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Type (λ^*, λ) reduced nonlocal integrable AKNS equations and their soliton solutions

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A B S T R A C T

We construct a class of novel reduced nonlocal reverse-spacetime integrable AKNS equations by taking two group reductions of the AKNS matrix spectral problems of arbitrary order. One reduction is local, replacing the spectral parameter with its complex conjugate while the other is nonlocal, replacing the spectral parameter with itself. Then based on the specific distribution of eigenvalues, we compute soliton solutions by using the Riemann-Hilbert problems with the identity jump matrix, where eigenvalues could equal adjoint eigenvalues.

1. Introduction

The Lax formulation of matrix spectral problems lays the foundations for using the inverse scattering transform to study integrable equations. It has been found that group reductions of matrix spectral problems can lead to nonlocal integrable equations [2,10,13]. For the Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems, taking one group reduction yields three kinds of nonlocal nonlinear Schrödinger (NLS) equations and two kinds of nonlocal modified Korteweg-de Vries (mKdV) equations [2,15], and conducting two group reductions generates different kinds of novel nonlocal integrable equations. The inverse scattering transform can still be used to construct soliton solutions to nonlocal integrable equations (see, e.g., [1,19]).

Moreover, Darboux transformation, the Hirota bilinear method and Riemann-Hilbert problems have been proved to be other powerful approaches to integrable equations, both local and nonlocal, and especially to their soliton solutions. Indeed, numerous integrable equations have been studied through those methods (see, e.g., [5,9,20,21,24]). Particularly, Riemann-Hilbert problems are formulated and used to solve nonlocal integrable NLS and mKdV equations. We refer the interested readers to the recent references [3,22,23] for the local case and [10,14,16,25] for the nonlocal case, regarding the application of Riemann-Hilbert problems. In this paper, we would like to present a kind of novel reduced nonlocal integrable AKNS equations by taking two group reductions and compute their soliton solutions through special Riemann-Hilbert problems with the identity jump matrix.

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The rest of this paper is structured as follows. In Section 2, we make two group reductions of the AKNS matrix spectral problems to present type (λ^*, λ) reduced nonlocal integrable AKNS equations, where λ^* denotes the complex conjugate of λ. Two paradigmatic examples of scalar nonlocal integrable mKdV equations are

$$p_{1,t} = p_{1,xxx} + 6\sigma |p_1|^2 p_{1,x} + 3\sigma p_1(-x,-t)(p_1 p_1^*(-x,-t))_x,$$

and

$$p_{1,t} = p_{1,xxx} + 6\delta p_1 p_1(-x,-t) p_{1,x} + 3\delta p_1^*(p_1 p_1^*(-x,-t))_x,$$

where $\sigma = \delta = \pm 1$ (see how σ and δ are involved later in Section 2). In Section 3, based on the specific distribution of eigenvalues, we formulate solutions to the corresponding Riemann-Hilbert problems with the identity jump matrix, where eigenvalues could equal adjoint eigenvalues, and construct soliton solutions to the resulting reduced nonlocal integrable AKNS equations. In the last section, we give a conclusion and a few concluding remarks.

2. Reduced nonlocal integrable AKNS equations

2.1. The matrix AKNS integrable hierarchies revisited

In order to make the subsequent analysis smoothly, let us recall the AKNS hierarchies of matrix integrable equations. As usual, let λ denote the spectral parameter, and assume that p and q are two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{2.1}$$

where $m, n \geq 1$ are two arbitrarily given integers. We consider the matrix AKNS spectral problems as follows:

$$\begin{cases} -i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda\Lambda + P)\phi, \\ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi = (\lambda^r\Omega + Q^{[r]})\phi,\ r \geq 0, \end{cases} \tag{2.2}$$

where the $(m+n)$-th order square matrices, Λ and Ω, read

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n),\ \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.3}$$

in which I_s denotes the identity matrix of size s, and α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct real constants, and where the other two $(m+n)$-th order square matrices, P and $Q^{[r]}$, read

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.4}$$

called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.5}$$

in which $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ will be defined recursively by

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a^{[0]} = \beta_1 I_m,\ d^{[0]} = \beta_2 I_n, \tag{2.6a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p),\ s \geq 0, \tag{2.6b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q),\ s \geq 0, \tag{2.6c}$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q),\ d_x^{[s]} = i(qb^{[s]} - c^{[s]}p),\ s \geq 1, \tag{2.6d}$$

with zero constants of integration being taken. In particular, we can obtain

$$Q^{[1]} = \frac{\beta}{\alpha}P,\ Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + iP_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3),$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. The recursive relations in (2.6) also tell that

$$W=\sum_{s\ge 0}\lambda^{-s}\begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]}\end{bmatrix} \tag{2.7}$$

solves the stationary zero curvature equation

$$W_x=i[U,W]. \tag{2.8}$$

The compatibility conditions of the two matrix spectral problems in (2.2), i.e., the zero curvature equations

$$U_t-V_x^{[r]}+i[U,V^{[r]}]=0,\ r\ge 0, \tag{2.9}$$

give us one matrix AKNS integrable hierarchy (see, e.g., [17] for more details):

$$p_t=i\alpha b^{[r+1]},\ q_t=-i\alpha c^{[r+1]},\ r\ge 0, \tag{2.10}$$

which has a bi-Hamiltonian structure and so infinitely many symmetries and conservation laws. The first and second nonlinear integrable equations in the hierarchy give us the AKNS matrix NLS equations:

$$p_t=-\frac{\beta}{\alpha^2}i(p_{xx}+2pqp),\ q_t=\frac{\beta}{\alpha^2}i(q_{xx}+2qpq), \tag{2.11}$$

and the AKNS matrix mKdV equations:

$$p_t=-\frac{\beta}{\alpha^3}(p_{xxx}+3pqp_x+3p_xqp),\ q_t=-\frac{\beta}{\alpha^3}(q_{xxx}+3q_xpq+3qpq_x), \tag{2.12}$$

where the two matrix potentials, p and q, are defined by (2.1).

2.2. Reduced nonlocal integrable AKNS equations

We would like to construct a class of novel reduced nonlocal reverse-spacetime integrable AKNS equations by taking two group reductions for the matrix AKNS spectral problems in (2.2). One reduction is local while the other is nonlocal (see also [11] for the basic idea of conducting group reductions).

Let Σ_1,Σ_2 be a pair of constant invertible Hermitian matrices of sizes m and n, respectively, and Δ_1,Δ_2 be another pair of constant invertible symmetric matrices of sizes m and n, respectively. We now introduce two group reductions for the spectral matrix U:

$$U^\dagger(x,t,\lambda^*)=(U(x,t,\lambda^*))^\dagger=\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{2.13}$$

and

$$U^T(-x,-t,\lambda)=(U(-x,-t,\lambda))^T=\Delta U(x,t,\lambda)\Delta^{-1}, \tag{2.14}$$

where † denotes the Hermitian transpose and the two constant invertible matrices, Σ and Δ, are defined by

$$\Sigma=\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix},\ \Delta=\begin{bmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{bmatrix}. \tag{2.15}$$

These two group reductions equivalently generate

$$P^\dagger(x,t)=\Sigma P(x,t)\Sigma^{-1}, \tag{2.16}$$

and

$$P^T(-x,-t)=\Delta P(x,t)\Delta^{-1}, \tag{2.17}$$

respectively. More precisely, they allow us to make the reductions for the matrix potentials:

$$q(x,t)=\Sigma_2^{-1}p^\dagger(x,t)\Sigma_1, \tag{2.18}$$

and

$$q(x,t)=\Delta_2^{-1}p^T(-x,-t)\Delta_1, \tag{2.19}$$

respectively. Therefore, an additional constraint is required for the matrix potential p:

$$\Sigma_2^{-1}p^\dagger(x,t)\Sigma_1=\Delta_2^{-1}p^T(-x,-t)\Delta_1, \tag{2.20}$$

to satisfy both group reductions in (2.13) and (2.14). Moreover, we point out that under the group reductions in (2.13) and (2.14), we have that

$$\begin{cases} W^{\dagger}(x,t,\lambda^*) = (W(x,t,\lambda^*))^{\dagger} = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(-x,-t,\lambda) = (W(-x,-t,\lambda))^T = \Delta W(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.21}$$

which implies that

$$\begin{cases} V^{[r]\dagger}(x,t,\lambda^*) = (V^{[r]}(x,t,\lambda^*))^{\dagger} = \Sigma V^{[r]}(x,t,\lambda)\Sigma^{-1}, \\ V^{r]T}(-x,-t,\lambda) = (V^{[r]}(-x,-t,\lambda))^T = \Delta V^{[r]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.22}$$

and

$$\begin{cases} Q^{[r]\dagger}(x,t,\lambda^*) = (Q^{[r]}(x,t,\lambda^*))^{\dagger} = \Sigma Q^{[r]}(x,t,\lambda)\Sigma^{-1}, \\ Q^{[r]T}(-x,-t,\lambda) = (Q^{[r]}(-x,-t,\lambda))^T = \Delta Q^{[r]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.23}$$

where $r \geq 0$.

As a consequence, we see that under the potential reductions (2.18) and (2.19), the integrable matrix AKNS equations in (2.10) reduce to a hierarchy of nonlocal reverse-spacetime integrable matrix AKNS equations:

$$p_t = i\alpha b^{[r+1]}|_{q=\Sigma_2^{-1}p^{\dagger}\Sigma_1=\Delta_2^{-1}p^T(-x,-t)\Delta_1}, \ r \geq 0, \tag{2.24}$$

where p is an $m \times n$ matrix potential satisfying (2.20), Σ_1, Σ_2 are a pair of arbitrary invertible Hermitian matrices of sizes m and n, respectively, and Δ_1, Δ_2 are a pair of arbitrary invertible symmetric matrices of sizes m and n, respectively. Each reduced equation in the hierarchy (2.24) possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in (2.2) and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (2.10).

If we fix $r = 3$, then the reduced matrix integrable AKNS equations in (2.24) give a kind of reduced nonlocal integrable matrix mKdV equations:

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Sigma_2^{-1}p^{\dagger}\Sigma_1 p_x + 3p_x\Sigma_2^{-1}p^{\dagger}\Sigma_1 p) \\ &= -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Delta_2^{-1}p^T(-x,-t)\Delta_1 p_x + 3p_x\Delta_2^{-1}p^T(-x,-t)\Delta_1 p), \end{aligned} \tag{2.25}$$

where p is an $m \times n$ matrix potential satisfying (2.20).

In what follows, we would like to show the richness of these novel reduced nonlocal integrable matrix mKdV equations, by a few examples with different values for m, n and appropriate choices for Σ, Δ.

Let us first consider $m = 1$ and $n = 2$. We take

$$\Sigma_1 = 1, \ \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \ \Delta_1 = 1, \ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{2.26}$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$. Then the potential constraint (2.20) requires

$$p_2 = \sigma\delta p_1^*(-x,-t), \tag{2.27}$$

where $p = (p_1, p_2)$, and thus, the corresponding potential matrix P reads

$$P = \begin{bmatrix} 0 & p_1 & \sigma\delta p_1^*(-x,-t) \\ \sigma p_1^* & 0 & 0 \\ \delta p_1(-x,-t) & 0 & 0 \end{bmatrix}. \tag{2.28}$$

Further, the corresponding novel reduced nonlocal integrable mKdV equations become

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\sigma|p_1|^2 p_{1,x} + 3\sigma p_1(-x,-t)(p_1 p_1^*(-x,-t))_x], \tag{2.29}$$

where $\sigma = \pm 1$ and $|z|$ denotes the absolute value of z. These two equations are totally different from the ones studied in [2,6–8], in which only one nonlocal factor and one nonlinear term appear. A similar argument with

$$\Sigma_1 = 1, \ \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \ \Delta_1 = 1, \ \Delta_2^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \tag{2.30}$$

where σ and δ are real constants and satisfy $\sigma^2=\delta^2=1$, can generate the following different type reduced nonlocal integrable mKdV equations:

$$p_{1,t}=-\frac{\beta}{\alpha^3}[p_{1,xxx}+6\delta p_1 p_1(-x,-t)p_{1,x}+3\delta p_1^*(p_1 p_1^*(-x,-t))_x], \tag{2.31}$$

where $\delta=\pm 1$.

Let us second consider $m=1$ and $n=4$. We take

$$\Sigma_1=1,\ \Sigma_2^{-1}=\begin{bmatrix}\sigma_1&0&0&0\\0&\sigma_1&0&0\\0&0&\sigma_2&0\\0&0&0&\sigma_2\end{bmatrix},\ \Delta_1=1,\ \Delta_2^{-1}=\begin{bmatrix}0&\delta_1&0&0\\\delta_1&0&0&0\\0&0&0&\delta_2\\0&0&\delta_2&0\end{bmatrix}, \tag{2.32}$$

where σ_j and δ_j are real constants and satisfy $\sigma_j^2=\delta_j^2=1,\ j=1,2$. Then the potential constraint (2.20) generates

$$p_2=\sigma_1\delta_1 p_1^*(-x,-t),\ p_4=\sigma_2\delta_2 p_3^*(-x,-t), \tag{2.33}$$

where $p=(p_1,p_2,p_3,p_4)$, and so the corresponding potential matrix P becomes

$$P=\begin{bmatrix}0&p_1&\sigma_1\delta_1 p_1^*(-x,-t)&p_3&\sigma_2\delta_2 p_3^*(-x,-t)\\\sigma_1 p_1^*&0&0&0&0\\\delta_1 p_1(-x,-t)&0&0&0&0\\\sigma_2 p_3^*&0&0&0&0\\\delta_2 p_3(-x,-t)&0&0&0&0\end{bmatrix}. \tag{2.34}$$

This enables us to obtain a class of two-component reduced nonlocal integrable mKdV equations:

$$\begin{cases}p_{1,t}=-\frac{\beta}{\alpha^3}[p_{1,xxx}+6\sigma_1|p_1|^2p_{1,x}+3\sigma_1 p_1(-x,-t)(p_1p_1^*(-x,-t))_x\\\qquad +3\sigma_2 p_3^*(p_1p_3)_x+3\sigma_2 p_3(-x,-t)(p_1p_3^*(-x,-t))_x],\\p_{3,t}=-\frac{\beta}{\alpha^3}[p_{3,xxx}+3\sigma_1 p_1^*(p_1p_3)_x+3\sigma_1 p_1(-x,-t)(p_1^*(-x,-t)p_3)_x\\\qquad +6\sigma_2|p_3|^2p_{3,x}+3\sigma_2 p_3(-x,-t)(p_3p_3^*(-x,-t))_x],\end{cases} \tag{2.35}$$

where σ_j are real constants which satisfy $\sigma_j^2=1,\ j=1,2$.

Let us third consider $m=2$ and $n=2$. We take

$$\Sigma_1=\begin{bmatrix}0&\sigma_1\\\sigma_1&0\end{bmatrix},\ \Sigma_2^{-1}=\begin{bmatrix}\sigma_2&0\\0&\sigma_2\end{bmatrix},\ \Delta_1=\begin{bmatrix}0&\delta_1\\\delta_1&0\end{bmatrix},\ \Delta_2^{-1}=\begin{bmatrix}0&\delta_2\\\delta_2&0\end{bmatrix}, \tag{2.36}$$

where σ_j and δ_j are real constants and satisfy $\sigma_j^2=\delta_j^2=1$. Then the potential constraint (2.20) tells

$$p_{12}=\sigma_1\delta_1\sigma_2\delta_2 p_{11}^*(-x,-t),\ p_{22}=\sigma_1\delta_1\sigma_2\delta_2 p_{12}^*(-x,-t), \tag{2.37}$$

and so the corresponding matrix potentials read

$$p=\begin{bmatrix}p_{11}&\sigma_1\delta_1\sigma_2\delta_2 p_{11}^*(-x,-t)\\p_{12}&\sigma_1\delta_1\sigma_2\delta_2 p_{12}^*(-x,-t)\end{bmatrix},\ q=\begin{bmatrix}\sigma_1\sigma_2 p_{12}^*&\sigma_1\sigma_2 p_{11}^*\\\delta_1\delta_2 p_{12}(-x,-t)&\delta_1\delta_2 p_{11}(-x,-t)\end{bmatrix}. \tag{2.38}$$

This enables us to get another class of two-component reduced nonlocal integrable mKdV equations:

$$\begin{cases}p_{11,t}=-\frac{\beta}{\alpha^3}[p_{11,xxx}+6\sigma p_{11}p_{12}^*p_{11,x}+3\sigma p_{12}(-x,-t)(p_{11}p_{11}^*(-x,-t))_x\\\qquad +3\sigma p_{11}^*(p_{11}p_{12})_x+3\sigma p_{11}(-x,-t)(p_{11}^*(-x,-t)p_{12})_x],\\p_{12,t}=-\frac{\beta}{\alpha^3}[p_{12,xxx}+3\sigma p_{12}^*(p_{11}p_{12})_x+3\sigma p_{12}(-x,-t)(p_{11}p_{12}^*(-x,-t))_x\\\qquad +6\sigma p_{11}^*p_{12}p_{12,x}+3\sigma p_{11}(-x,-t)(p_{12}p_{12}^*(-x,-t))_x],\end{cases} \tag{2.39}$$

where $\sigma=\sigma_1\sigma_2=\pm 1$. The pattern of nonlinear terms in these two equations is different from the one in (2.35).

3. Soliton solutions

3.1. Distribution of eigenvalues

It is easy to see that the group reduction in (2.13) (or (2.14)) guarantees that λ is an eigenvalue of the matrix spectral problems in (2.2) if and only if $\hat{\lambda}=\lambda^*$ (or $\hat{\lambda}=\lambda$) is an adjoint eigenvalue, i.e., it satisfies the adjoint matrix spectral problems:

$$i\tilde{\phi}_x=\tilde{\phi}U=\tilde{\phi}U(u,\hat{\lambda}),\ i\tilde{\phi}_t=\tilde{\phi}V^{[r]}=\tilde{\phi}V^{[r]}(u,\hat{\lambda}), \tag{3.1}$$

where $r\geq 0$. Accordingly, we can assume that we have eigenvalues $\lambda:\ \mu,\ \mu^*$, and adjoint eigenvalues $\hat{\lambda}:\ \mu^*,\ \mu$, where $\mu\in\mathbb{C}$.

Moreover, under the group reductions in (2.13) and (2.14), $\phi^\dagger(\lambda^*)\Sigma$ and $\phi^T(-x,-t,\lambda)\Delta$ present two adjoint eigenfunctions associated with the same eigenvalue λ, when $\phi(\lambda)$ is an eigenfunction of the matrix spectral problems in (2.2) associated with an eigenvalue λ.

3.2. Solution formulation of special Riemann-Hilbert problems

We would like to present a general formulation of solutions to special Riemann-Hilbert problems with the identity jump matrix. Let $N_1,N_2\geq 0$ be two integers such that $N=2N_1+N_2\geq 1$. First, we take N eigenvalues λ_k and N adjoint eigenvalues $\hat{\lambda}_k$ as follows:

$$\lambda_k,\ 1\leq k\leq N:\ \mu_1,\ \cdots,\ \mu_{N_1},\mu_1^*,\ \cdots,\ \mu_{N_1}^*,\ \nu_1,\ \cdots,\ \nu_{N_2}, \tag{3.2}$$

and

$$\hat{\lambda}_k,\ 1\leq k\leq N:\ \mu_1^*,\ \cdots,\ \mu_{N_1}^*,\mu_1,\ \cdots,\ \mu_{N_1},\ \nu_1^*,\ \cdots,\ \nu_{N_2}^*, \tag{3.3}$$

where $\mu_k\in\mathbb{C},\ 1\leq k\leq N_1$, and $\nu_k\in\mathbb{C},\ 1\leq k\leq N_2$, and assume that their corresponding eigenfunctions and adjoint eigenfunctions are given by

$$v_k,\ 1\leq k\leq N,\ \text{and}\ \hat{v}_k,\ 1\leq k\leq N, \tag{3.4}$$

respectively. We point out that in the current nonlocal case, the condition

$$\{\lambda_k\,|\,1\leq k\leq N\}\cap\{\hat{\lambda}_k\,|\,1\leq k\leq N\}=\emptyset,$$

is not satisfied. Next, let us introduce two matrices:

$$G^+(\lambda)=I_{m+n}-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\ (G^-)^{-1}(\lambda)=I_{m+n}+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_k}, \tag{3.5}$$

where M is a square matrix $M=(m_{kl})_{N\times N}$ with its entries defined by

$$m_{kl}=\begin{cases}\dfrac{\hat{v}_k v_l}{\lambda_l-\hat{\lambda}_k}, & \text{if } \lambda_l\neq\hat{\lambda}_k,\\ 0, & \text{if } \lambda_l=\hat{\lambda}_k,\end{cases}\quad \text{where } 1\leq k,l\leq N. \tag{3.6}$$

It has been shown in [14] that these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the corresponding special Riemann-Hilbert problem with the identity jump matrix:

$$(G^-)^{-1}(\lambda)G^+(\lambda)=I_{m+n},\ \lambda\in\mathbb{R}, \tag{3.7}$$

when an orthogonal condition:

$$\hat{v}_k v_l=0\ \text{if}\ \lambda_l=\hat{\lambda}_k,\ \text{when}\ 1\leq k,l\leq N, \tag{3.8}$$

is satisfied.

When zero potentials are taken, the matrix spectral problems in (2.2) yield

$$v_k=v_k(x,t,\lambda_k)=\mathrm{e}^{i\lambda_k\Lambda x+i\lambda_k^r\Omega t}w_k,\ 1\leq k\leq N, \tag{3.9}$$

where w_k, $1\leq k\leq N$, are constant column vectors. Following the preceding analysis, we can take

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(x,t,\lambda_k)\Sigma = \hat{w}_k e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^r \Omega t},\ \hat{w}_k = w_k^\dagger \Sigma,\ 1 \le k \le N. \tag{3.10}$$

The orthogonal condition (3.8) leads equivalently to

$$w_k^\dagger \Sigma w_l = 0 \text{ if } \lambda_l = \hat{\lambda}_k,\ \text{where } 1 \le k,l \le N. \tag{3.11}$$

Now, upon making an asymptotic expansion

$$G^+(\lambda) = I_{m+n} + \frac{1}{\lambda} G_1^+ + O(\frac{1}{\lambda^2}), \tag{3.12}$$

as $\lambda \to \infty$, we get

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{3.13}$$

and further, substituting this into the matrix spatial spectral problems in (2.2), we obtain

$$P = -[\Lambda, G_1^+] = \lim_{\lambda\to\infty} [G^+(\lambda), \Lambda]. \tag{3.14}$$

This produces the N-soliton solution to the matrix AKNS equations (2.10):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2,\ q = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl} \hat{v}_l^1, \tag{3.15}$$

where for each $1 \le k \le N$, we have made the splittings, $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, where v_k^1 and $\hat{v}_k^1$ are column and row vectors of dimension m, respectively, and v_k^2 and $\hat{v}_k^2$ are column and row vectors of dimension n, respectively.

To present N-soliton solutions for the reduced nonlocal matrix integrable AKNS equations (2.24), we need to check whether G_1^+ determined by (3.13) satisfies the involution properties:

$$(G_1^+)^\dagger = -\Sigma G_1^+ \Sigma^{-1},\ (G_1^+)^T(-x,-t) = -\Delta G_1^+ \Delta^{-1}, \tag{3.16}$$

which mean that the resulting potential matrix P defined by (3.14) will satisfy the two group reduction conditions in (2.16) and (2.17). Consequently, under these conditions, the above N-soliton solution to the matrix AKNS equations (2.10) reduces to the following N-soliton solution:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{3.17}$$

to the reduced nonlocal matrix integrable AKNS equations (2.24).

3.3. Realizing the involution conditions

We would now like to check how to realize the involution properties in (3.16).

First, based on the preceding analysis in subsection 3.1, all adjoint eigenfunctions $\hat{v}_k,\ 1 \le k \le 2N_1$, can be taken as follows:

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(\lambda_k)\Sigma = v_{N_1+k}^T(-x,-t,\lambda_k^*)\Delta,\ 1 \le k \le N_1, \tag{3.18}$$

and

$$\hat{v}_{N_1+k} = \hat{v}_{N_1+k}(x,t,\hat{\lambda}_{N_1+k}) = v_{N_1+k}^\dagger(\lambda_{N_1+k})\Sigma = v_k^T(-x,-t,\lambda_k)\Delta,\ 1 \le k \le N_1. \tag{3.19}$$

These selections in (3.18) and (3.19) require the conditions on $w_k,\ 1 \le k \le N$:

$$\begin{cases} w_k^T(\Sigma^* \Delta^{*-1} - \Delta \Sigma^{-1}) = 0,\ 1 \le k \le N_1, \\ w_k = \Delta^{-1} \Sigma^* w_{k-N_1}^*,\ N_1 + 1 \le k \le 2N_1, \end{cases} \tag{3.20}$$

where $*$ denotes the complex conjugate of a matrix. Note that all these conditions aim to satisfy the reduction conditions in (2.16) and (2.17).

Next, note that when the solutions to the Riemann-Hilbert problems with the identity jump matrix, defined by (3.5) and (3.6), possess the involution properties

$$(G^+)^\dagger(\lambda^*) = \Sigma (G^-)^{-1}(\lambda)\Sigma^{-1},\ (G^+)^T(-x,-t,\lambda) = \Delta (G^-)^{-1}(\lambda)\Delta^{-1}, \tag{3.21}$$

the corresponding relevant matrix G_1^+ will satisfy the involution properties in (3.16), which are consequences of the group reductions in (2.13) and (2.14). Accordingly, when the conditions in (3.20) and the orthogonal condition in (3.11) are satisfied for w_k, $1 \le k \le N$, the formula (3.17), together with (3.5), (3.6), (3.9) and (3.10), gives rise to N-soliton solutions to the reduced nonlocal matrix integrable AKNS equations (2.24).

Finally, we present an example of solutions to the reduced nonlocal mKdV equations in the case of $m = n = N = 1$. We take $\lambda_1 = i\nu,\ \hat{\lambda}_1 = -i\nu,\ \nu \in \mathbb{R}$, and choose

$$w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^T, \tag{3.22}$$

where $w_{1,1}, w_{1,2}, w_{1,3}$ are arbitrary real numbers with $w_{1,3}^2 = w_{1,2}^2$. These selections lead to a class of one-soliton solutions to the reduced nonlocal integrable mKdV equation (2.29):

$$p_1 = -\frac{2i\sigma\nu(\alpha_1-\alpha_2)w_{1,1}w_{1,2}}{w_{1,1}^2 e^{-(\alpha_1-\alpha_2)\nu x-(\beta_1-\beta_2)\nu^3 t} + 2\sigma w_{1,2}^2 e^{(\alpha_1-\alpha_2)\nu x+(\beta_1-\beta_2)\nu^3 t}}, \tag{3.23}$$

where $\nu \in \mathbb{R}$ is arbitrary and $w_{1,1}, w_{1,2} \in \mathbb{R}$ are arbitrary but need to satisfy $w_{1,1}^2 = 2w_{1,2}^2$, which comes from the involution properties in (3.16).

4. Concluding remarks

Type (λ^*, λ) reduced nonlocal reverse-spacetime integrable AKNS equations were presented and their soliton solutions were formulated through special Riemann-Hilbert problems with the identity jump matrix. The analysis is based on two group reductions of the AKNS matrix spectral problems, one of which is local while the other is nonlocal. The resulting nonlocal integrable AKNS equations are a type of novel nonlocal reverse-spacetime integrable equations.

We remark that it would also be interesting to search for other kinds of reduced nonlocal integrable equations by different kinds of pairs of group reductions, both local and nonlocal. Moreover, it is very interesting to study dynamical properties of exact solutions in the nonlocal case, including lump solutions [18], solitonless solutions [12] and algebro-geometric solutions [4], from a perspective of Riemann-Hilbert problems. All this will greatly enrich the mathematical theory of nonlocal integrable equations.

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