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Integrable Couplings and Two-Dimensional Unital Algebras

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Abstract: The paper aims to demonstrate that a linear expansion in a unital two-dimensional algebra can generate integrable couplings, proposing a novel approach for their construction. The integrable couplings presented encompass a range of perturbation equations and nonlinear integrable couplings. Their corresponding Lax pairs and hereditary recursion operators are explicitly detailed. Concrete applications to the KdV equation and the AKNS system of nonlinear Schrödinger equations are extensively explored.

Keywords: integrable couplings; two-dimensional algebras; Lax pairs; recursion operators; integrability

MSC: 37K15; 35Q55

1. Introduction

Integrable equations represent a crucial class of partial differential equations in mathematical physics. Derived from Lax pairs of matrix spectral problems, they exhibit distinctive features including infinitely many symmetries, conservation laws, and hereditary recursion operators. Integrable couplings [1], a specific type of integrable equation, are characterized by their matrix spectral problems being associated with non-semisimple Lie algebras [2].

Researchers have explored integrable couplings across different types of nonlinear wave models, including both discrete and continuous ones (see, e.g., [3–9]). A particular class of integrable couplings are perturbation equations [1]. It is known in various previous studies (see, e.g., [1,3–9]) that a spectral matrix of the form

$$\hat{U} = \begin{bmatrix} U(u) & U'(u)[v] \\ 0 & U(u) \end{bmatrix}, \tag{1}$$

or equivalently,

$$\hat{U} = \begin{bmatrix} U(u) & 0 \\ U'(u)[v] & U(u) \end{bmatrix}, \tag{2}$$

where the block U is a spectral matrix associated with a given integrable equation $u_t = K(u)$ and the block U' denotes its Gateaux derivative, generates an integrable coupling of the perturbation type:

$$u_t = K(u),\ v_t = K'(u)[v]. \tag{3}$$

In this perturbation-type coupling (3), the second equation for the variable v is linear with respect to v, while u is fixed. If the second equation in an integrable coupling,

$$u_t = K(u),\ v_t = S(u,v), \tag{4}$$

defines a nonlinear equation for v, then the system (4) is called a nonlinear integrable coupling [10].

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Linear integrable couplings contain extensions of symmetry equations (see, e.g., [1,11–13]) and are important in classifying integrable equations, but definitely, nonlinear ones possess much richer structures. There are a few systematical ways to construct linear integrable couplings, stemming from the perturbed spectral matrices defined as before and the enlarged spectral matrices [14]:

$$\hat{U} = \begin{bmatrix} U(u) & U_{1,a}(v) \\ 0 & 0 \end{bmatrix} \text{ and } \hat{U} = \begin{bmatrix} U(u) & 0 \\ U_{2,a}(v) & 0 \end{bmatrix}, \tag{5}$$

where $U_{1,a}$ and $U_{2,a}$ may not be square matrices. There is a feasible way which enables us to construct nonlinear integrable couplings from the choices of spectral matrices [10,15]:

$$\hat{U} = \begin{bmatrix} U(u) & U_a(v) \\ 0 & U(u) + U_a(v) \end{bmatrix} \text{ or } \hat{U} = \begin{bmatrix} U(u) & 0 \\ U_a(v) & U(u) + U_a(v) \end{bmatrix}. \tag{6}$$

In this paper, we consider the connection between integrable couplings and two-dimensional unital algebras. We demonstrate that all unital algebras of dimension two yield two classes of integrable couplings, one of which is of perturbation type and the other of nonlinear type. This provides an approach for constructing integrable couplings. We illustrate the general idea with the KdV equation and the AKNS system of nonlinear Schrödinger equations. All results will enrich the existing theories of integrable couplings.

2. Integrable Couplings via Two-Dimensional Algebras

There are two unital associative algebras of dimension two over the complex number field [16]. Let the identity element be denoted by **1**. Each of those two algebras consists of linear combinations of two basis elements: **1** and a. According to the definition of the identity element,

$$\mathbf{1} \cdot \mathbf{1} = \mathbf{1},\ \mathbf{1} \cdot a = a,\ a \cdot \mathbf{1} = a. \tag{7}$$

The other properties are

$$aa = 0, \tag{8}$$

for the first algebra and

$$aa = a \tag{9}$$

for the second algebra, where we denote $a \cdot a$ by aa for convenience.

Let us take an integrable equation

$$u_t = K(u, u_x, \ldots, u^{(n)}), \tag{10}$$

where u is a column vector of dependent variables and $u^{(n)}$ means the n-th derivative with respect to x. Assume that it possesses a Lax pair of matrix spectral problems:

$$\phi_x = U(u, \lambda)\phi,\ \phi_t = V(u, \lambda)\phi, \tag{11}$$

and a hereditary recursion operator $\Phi(u)$:

$$L_{\Phi X}\Phi = \Phi L_X \Phi,\ L_K \Phi = 0, \tag{12}$$

where X is an arbitrary vector field and L_X is the Lie derivative. The hereditary property [17] allows us to define a bidifferential calculus, which yields a commuting hierarchy of conserved currents [18].

2.1. Perturbation Equations

We first make use of the first algebra and introduce linear expansions

$$u = u_1\mathbf{1} + u_2 a,\ \phi = \phi_1\mathbf{1} + \phi_2 a. \tag{13}$$

Then, based on (7) and (8), we can observe

$$X(u_1\mathbf{1}+u_2a)=\hat{X}_1\mathbf{1}+\hat{X}_2a=X(u_1)\mathbf{1}+X'(u_1)[u_2]a, \tag{14}$$

for a vector or operator $X(u)$ and, further, obtain an integrable coupling of perturbation type [1]:

$$\hat{u}_t=\hat{K}(\hat{u},\hat{u}_x,\ldots,\hat{u}^{(n)}),\ \text{i.e., } u_{1,t}=K(u_1),\ u_{2,t}=K'(u_1)[u_2], \tag{15}$$

where $\hat{u}=(u_1^T,u_2^T)^T$ and $\hat{K}=(K^T,K'^T)^T$, and its corresponding associated Lax pair:

$$\hat{\phi}_x=\hat{U}\hat{\phi},\ \hat{U}=\begin{bmatrix} U(u_1,\lambda) & 0 \\ U'(u_1)[u_2] & U(u_1,\lambda)\end{bmatrix}, \tag{16}$$

and

$$\hat{\phi}_t=\hat{V}\hat{\phi},\ \hat{V}=\begin{bmatrix} V(u_1,\lambda) & 0 \\ V'(u_1)[u_2] & V(u_1,\lambda)\end{bmatrix}, \tag{17}$$

where $\hat{\phi}=(\phi_1^T,\phi_2^T)^T$. Moreover, similarly based on (14), we can see that the perturbation Equation (15) has the following hereditary recursion operator

$$\hat{\Phi}(\hat{u})=\begin{bmatrix} \Phi(u_1) & 0 \\ \Phi'(u_1)[u_2] & \Phi(u_1)\end{bmatrix}, \tag{18}$$

where $\Phi(u)$ is a hereditary recursion operator for the original Equation (10).

The system (15) is a generalization of the symmetry problem. The second equation in (15) is the linearized equation of the original Equation (10). When u_2 satisfies the second equation for all solutions u_1 of the first equation, then u_2 presents a symmetry of the first equation.

Let us illustrate the above generating idea by a specific example. Consider the KdV equation

$$u_t=6uu_x+u_{xxx}, \tag{19}$$

and its Lax pair:

$$U(u,\lambda)=\begin{bmatrix} 0 & 1 \\ \lambda-u & 0\end{bmatrix},\ V(u,\lambda)=\begin{bmatrix} -u_x & 4\lambda+2u \\ -2(2\lambda+u)(u-\lambda)-u_{xx} & u_x\end{bmatrix}. \tag{20}$$

Then the resulting integrable couplings in (15) appear as

$$\begin{cases} u_{1,t}=6u_1u_{1,x}+u_{1,xxx}, \\ u_{2,t}=6(u_1u_2)_x+u_{2,xxx}, \end{cases} \tag{21}$$

which is exactly the first-order perturbation equation of the KdV equation [1]. Its corresponding Lax pair reads

$$\hat{U}=\begin{bmatrix} U(u_1,\lambda) & 0 \\ U'(u_1)[u_2] & U(u_1,\lambda)\end{bmatrix},\ \hat{V}=\begin{bmatrix} V(u_1,\lambda) & 0 \\ V'(u_1)[u_2] & V(u_1,\lambda)\end{bmatrix}, \tag{22}$$

where

$$U'(u_1)[u_2]=\begin{bmatrix} 0 & 0 \\ -u_2 & 0\end{bmatrix},\ V'(u_1)[u_2]=\begin{bmatrix} -u_{2,x} & 2u_2 \\ -2u_2(u_1-\lambda)-2(2\lambda+u_1)u_2-u_{2,xx} & u_{2,x}\end{bmatrix}. \tag{23}$$

The KdV Equation (19) has the hereditary recursion operator

$$\Phi(u)=\partial^2+4u+2u_x\partial^{-1}, \tag{24}$$

and by (18), this leads to a hereditary recursion operator for the perturbation Equation (21):

$$\hat{\Phi} = \begin{bmatrix} \Phi(u_1) & 0 \\ \Phi'(u_1)[u_2] & \Phi(u_1) \end{bmatrix} = \begin{bmatrix} \partial^2 + 4u_1 + 2u_{1,x}\partial^{-1} & 0 \\ 4u_2 + 2u_{2,x}\partial^{-1} & \partial^2 + 4u_1 + 2u_{1,x}\partial^{-1} \end{bmatrix}. \tag{25}$$

More generally, such hereditary recursion operators can be found in [19].

2.2. M_2-Extensions

Secondly, we make use of the second algebra and introduce the same linear expansions

$$u = u_1\mathbf{1} + u_2 a,\ \phi = \phi_1\mathbf{1} + \phi_2 a. \tag{26}$$

Then, based on (7) and (9), we obtain an integrable coupling

$$u_{1,t} = K(u_1),\ u_{2,t} = S(u_1, u_2), \tag{27}$$

where S is determined through

$$K(u_1\mathbf{1} + u_2 a) = K(u_1)\mathbf{1} + S(u_1, u_2)a. \tag{28}$$

It is also efficient to show that its corresponding associated Lax pair is given by

$$\hat{\phi}_x = \hat{U}\hat{\phi},\ \hat{U} = \begin{bmatrix} \hat{U}_1 & 0 \\ \hat{U}_2 & \hat{U}_1 + \hat{U}_2 \end{bmatrix}, \tag{29}$$

and

$$\hat{\phi}_t = \hat{V}\hat{\phi},\ \hat{V} = \begin{bmatrix} \hat{V}_1 & 0 \\ \hat{V}_2 & \hat{V}_1 + \hat{V}_2 \end{bmatrix}, \tag{30}$$

where $\hat{\phi} = (\phi_1^T, \phi_2^T)^T$ and

$$\begin{cases} U(u_1\mathbf{1} + u_2 a, \lambda) = \hat{U}_1\mathbf{1} + \hat{U}_2 a = U(u_1)\mathbf{1} + \hat{U}_2(u_1, u_2)a, \\ V(u_1\mathbf{1} + u_2 a, \lambda) = \hat{V}_1\mathbf{1} + \hat{V}_2 a = V(u_1)\mathbf{1} + \hat{V}_2(u_1, u_2)a. \end{cases} \tag{31}$$

Moreover, its resulting hereditary recursion operator reads

$$\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_1 & 0 \\ \hat{\Phi}_2 & \hat{\Phi}_1 + \hat{\Phi}_2 \end{bmatrix}, \tag{32}$$

where $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are determined by

$$\Phi(u_1\mathbf{1} + u_2 a) = \tilde{\Phi}_1\mathbf{1} + \tilde{\Phi}_2 a = \Phi(u_1)\mathbf{1} + \tilde{\Phi}_2(u_1, u_2)a. \tag{33}$$

Other integrable properties can also be obtained for the resulting integrable coupling (27).

Let us give an example. Again, start from the KdV Equation (19). Upon working out S by (28), we see that the corresponding resultant integrable coupling is

$$\begin{cases} u_{1,t} = 6u_1u_{1,x} + u_{1,xxx}, \\ u_{2,t} = 6(u_1u_2)_x + u_{2,xxx} + 6u_2u_{2,x}, \end{cases} \tag{34}$$

which is a nonlinear integrable coupling of the KdV equation and can be presented by an approach in [10]. As a result of (29) and (30), it possesses a Lax pair defined by (29) and (30) with

$$\hat{U}_1 = \begin{bmatrix} 0 & 1 \\ \lambda - u_1 & 0 \end{bmatrix},\ \hat{U}_2 = \begin{bmatrix} 0 & 0 \\ -u_2 & 0 \end{bmatrix}, \tag{35}$$

and

$$\begin{cases} \hat{V}_1 = \begin{bmatrix} -u_{1,x} & 4\lambda + 2u_1 \\ -2(2\lambda + u_1)(u_1 - \lambda) - u_{1,xx} & u_{1,x} \end{bmatrix}, \\ \hat{V}_2 = \begin{bmatrix} -u_{2,x} & 2u_2 \\ -2(\lambda + 2u_1)u_2 - 2u_2^2 - u_{2,xx} & u_{2,x} \end{bmatrix}. \end{cases} \tag{36}$$

Furthermore, it has a hereditary recursion operator defined by (32) with

$$\hat{\Phi}_1 = \partial^2 + 4u_1 + 2u_{1,x}\partial^{-1},\ \hat{\Phi}_2 = 4u_2 + 2u_{2,x}\partial^{-1}. \tag{37}$$

An M_2-extension [20] comes from an application of the second algebra as well, and it is [21] associated with

$$bb = \mathbf{1}, \tag{38}$$

where b can be taken as

$$b = \mathbf{1} - 2a. \tag{39}$$

With the linear expansions,

$$u = r\mathbf{1} + sb,\ \phi = \psi_1\mathbf{1} + \psi_2 b, \tag{40}$$

we obtain an M_2-integrable extension

$$r_t = \tilde{K}_1(r,s),\ s_t = \tilde{K}_2(r,s), \tag{41}$$

where $\tilde{K}_1$ and $\tilde{K}_2$ are generated from K as follows:

$$K(r\mathbf{1} + sb) = \tilde{K}_1(r,s)\mathbf{1} + \tilde{K}_2(r,s)b. \tag{42}$$

Its corresponding associated Lax pair reads

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi},\ \tilde{U} = \begin{bmatrix} \tilde{U}_1(r,s) & \tilde{U}_2(r,s) \\ \tilde{U}_2(r,s) & \tilde{U}_1(r,s) \end{bmatrix}, \tag{43}$$

and

$$\tilde{\phi}_t = \tilde{V}\tilde{\phi},\ \tilde{V} = \begin{bmatrix} \tilde{V}_1(r,s) & \tilde{V}_2(r,s) \\ \tilde{V}_2(r,s) & \tilde{V}_1(r,s) \end{bmatrix}. \tag{44}$$

where $\tilde{\phi} = (\psi_1^T, \psi_2^T)^T$, and

$$\begin{cases} U(r\mathbf{1} + sb, \lambda) = \tilde{U}_1(r,s)\mathbf{1} + \tilde{U}_2(r,s)b, \\ V(r\mathbf{1} + sb, \lambda) = \tilde{V}_1(r,s)\mathbf{1} + \tilde{V}_2(r,s)b. \end{cases} \tag{45}$$

Likewise, we can show that the extended system (41) has a hereditary recursion operator

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_1(r,s) & \tilde{\Phi}_2(r,s) \\ \tilde{\Phi}_2(r,s) & \tilde{\Phi}_1(r,s) \end{bmatrix}, \tag{46}$$

where

$$\Phi(r\mathbf{1} + sb) = \tilde{\Phi}_1(r,s)\mathbf{1} + \tilde{\Phi}_2(r,s)b. \tag{47}$$

We remark that the system (41) can be transformed into the one in (27), under

$$u_1 = r + s,\ u_2 = -2s. \tag{48}$$

There is also a similarity transformation between the two extended Lax pairs:

$$\tilde{U} = T^{-1}\hat{U}T,\ \tilde{V} = T^{-1}\hat{V}T, \tag{49}$$

since

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = T\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},\ T = \begin{bmatrix} I & I \\ 0 & -2I \end{bmatrix}, \tag{50}$$

where I denotes the identity matrix.

Let us illustrate the general scheme above by considering the AKNS system of nonlinear Schrödinger Equations [22]:

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q, \\ q_t = \frac{1}{2}q_{xx} - pq^2. \end{cases} \tag{51}$$

This system has a Lax pair:

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix},\ V = \begin{bmatrix} -\lambda^2 + \frac{1}{2}pq & -\lambda p - \frac{1}{2}p_x \\ -\lambda q + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq \end{bmatrix}, \tag{52}$$

and a hereditary recursion operator:

$$\Phi = \begin{bmatrix} \frac{1}{2}\partial - p\partial^{-1}q & -p\partial^{-1}p \\ q\partial^{-1}q & -\frac{1}{2}\partial + q\partial^{-1}p \end{bmatrix}. \tag{53}$$

By virtue of (41) and (42), the corresponding M_2-extended equation of the AKNS system of nonlinear Schrödinger equations reads

$$\left[\begin{array}{l} r_{1,t} = -\frac{1}{2}r_{1,xx} + r_1^2 r_2 + 2r_1 s_1 s_2 + r_2 s_1^2, \\ r_{2,t} = \frac{1}{2}r_{2,xx} - r_1 r_2^2 - 2r_2 s_1 s_2 - r_1 s_2^2, \\ s_{1,t} = -\frac{1}{2}s_{1,xx} + r_1^2 s_2 + 2r_1 r_2 s_1 + s_1^2 s_2, \\ s_{2,t} = \frac{1}{2}s_{2,xx} - r_2^2 s_1 - 2r_1 r_2 s_2 - s_1 s_2^2, \end{array}\right. \tag{54}$$

and based on (43)–(45), its corresponding Lax pair is given by

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi},\ \tilde{\phi}_t = \tilde{V}\tilde{\phi},\ \tilde{U} = \begin{bmatrix} \tilde{U}_1 & \tilde{U}_2 \\ \tilde{U}_2 & \tilde{U}_1 \end{bmatrix},\ \tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \\ \tilde{V}_2 & \tilde{V}_1 \end{bmatrix}, \tag{55}$$

where

$$\tilde{U}_1 = \begin{bmatrix} \lambda & r_1 \\ r_2 & -\lambda \end{bmatrix},\ \tilde{U}_2 = \begin{bmatrix} 0 & s_1 \\ s_2 & 0 \end{bmatrix}, \tag{56}$$

and

$$\begin{cases} \tilde{V}_1 = \begin{bmatrix} -\lambda^2 + \frac{1}{2}(r_1 r_2 + s_1 s_2) & -\lambda r_1 - \frac{1}{2}r_{1,x} \\ -\lambda r_2 + \frac{1}{2}r_{2,x} & \lambda^2 - \frac{1}{2}(r_1 r_2 + s_1 s_2) \end{bmatrix}, \\ \tilde{V}_2 = \begin{bmatrix} \frac{1}{2}(r_1 s_2 + r_2 s_1) & -\lambda s_1 - \frac{1}{2}s_{1,x} \\ -\lambda s_2 + \frac{1}{2}s_{2,x} & -\frac{1}{2}(r_1 s_2 + r_2 s_1) \end{bmatrix}. \end{cases} \tag{57}$$

Similarly, via (46) and (47), we can work out the resulting hereditary recursion operator:

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_1 & \tilde{\Phi}_2 \\ \tilde{\Phi}_2 & \tilde{\Phi}_1 \end{bmatrix}, \tag{58}$$

where

$$\begin{cases} \tilde{\Phi}_1 = \begin{bmatrix} \frac{1}{2}\partial - r_1\partial^{-1}r_2 - s_1\partial^{-1}s_2 & -r_1\partial^{-1}r_1 - s_1\partial^{-1}s_1 \\ r_2\partial^{-1}r_2 + s_2\partial^{-1}s_2 & -\frac{1}{2}\partial + r_2\partial^{-1}r_1 + s_2\partial^{-1}s_1 \end{bmatrix}, \\ \tilde{\Phi}_2 = \begin{bmatrix} -r_1\partial^{-1}s_2 - s_1\partial^{-1}r_2 & -r_1\partial^{-1}s_1 - s_1\partial^{-1}r_1 \\ r_2\partial^{-1}s_2 + s_2\partial^{-1}r_2 & r_2\partial^{-1}s_1 + s_2\partial^{-1}r_1 \end{bmatrix}. \end{cases} \tag{59}$$

By using such M_2-extended integrable equations, Alice–Bob models (see, e.g., [20,21]), both local and nonlocal, can also be generated, which enrich multicomponent integrable equations (see, e.g., [23,24]).

3. Conclusions and Discussion

We proved that expansions over two-dimensional algebras lead to integrable couplings, providing an idea of generating a class of perturbation equations and a class of nonlinear integrable couplings. The analysis explored the algebraic structures of the resulting integrable equations and their corresponding Lax pairs and hereditary recursion operators. Illustrative examples were presented in the cases of the KdV equation and the AKNS system of nonlinear Schrödinger equations. All results enrich the existing studies on integrable couplings [1,25].

It is clear that by using the block-type matrix algebras for Lax pairs, we are able to generate larger classes of integrable couplings. Combining the considered form of spectral matrices with the other forms in the literature will lead to more diverse integrable couplings. The obtained integrable couplings can also possess other integrable properties such as Hirota bilinear forms [26] and τ-symmetry algebras [27]. One can also observe that M_2-extensions can be separated through linear transformations of the dependent variables. All such analyses will help us understand multicomponent integrable equations better to work towards classification of integrable equations from a Lie algebra perspective. Concerning these integrable couplings, the issue of exact controllability (see, for example, [28,29]) warrants attention.

Integrable couplings can often be classified into different categories based on the nature and structure of the coupling. Two such classifications are bi-integrable couplings [30] and tri-integrable couplings [31,32]. A bi-integrable coupling takes a specific form,

$$u_t = K(u),\ v_t = S(u,v),\ w_t = T(u,v,w). \tag{60}$$

An interesting question is whether there exists an integrable coupling which contains two given integrable equations. That is, if we are given two integrable equations $u_t = K(u)$ and $v_t = S(v)$, can we construct a bigger system of the form,

$$u_t = K(u),\ v_t = S(v),\ w_t = T(u,v,w) \tag{61}$$

in a way that maintains overall integrability? Here, the Gateaux derivatives $T'(u)$ and $T'(v)$ are assumed to be nonzero. This is a special bi-integrable coupling. Further research on integrable couplings holds the promise of uncovering new mathematical structures and physical phenomena.

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