e & f & a & b \\ g & -e & c & -a \end{bmatrix} = \sum_{n\geq 0} k^{-n}\mathcal{R}^{\{n\}}, \tag{8}$$

with six fundamental components assumed to be expanded in Laurent series of the eigenvalue parameter k:

$$\begin{array}{l} a = \sum_{n\geq 0} k^{-n}a^{\{n\}},\ b = \sum_{n\geq 0} k^{-n}b^{\{n\}},\ c = \sum_{n\geq 0} k^{-n}c^{\{n\}}, \\ e = \sum_{n\geq 0} k^{-n}e^{\{n\}},\ f = \sum_{n\geq 0} k^{-n}f^{\{n\}},\ g = \sum_{n\geq 0} k^{-n}g^{\{n\}}. \end{array} \tag{9}$$

It is evident that the corresponding associated stationary zero-curvature Equation (4) leads to the following relations:

$$\begin{cases} a_x = u_1 c - u_2 b + u_3 g - u_4 f, \\ b_x = \xi k b + \eta k f - 2u_1 a - 2u_3 e, \\ c_x = -\xi k c - \eta k g + 2u_2 a + 2u_4 e, \\ e_x = u_1 g - u_2 f + u_3 c - u_4 b, \\ f_x = \eta k b + \xi k f - 2u_1 e - 2u_3 a, \\ g_x = -\eta k c - \xi k g + 2u_2 e + 2u_4 a. \end{cases} \tag{10}$$

This gives

$$k\begin{bmatrix} \xi & \eta \\ \eta & \xi \end{bmatrix}\begin{bmatrix} b \\ f \end{bmatrix} = \begin{bmatrix} b_x + 2u_1 a + 2u_3 e \\ f_x + 2u_1 e + 2u_3 a \end{bmatrix}$$

and

$$k\begin{bmatrix} \xi & \eta \\ \eta & \xi \end{bmatrix}\begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} -c_x + 2u_2 a + 2u_4 e \\ -g_x + 2u_2 e + 2u_4 a \end{bmatrix}.$$

Therefore, the condition, Equation (7), which guarantees the invertibility of the coefficient matrix in the above two systems, is both necessary and sufficient to ensure that we can recursively determine a Laurent series solution $\mathcal{R}$. Furthermore, we observe that the system, Equation (10), yields the initial requirements

$$a_x^{\{0\}} = e_x^{\{0\}} = 0,\ b^{\{0\}} = c^{\{0\}} = f^{\{0\}} = g^{\{0\}} = 0, \tag{11}$$

and the recursion relations used to define the Laurent series solution:

$$\begin{cases} b^{\{n+1\}} = \frac{\xi}{\xi^2-\eta^2}(b_x^{\{n\}} + 2u_1a^{\{n\}} + 2u_3e^{\{n\}}) - \frac{\eta}{\xi^2-\eta^2}(f_x^{\{n\}} + 2u_1e^{\{n\}} + 2u_3a^{\{n\}}), \\ f^{\{n+1\}} = -\frac{\eta}{\xi^2-\eta^2}(b_x^{\{n\}} + 2u_1a^{\{n\}} + 2u_3e^{\{n\}}) + \frac{\xi}{\xi^2-\eta^2}(f_x^{\{n\}} + 2u_1e^{\{n\}} + 2u_3a^{\{n\}}), \end{cases} \tag{12}$$

$$\begin{cases} c^{\{n+1\}} = \frac{\xi}{\xi^2-\eta^2}(-c_x^{\{n\}} + 2u_2a^{\{n\}} + 2u_4e^{\{n\}}) - \frac{\eta}{\xi^2-\eta^2}(-g_x^{\{n\}} + 2u_2e^{\{n\}} + 2u_4a^{\{n\}}), \\ g^{\{n+1\}} = -\frac{\eta}{\xi^2-\eta^2}(-c_x^{\{n\}} + 2u_2a^{\{n\}} + 2u_4e^{\{n\}}) + \frac{\xi}{\xi^2-\eta^2}(-g_x^{\{n\}} + 2u_2e^{\{n\}} + 2u_4a^{\{n\}}), \end{cases} \tag{13}$$

$$\begin{cases} a_x^{\{n+1\}} = u_1c^{\{n+1\}} - u_2b^{\{n+1\}} + u_3g^{\{n+1\}} - u_4f^{\{n+1\}}, \\ e_x^{\{n+1\}} = u_1g^{\{n+1\}} - u_2f^{\{n+1\}} + u_3c^{\{n+1\}} - u_4b^{\{n+1\}}, \end{cases} \tag{14}$$

where $n \geq 0$. As usual, to determine a specific Laurent series solution, we introduce arbitrary constant initial data

$$a^{\{0\}} = \frac{1}{2}\mu,\ e^{\{0\}} = \frac{1}{2}\nu, \tag{15}$$

and assume the integration constants to be zero:

$$a^{\{n\}}|_{u=0} = 0,\ e^{\{n\}}|_{u=0} = 0,\ n \geq 1. \tag{16}$$

Through these conditions, one can derive all sequences of $\{a^{\{n\}}, b^{\{n\}}, c^{\{n\}}, e^{\{n\}}, f^{\{n\}}, g^{\{n\}}\}$ for $n \geq 1$. The first sequence reads

$$\begin{cases} b^{\{1\}} = \frac{1}{\xi^2-\eta^2}[\xi(\mu u_1 + \nu u_3) - \eta(\nu u_1 + \mu u_3)], \\ f^{\{1\}} = \frac{1}{\xi^2-\eta^2}[\xi(\nu u_1 + \mu u_3) - \eta(\mu u_1 + \nu u_3)], \\ c^{\{1\}} = \frac{1}{\xi^2-\eta^2}[\xi(\mu u_2 + \nu u_4) - \eta(\nu u_2 + \mu u_4)], \\ g^{\{1\}} = \frac{1}{\xi^2-\eta^2}[\xi(\nu u_2 + \mu u_4) - \eta(\mu u_2 + \nu u_4)], \\ a^{\{1\}} = e^{\{1\}} = 0. \end{cases}$$

The second sequence reads

$$\begin{cases} b^{\{2\}} = \frac{1}{(\xi^2-\eta^2)^2}(p_{2,1}u_{1,x} - p_{2,2}u_{3,x}), \\ f^{\{2\}} = \frac{1}{(\xi^2-\eta^2)^2}(-p_{2,2}u_{1,x} + p_{2,1}u_{3,x}), \\ c^{\{2\}} = \frac{1}{(\xi^2-\eta^2)^2}(-p_{2,1}u_{2,x} - p_{2,2}u_{4,x}), \\ g^{\{2\}} = \frac{1}{(\xi^2-\eta^2)^2}(-p_{2,2}u_{2,x} - p_{2,1}u_{4,x}), \end{cases}$$

$$\begin{cases} a^{\{2\}} = -\frac{1}{(\xi^2-\eta^2)^2}[(p_{2,1}u_2 + p_{2,2}u_4)u_1 + (p_{2,2}u_2 + p_{2,1}u_4)u_3], \\ e^{\{2\}} = -\frac{1}{(\xi^2-\eta^2)^2}[(p_{2,2}u_2 + p_{2,1}u_4)u_1 + (p_{2,1}u_2 + p_{2,2}u_4)u_3], \end{cases}$$

where $p_{2,1}$ and $p_{2,2}$ are two special polynomials of second order in terms of ξ and η:

$$p_{2,1} = \xi^2\mu - 2\xi\eta\nu + \eta^2\mu,\ p_{2,2} = \xi^2\nu - 2\xi\eta\mu + \eta^2\nu. \tag{17}$$

The third sequence reads

$$\begin{cases} b^{\{3\}} = \frac{1}{(\xi^2-\eta^2)^3}[p_{3,1}u_{1,xx} + p_{3,2}u_{3,xx} - 2(p_{3,1}u_2 + p_{3,2}u_4)u_1^2 \\ \qquad -4(p_{3,2}u_2 + p_{3,1}u_4)u_1u_3 - 2(p_{3,1}u_2 + p_{3,2}u_4)u_3^2], \\ f^{\{3\}} = \frac{1}{(\xi^2-\eta^2)^3}[p_{3,2}u_{1,xx} + p_{3,1}u_{3,xx} - 2(p_{3,2}u_2 + p_{3,1}u_4)u_1^2 \\ \qquad -4(p_{3,1}u_2 + p_{3,2}u_4)u_1u_3 - 2(p_{3,2}u_2 + p_{3,1}u_4)u_3^2], \end{cases}$$

$$\left\{\begin{array}{l} c^{\{3\}} = \frac{1}{(\xi^2-\eta^2)^3}[p_{3,1}u_{2,xx} + p_{3,2}u_{4,xx} - 2(p_{3,1}u_1 + p_{3,2}u_3)u_2^2 \\ \qquad -4(p_{3,2}u_1 + p_{3,1}u_3)u_2u_4 - 2(p_{3,1}u_1 + p_{3,2}u_3)u_4^2], \\ g^{\{3\}} = \frac{1}{(\xi^2-\eta^2)^3}[p_{3,2}u_{2,xx} + p_{3,1}u_{4,xx} - 2(p_{3,2}u_1 + p_{3,1}u_3)u_2^2 \\ \qquad -4(p_{3,1}u_1 + p_{3,2}u_3)u_2u_4 - 2(p_{3,2}u_1 + p_{3,1}u_3)u_4^2], \end{array}\right.$$

$$\left\{\begin{array}{l} a^{\{3\}} = \frac{1}{(\xi^2-\eta^2)^3}[-(p_{3,1}u_2 + p_{3,2}u_4)u_{1,x} + (p_{3,1}u_1 + p_{3,2}u_3)u_{2,x} \\ \qquad -(p_{3,2}u_2 + p_{3,1}u_4)u_{3,x} + (p_{3,2}u_1 + p_{3,1}u_3)u_{4,x}], \\ e^{\{3\}} = \frac{1}{(\xi^2-\eta^2)^3}[(-p_{3,2}u_2 - p_{3,1}u_4)u_{1,x} + (p_{3,2}u_1 + p_{3,1}u_3)u_{2,x} \\ \qquad -(p_{3,1}u_2 + p_{3,2}u_4)u_{3,x} + (p_{3,1}u_1 + p_{3,2}u_3)u_{4,x}], \end{array}\right.$$

where $p_{3,1}$ and $p_{3,2}$ are two special polynomials of third order in terms of ξ and η:

$$p_{3,1} = \xi^3\mu - 3\xi^2\eta\nu + 3\xi\eta^2\mu - \eta^3\nu,\ p_{3,2} = \xi^3\nu - 3\xi^2\eta\mu + 3\xi\eta^2\nu - \eta^3\mu. \tag{18}$$

The fourth sequence reads

$$\left\{\begin{array}{l} b^{\{4\}} = \frac{1}{(\xi^2-\eta^2)^4}[p_{4,1}u_{1,xxx} + p_{4,2}u_{3,xxx} \\ \qquad -6(p_{4,1}u_1u_2 + p_{4,2}u_1u_4 + p_{4,2}u_2u_3 + p_{4,1}u_3u_4)u_{1,x} \\ \qquad -6(p_{4,2}u_1u_2 + p_{4,1}u_1u_4 + p_{4,1}u_2u_3 + p_{4,2}u_3u_4)u_{3,x}], \\ f^{\{4\}} = \frac{1}{(\xi^2-\eta^2)^4}[p_{4,2}u_{1,xxx} + p_{4,1}u_{3,xxx} \\ \qquad -6(p_{4,2}u_1u_2 + p_{4,1}u_1u_4 + p_{4,1}u_2u_3 + p_{4,2}u_3u_4)u_{1,x} \\ \qquad -6(p_{4,1}u_1u_2 + p_{4,2}u_1u_4 + p_{4,2}u_2u_3 + p_{4,1}u_3u_4)u_{3,x}], \end{array}\right.$$

$$\left\{\begin{array}{l} c^{\{4\}} = \frac{1}{(\xi^2-\eta^2)^4}[-p_{4,1}u_{2,xxx} - p_{4,2}u_{4,xxx} \\ \qquad +6(p_{4,1}u_1u_2 + p_{4,2}u_1u_4 + p_{4,2}u_2u_3 + p_{4,1}u_3u_4)u_{2,x} \\ \qquad +6(p_{4,2}u_1u_2 + p_{4,1}u_1u_4 + p_{4,1}u_2u_3 + p_{4,2}u_3u_4)u_{4,x}], \\ g^{\{4\}} = \frac{1}{(\xi^2-\eta^2)^4}[-p_{4,2}u_{2,xxx} - p_{4,1}u_{4,xxx} \\ \qquad +6(p_{4,2}u_1u_2 + p_{4,1}u_1u_4 + p_{4,1}u_2u_3 + p_{4,2}u_3u_4)u_{2,x} \\ \qquad +6(p_{4,1}u_1u_2 + p_{4,2}u_1u_4 + p_{4,2}u_2u_3 + p_{4,1}u_3u_4)u_{4,x}], \end{array}\right.$$

$$\begin{array}{l} a^{\{4\}} = \frac{1}{(\xi^2-\eta^2)^4}[-(p_{4,1}u_2 + p_{4,2}u_4)u_{1,xx} - (p_{4,1}u_1 + p_{4,2}u_3)u_{2,xx} \\ \quad -(p_{4,2}u_2 + p_{4,1}u_4)u_{3,xx} - (p_{4,2}u_1 + p_{4,1}u_3)u_{4,xx} \\ \quad +p_{4,1}u_{1,x}u_{2,x} + p_{4,2}u_{1,x}u_{4,x} + p_{4,2}u_{2,x}u_{3,x} + p_{4,1}u_{3,x}u_{4,x} \\ \quad +3(p_{4,1}u_2^2 + 2p_{4,2}u_2u_4 + p_{4,1}u_4^2)u_1^2 \\ \quad +6(p_{4,2}u_2^2 + 2p_{4,1}u_2u_4 + p_{4,2}u_4^2)u_1u_3 \\ \quad -3(p_{4,1}u_2^2 - 2p_{4,2}u_2u_4 - p_{4,1}u_4^2)u_3^2], \end{array}$$

$$\begin{array}{l} e^{\{4\}} = \frac{1}{(\xi^2-\eta^2)^4}[-(p_{4,2}u_2 + p_{4,1}u_4)u_{1,xx} - (p_{4,2}u_1 + p_{4,1}u_3)u_{2,xx} \\ \quad -(p_{4,1}u_2 + p_{4,2}u_4)u_{3,xx} - (p_{4,1}u_1 + p_{4,2}u_3)u_{4,xx} \\ \quad +p_{4,2}u_{1,x}u_{2,x} + p_{4,1}u_{1,x}u_{4,x} + p_{4,1}u_{2,x}u_{3,x} + p_{4,2}u_{3,x}u_{4,x} \\ \quad +3(p_{4,2}u_2^2 + 2p_{4,1}u_2u_4 + p_{4,2}u_4^2)u_1^2 \\ \quad +6(p_{4,1}u_2^2 + 2p_{4,2}u_2u_4 + p_{4,1}u_4^2)u_1u_3 \\ \quad +3(p_{4,2}u_2^2 + 2p_{4,1}u_2u_4 + p_{4,2}u_4^2)u_3^2], \end{array}$$

where $p_{4,1}$ and $p_{4,2}$ are two special polynomials of fourth order in terms of ξ and η:

$$\left\{\begin{array}{l} p_{4,1} = \xi^4\mu - 4\xi^3\eta\nu + 6\xi^2\eta^2\mu - 4\xi\eta^3\nu + \eta^4\mu, \\ p_{4,2} = \xi^4\nu - 4\xi^3\eta\mu + 6\xi^2\eta^2\nu - 4\xi\eta^3\mu + \eta^4\nu. \end{array}\right. \tag{19}$$

On the basis of these computations, we can set $\Delta_m = 0,\ m \geq 0$, to formulate

$$\phi_{t_m} = \mathcal{Q}^{\{m\}}\phi = \mathcal{Q}^{\{m\}}(u,k)\phi,\ \mathcal{Q}^{\{m\}} = (k^m\mathcal{R})_+ = \sum_{n=0}^{m} k^n\mathcal{R}^{\{m-n\}},\ m \geq 0. \tag{20}$$

These are the temporal matrix eigenvalue problems within the Lax pair formulation. The conditions ensuring solvability for the spatial and temporal matrix eigenvalue problems in Equations (5) and (20) are given by the following zero-curvature equations:

$$\mathcal{P}_{t_m} - \mathcal{Q}_x^{\{m\}} + [\mathcal{P}, \mathcal{Q}^{\{m\}}] = 0,\ m \geq 0. \tag{21}$$

These compatibility equations engender a hierarchy of integrable models with four dependent variables:

$$\begin{aligned} u_{t_m} = \mathcal{Z}^{\{m\}} = (&\xi b^{\{m+1\}} + \eta f^{\{m+1\}}, -\xi c^{\{m+1\}} - \eta g^{\{m+1\}}, \\ &\eta b^{\{m+1\}} + \xi f^{\{m+1\}}, -\eta c^{\{m+1\}} - \xi g^{\{m+1\}})^T, \end{aligned} \tag{22}$$

or more concretely,

$$\begin{cases} u_{1,t_m} = \xi b^{\{m+1\}} + \eta f^{\{m+1\}}, \\ u_{2,t_m} = -\xi c^{\{m+1\}} - \eta g^{\{m+1\}}, \\ u_{3,t_m} = \eta b^{\{m+1\}} + \xi f^{\{m+1\}}, \\ u_{4,t_m} = -\eta c^{\{m+1\}} - \xi g^{\{m+1\}}, \end{cases} \tag{23}$$

in which $m \geq 0$.

As particular examples, this soliton hierarchy contains various coupled systems of integrable NLS equations and coupled systems of integrable mKdV equations. If taking

$$\xi = 1,\ \eta = 0,\ \mu = 1,\ \nu = 0, \tag{24}$$

one obtains a coupled system of the following integrable NLS equations:

$$\begin{cases} u_{1,t_2} = u_{1,xx} - 2u_1^2 u_2 - 4u_1 u_3 u_4 - 2u_2 u_3^2, \\ u_{2,t_2} = -u_{2,xx} + 2u_1 u_2^2 + 2u_1 u_4^2 + 4u_2 u_3 u_4, \\ u_{3,t_2} = u_{3,xx} - 2u_1^2 u_4 - 4u_1 u_2 u_3 - 2u_3^2 u_4, \\ u_{4,t_2} = -u_{4,xx} + 4u_1 u_2 u_4 + 2u_2^2 u_3 + 2u_3 u_4^2, \end{cases} \tag{25}$$

and a coupled system of integrable mKdV equations:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} - 6(u_1 u_2 + u_3 u_4) u_{1,x} - 6(u_1 u_4 + u_2 u_3) u_{3,x}, \\ u_{2,t_3} = u_{2,xxx} - 6(u_1 u_2 + u_3 u_4) u_{2,x} - 6(u_1 u_4 + u_2 u_3) u_{4,x}, \\ u_{3,t_3} = u_{3,xxx} - 6(u_1 u_4 + u_2 u_3) u_{1,x} - 6(u_1 u_2 + u_3 u_4) u_{3,x}, \\ u_{4,t_3} = u_{4,xxx} - 6(u_1 u_4 + u_2 u_3) u_{2,x} - 6(u_1 u_2 + u_3 u_4) u_{4,x}. \end{cases} \tag{26}$$

If taking

$$\xi = 1,\ \eta = 0,\ \mu = 1,\ \nu = 1, \tag{27}$$

one obtains a coupled system of the following combined integrable NLS equations:

$$\begin{cases} u_{1,t_2} = u_{1,xx} + u_{3,xx} - 2(u_2 + u_4) u_1^2 - 4(u_2 + u_4) u_1 u_3 - 2(u_2 + u_4) u_3^2, \\ u_{2,t_2} = -u_{2,xx} - u_{4,xx} + 2(u_1 + u_3) u_2^2 + 4(u_1 + u_3) u_2 u_4 + 2(u_1 + u_3) u_4^2, \\ u_{3,t_2} = u_{1,xx} + u_{3,xx} - 2(u_2 + u_4) u_1^2 - 4(u_2 - u_4) u_1 u_3 - 2(u_2 + u_4) u_3^2, \\ u_{4,t_2} = -u_{2,xx} - u_{4,xx} + 2(u_1 + u_3) u_2^2 + 4(u_1 + u_3) u_2 u_4 + 2(u_1 + u_3) u_4^2, \end{cases} \tag{28}$$

and a coupled system of the following combined integrable mKdV equations:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} + u_{3,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{1,x} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{3,x}, \\ u_{2,t_3} = u_{2,xxx} + u_{4,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{2,x} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{4,x}, \\ u_{3,t_3} = u_{1,xxx} + u_{3,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{1,x} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{3,x}, \\ u_{4,t_3} = u_{2,xxx} + u_{4,xxx} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{2,x} - 6[u_1(u_2 + u_4) + u_3(u_2 + u_4)] u_{4,x}. \end{cases} \tag{29}$$

These four systems represent typical coupled integrable models, expanding the class of coupled integrable NLS equations and mKdV equations (see, e.g., [26,27]).

3. Bi-Hamiltonian Structures

The introduction of bi-Hamiltonian structures into the soliton hierarchy, Equation (23), can be achieved by employing the classical trace identity, Equation (3), on the spatial matrix eigenvalue problem, Equation (5).

The trace identity uses the solution $\mathcal{R}$ defined by Equation (8). One can then readily work out the Hamiltonian structures for the resultant hierarchy of soliton models. Through applying the classical trace identity to the spatial matrix eigenvalue problem, the Hamiltonian densities and the associated flows can be systematically derived. Concretely, we have

$$\mathrm{tr}\big(\mathcal{R}\frac{\partial \mathcal{P}}{\partial k}\big) = 2\xi a + 2\eta e,\ \mathrm{tr}\big(\mathcal{R}\frac{\partial \mathcal{P}}{\partial u}\big) = (2c, 2b, 2g, 2f)^T, \tag{30}$$

and consequently, the classical trace identity gives

$$\frac{\delta}{\delta u}\int k^{-(n+1)}(\xi a^{\{n+1\}} + \eta e^{\{n+1\}})\,dx = k^{-\tau}\frac{\partial}{\partial k}k^{\tau-n}(c^{\{n\}}, b^{\{n\}}, g^{\{n\}}, f^{\{n\}})^T,\ n \geq 0. \tag{31}$$

When checked with $n = 2$, it results in $\tau = 0$, and as a consequence, one obtains

$$\frac{\delta}{\delta u}\mathcal{H}^{\{n\}} = (c^{\{n+1\}}, b^{\{n+1\}}, g^{\{n+1\}}, f^{\{n+1\}})^T,\ n \geq 0, \tag{32}$$

in which the required Hamiltonian quantities are computed as follows:

$$\mathcal{H}^{\{n\}} = -\int \frac{\xi a^{\{n+2\}} + \eta e^{\{n+2\}}}{n+1}\,dx,\ n \geq 0. \tag{33}$$

This allows us to establish the Hamiltonian structures for the soliton hierarchy, Equation (23):

$$u_{t_m} = \mathcal{Z}^{\{m\}} = J_1\frac{\delta \mathcal{H}^{\{m\}}}{\delta u},\ J_1 = \begin{bmatrix} 0 & \xi & 0 & \eta \\ -\xi & 0 & -\eta & 0 \\ 0 & \eta & 0 & \xi \\ -\eta & 0 & -\xi & 0 \end{bmatrix},\ m \geq 0, \tag{34}$$

where J_1 is, obviously, Hamiltonian, and $\mathcal{H}^{[m]}$ are the functionals defined by Equation (33). It is important to note that Hamiltonian structures exhibit a significant property, namely, the interrelation $S = J_1\frac{\delta\mathcal{H}}{\delta u}$ between a conserved quantity $\mathcal{H}$ and a symmetry S within the same nonlinear model.

The standard soliton theory expresses that those vector fields $\mathcal{Z}^{\{n\}}$ commute

$$[\![\mathcal{Z}^{\{n_1\}}, \mathcal{Z}^{\{n_2\}}]\!] = \mathcal{Z}^{\{n_1\}}{}'(u)[\mathcal{Z}^{\{n_2\}}] - \mathcal{Z}^{\{n_2\}}{}'(u)[\mathcal{Z}^{\{n_1\}}] = 0,\ n_1, n_2 \geq 0, \tag{35}$$

which can bee seen from an algebra of temporal spectral matrices:

$$[\![\mathcal{Q}^{\{n_1\}}, \mathcal{Q}^{\{n_2\}}]\!] = \mathcal{Q}^{\{n_1\}}{}'(u)[\mathcal{Z}^{\{n_2\}}] - \mathcal{Q}^{\{n_2\}}{}'(u)[\mathcal{Z}^{\{n_1\}}] + [\mathcal{Q}^{\{n_1\}}, \mathcal{Q}^{\{n_2\}}] = 0,\ n_1, n_2 \geq 0. \tag{36}$$

One can also verify this property directly by analyzing the relationship between the isospectral zero-curvature equations.

Moreover, utilizing the recursion relation $\mathcal{Z}^{m+1} = \Phi\mathcal{Z}^m$, a straightforward yet lengthy computation results in a recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$, which is established as hereditary [28], for the soliton hierarchy, Equation (23). This hereditary recursion operator Φ reads

$$\begin{cases} \Phi_{11} = \frac{1}{\xi^2-\eta^2}(\xi\partial - 2\xi u_1\partial^{-1}u_2 - 2\xi u_3\partial^{-1}u_4 + 2\eta u_1\partial^{-1}u_4 + 2\eta u_3\partial^{-1}u_2), \\ \Phi_{12} = \frac{1}{\xi^2-\eta^2}(-2\xi u_1\partial^{-1}u_1 - 2\xi u_3\partial^{-1}u_3 + 2\eta u_1\partial^{-1}u_3 + 2\eta u_3\partial^{-1}u_1), \\ \Phi_{13} = \frac{1}{\xi^2-\eta^2}(-\eta\partial + 2\xi u_1\partial^{-1}u_4 - 2\xi u_3\partial^{-1}u_2 + 2\eta u_1\partial^{-1}u_2 + 2\eta u_3\partial^{-1}u_4), \\ \Phi_{14} = \frac{1}{\xi^2-\eta^2}(2\xi u_1\partial^{-1}u_3 - 2\xi u_3\partial^{-1}u_1 + 2\eta u_1\partial^{-1}u_1 + 2\eta u_3\partial^{-1}u_3); \end{cases} \tag{37}$$

$$\begin{cases} \Phi_{21} = \frac{1}{\xi^2-\eta^2}(2\xi u_2\partial^{-1}u_2 + 2\xi u_4\partial^{-1}u_4 - 2\eta u_2\partial^{-1}u_4 - 2\eta u_4\partial^{-1}u_2), \\ \Phi_{22} = \frac{1}{\xi^2-\eta^2}(-\xi\partial + 2\xi u_2\partial^{-1}u_1 + 2\xi u_4\partial^{-1}u_3 - 2\eta u_2\partial^{-1}u_3 - 2\eta u_4\partial^{-1}u_1), \\ \Phi_{23} = \frac{1}{\xi^2-\eta^2}(2\xi u_2\partial^{-1}u_4 + 2\xi u_4\partial^{-1}u_2 - 2\eta u_2\partial^{-1}u_2 - 2\eta u_4\partial^{-1}u_4), \\ \Phi_{24} = \frac{1}{\xi^2-\eta^2}(\eta\partial + 2\xi u_2\partial^{-1}u_3 + 2\xi u_4\partial^{-1}u_1 - 2\eta u_2\partial^{-1}u_1 - 2\eta u_4\partial^{-1}u_3); \end{cases} \tag{38}$$

$$\begin{cases} \Phi_{31} = \frac{1}{\xi^2-\eta^2}(-\eta\partial - 2\xi u_1\partial^{-1}u_4 - 2\xi u_3\partial^{-1}u_2 + 2\eta u_1\partial^{-1}u_2 + 2\eta u_3\partial^{-1}u_4), \\ \Phi_{32} = \frac{1}{\xi^2-\eta^2}(-2\xi u_1\partial^{-1}u_3 - 2\xi u_3\partial^{-1}u_1 + 2\eta u_1\partial^{-1}u_1 + 2\eta u_3\partial^{-1}u_3), \\ \Phi_{33} = \frac{1}{\xi^2-\eta^2}(\xi\partial - 2\xi u_1\partial^{-1}u_2 - 2\xi u_3\partial^{-1}u_4 + 2\eta u_1\partial^{-1}u_4 + 2\eta u_3\partial^{-1}u_2), \\ \Phi_{34} = \frac{1}{\xi^2-\eta^2}(-2\xi u_1\partial^{-1}u_1 - 2\xi u_3\partial^{-1}u_3 + 2\eta u_1\partial^{-1}u_3 + 2\eta u_3\partial^{-1}u_1); \end{cases} \tag{39}$$

and

$$\begin{cases} \Phi_{41} = \frac{1}{\xi^2-\eta^2}(2\xi u_2\partial^{-1}u_4 + 2\xi u_4\partial^{-1}u_2 - 2\eta u_2\partial^{-1}u_2 - 2\eta u_4\partial^{-1}u_4), \\ \Phi_{42} = \frac{1}{\xi^2-\eta^2}(\eta\partial + 2\xi u_2\partial^{-1}u_3 + 2\xi u_4\partial^{-1}u_1 - 2\eta u_2\partial^{-1}u_1 - 2\eta u_4\partial^{-1}u_3), \\ \Phi_{43} = \frac{1}{\xi^2-\eta^2}(2\xi u_2\partial^{-1}u_2 + 2\xi u_4\partial^{-1}u_4 - 2\eta u_2\partial^{-1}u_4 - 2\eta u_4\partial^{-1}u_2), \\ \Phi_{44} = \frac{1}{\xi^2-\eta^2}(-\xi\partial + 2\xi u_2\partial^{-1}u_1 + 2\xi u_4\partial^{-1}u_3 - 2\eta u_2\partial^{-1}u_3 - 2\eta u_4\partial^{-1}u_1). \end{cases} \tag{40}$$

Let us show the idea for computing this recursion operator using the recursion relations in Equations (12)–(14). Assume that $\mathcal{Z}^{\{m\}} = (\mathcal{Z}_1^{\{m\}}, \mathcal{Z}_2^{\{m\}}, \mathcal{Z}_3^{\{m\}}, \mathcal{Z}_4^{\{m\}})^T$, $m \geq 0$. Here, we focus solely on the third component of $\mathcal{Z}^{\{m+1\}}$ and perform the following computation:

$$\begin{aligned} &\mathcal{Z}_3^{\{m+1\}} = \eta b^{\{m+2\}} + \xi f^{\{m+2\}} \\ &= \tfrac{1}{\xi^2-\eta^2}[\xi\eta(b_x^{\{m+1\}} + 2u_1 a^{\{m+1\}} + 2u_3 e^{\{m+1\}}) - \eta^2(f_x^{\{m+1\}} + 2u_1 e^{\{m+1\}} + 2u_3 a^{\{m+1\}}) \\ &\qquad -\xi\eta(b_x^{\{m+1\}} + 2u_1 a^{\{m+1\}} + 2u_3 e^{\{m+1\}}) + \xi^2(f_x^{\{m+1\}} + 2u_1 e^{\{m+1\}} + 2u_3 a^{\{m+1\}})] \\ &= \tfrac{1}{\xi^2-\eta^2}[\xi\mathcal{Z}_{3,x}^{\{m\}} - \eta\mathcal{Z}_{1,x}^{\{m\}} + 2(\xi u_1 - \eta u_3)(\eta a^{\{m+1\}} + \xi e^{\{m+1\}}) \\ &\qquad +2(\xi u_3 - \eta u_1)(\eta e^{\{m+1\}} + \xi a^{\{m+1\}})]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \eta a^{\{m+1\}} + \xi e^{\{m+1\}} &= -\partial^{-1}(u_1\mathcal{Z}_4^{\{m\}} + u_2\mathcal{Z}_3^{\{m\}} + u_3\mathcal{Z}_2^{\{m\}} + u_4\mathcal{Z}_1^{\{m\}}), \\ \eta e^{\{m+1\}} + \xi a^{\{m+1\}} &= -\partial^{-1}(u_1\mathcal{Z}_2^{\{m\}} + u_2\mathcal{Z}_1^{\{m\}} + u_3\mathcal{Z}_4^{\{m\}} + u_4\mathcal{Z}_3^{\{m\}}). \end{aligned}$$

All this yields the third row of the recursion operator Φ, defined by Equation (39). The remaining rows can be derived in a completely similar manner.

The recursion operator, defined by Equations (37)–(40), involves two constant parameters, ξ and η, which are not simultaneously zero, exhibiting the diversity of the recursion structure in the integrable hierarchy. Despite the nonlocality of the recursion operator, the locality of the isospectral ($k_{t_m} = 0$) flows is maintained. This implies that each flow in the hierarchy preserves the integrable structure, ensuring that the derived soliton equations remain solvable by inverse scattering techniques and other methods applicable to local equations.

Further, with some detailed analysis, we can see that J_1 and $J_2 = \Phi J_1$ form a Hamiltonian pair. Therefore, the soliton hierarchy Equation (23) exhibits the following bi-Hamiltonian structures [29]:

$$u_{t_m} = \mathcal{Z}^{\{m\}} = J_1\frac{\delta\mathcal{H}^{\{m\}}}{\delta u} = J_2\frac{\delta\mathcal{H}^{\{m-1\}}}{\delta u}, \ \ m \geq 1. \tag{41}$$

It can then be observed that the resulting Hamiltonian quantities commute under their respective Poisson brackets:

$$\{\mathcal{H}^{\{n_1\}}, \mathcal{H}^{\{n_2\}}\}_{J_i} = 0, \ n_1, n_2 \geq 0, \ i = 1, 2, \tag{42}$$

where

$$\{\mathcal{H},\mathcal{K}\}_{J_i} = \int (\frac{\delta\mathcal{H}}{\delta u})^T J_i \frac{\delta\mathcal{K}}{\delta u}\, dx,\ i = 1,2. \tag{43}$$

The two equalities established above, Equations (35) and (42), imply that all isospectral flows possess infinitely many conserved quantities and symmetries inherent to the integrable system. Additionally, based on the recursion and bi-Hamiltonian structures, the resulting conserved quantities and symmetries can be effectively computed and utilized. This property is crucial for the practical application and analysis of these integrable models, as it ensures that the solutions exhibit well-defined physical behavior and can be systematically studied.

In summary, the soliton hierarchy, Equation (23), exhibits specific bi-Hamiltonian structures, demonstrating Liouville integrability. Each model features infinitely many commuting conserved quantities $\mathcal{H}^{\{n\}}{}_{n=0}^{\infty}$ and symmetries $\mathcal{Z}^{\{n\}}{}_{n=0}^{\infty}$. The concrete examples provided in Equations (25), (26), (28), and (29) highlight special nonlinear coupled Liouville integrable models with bi-Hamiltonian structures, contributing to the ongoing discourse in the literature (see, for instance, [30–35]).

4. Concluding Remarks

This research explores integrable hierarchies and their relationship to specific matrix eigenvalue problems formulated under zero-curvature. Generating integrable models with bi-Hamiltonian structures is essential for comprehending the dynamics inherent in these systems.

Employing Laurent series solutions for solving the stationary zero-curvature equation proves to be a robust method, enabling researchers to reveal the integrability characteristics of the models under investigation. Furthermore, applying the trace identity to the matrix isospectral eigenvalue problem provides deeper insights into the bi-Hamiltonian structures embedded within these systems.

The concrete examples presented provide specific coupled systems of nonlinear uncombined and combined integrable models. These examples, which belong to M_2-extensions [36], demonstrate the practical application of the theoretical framework discussed earlier and highlight the integrability and rich structure of the resulting equations.

Exploring the structures of explicit soliton solutions in the resulting integrable models is of interest, employing advanced methods in soliton theory such as the Zakharov–Shabat dressing method [37], the Riemann–Hilbert technique [38], the determinant approach [39], and the Darboux transformation (see, e.g., [40–44]). Additionally, other significant solutions including breather, kink, anti-kink, lump and rogue wave solutions, as well as the corresponding mixed solutions (see, e.g., [45–52]), can be derived from specific wave number reductions of solitons. Novel reduced integrable equations involving reflection points can also be obtained through nonlocal reduced matrix eigenvalue problems under similarity transformations (see, e.g., [53]).

Most certainly, increasing the number of dependent variables in the spatial spectral matrix can indeed lead to the generation of larger integrable models (see, e.g., [54–56]). However, it is worth noting that as the number of dependent variables increases, the complexity of the resulting equations also grows. This can make the analysis and understanding of the system more challenging. Nevertheless, the study of larger integrable models remains a fruitful area of research [36,53], offering insights into the fundamental principles governing nonlinear dynamics and integrability in mathematical physics.

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