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NOVEL INTEGRABLE HAMILTONIAN HIERARCHIES WITH SIX POTENTIALS*

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Abstract This paper aims to construct six-component integrable hierarchies from a kind of matrix spectral problems within the zero curvature formulation. Their Hamiltonian formulations are furnished by the trace identity, which guarantee the commuting property of infinitely many symmetries and conserved Hamiltonian functionals. Illustrative examples of the resulting integrable equations of second and third orders are explicitly computed.

Key words matrix spectral problem; zero curvature equation; integrable hierarchy; NLS equations; mKdV equations

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1 Introduction

The zero curvature equation is a nonlinear partial differential equation that describes the compatibility conditions for a pair of linear differential equations, which arises in the theory of integrable equations and has important applications in mathematical physics [1]. It allows for the construction of infinitely many symmetries and conserved quantities for certain nonlinear equations, such as nonlinear Schrödinger equations and modified Korteweg-de Vries eqautions. This leads to the integrability of these equations and allows for the explicit construction of their solutions.

To generate integrable equations by the zero curvature equation, it is crucial to formulate an appropriate matrix spatial spectral problem. As usal, let u and λ denote a q-dimensional potential: $u=(u_1,\cdots,u_q)^T$ and the spectral parameter, respectively. First, use a loop algebra $\tilde{g}$ to form a spatial spectral matrix:

$$U=U(u,\lambda)=e_0(\lambda)+u_1e_1(\lambda)+\cdots+u_qe_q(\lambda), \tag{1.1}$$

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where $e_1, \cdots, e_q$ are linear independent elements in $\tilde{g}$ and e_0 is required to be a pseudo-regular element in $\tilde{g}$:

$$\mathrm{Ker}\,\mathrm{ad}_{e_0} \oplus \mathrm{Im}\,\mathrm{ad}_{e_0} = \tilde{g}, \text{ and } \mathrm{Ker}\,\mathrm{ad}_{e_0} \text{ is commutative.} \tag{1.2}$$

This characterizes the solvability of the stationary zero curvature equation:

$$W_x = \mathrm{i}[U, W], \tag{1.3}$$

among Laurent series

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]}.$$

Second, consider the spatial and temporal matrix spectral problems:

$$-\mathrm{i}\phi_x = U\phi, \ -\mathrm{i}\phi_t = V^{[r]}\phi, \ r \geq 0, \tag{1.4}$$

where $V^{[r]}$ is detemined by using the Laurent series solution W. Subsequently, an integrable hierarchy is presented through a hierarchy of zero curvature equations:

$$U_t - V_x^{[r]} + \mathrm{i}[U, V^{r]}] = 0, \ r \geq 0, \tag{1.5}$$

which are the compatibility conditions of the matrix spectral problems in (1.4). Hamiltonian formulations of the resulting integrable equations could be established by applying the trace identity [2, 3]:

$$\frac{\delta}{\delta u} \int \mathrm{tr}\big(W \frac{\partial U}{\partial \lambda}\big)\, \mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \mathrm{tr}\big(W \frac{\partial U}{\partial u}\big), \tag{1.6}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u and γ is the constant, independent of the spectral parameter. Usually, a bi-Hamiltonian formulation can also be furnished, which directly yields the Liouville integrability of the presented integrable equations [4].

Many integrable Hamiltonian hierarchies are computed through the above zero curvature formulation, based on the special linear algebras (see, e.g., [5–10]) and the special orthogonal algebras (see, e.g., [11–17]). The case of two components, p and q, is of great importance, The four well-known integrable hierarchies of two components are associated with the folllowing spectral matrices:

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, \ U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, \ U = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix}, \ U = \begin{bmatrix} \lambda v & \lambda p \\ \lambda q & -\lambda v \end{bmatrix}, \tag{1.7}$$

where $pq + v^2 = 1$. The corresponding integrable hierarchies are called the Ablowitz-Kaup-Newell-Segur hierarchy [5], the Kaup-Newell hierarchy [18], the Wadati-Konno-Ichikawa hierarchy [19] and the Heisenberg hierarchy [20], respectively.

This paper aims to present integrable Hamiltonian hierarchies of six components by applying the zero curvature formulation. We use the trace identity to furnish Hamiltonian formulations for the resulting hierarchies. Two illustrative examples are six-component integrable coupled nonlinear Schrödinger equations and six-component integrable coupled modified Korteweg-de Vries equations. The final section is devoted to the conclusion and some concluding remarks.

2 An Integrable Hierarchy with Six Potentials

To construct integrable equations by the zero curvature equation, we begin with a matrix spectral problem of the form:

$$-\mathrm{i}\phi_x = U\phi = U(u,\lambda)\phi,\ U = \begin{bmatrix} \lambda & p_1 & p_2 & p_3 & p_1 & 0 \\ q_1 & 0 & 0 & 0 & 0 & p_1 \\ q_2 & 0 & 0 & 0 & 0 & p_2 \\ q_3 & 0 & 0 & 0 & 0 & p_3 \\ q_1 & 0 & 0 & 0 & 0 & p_1 \\ 0 & q_1 & q_2 & q_3 & q_1 & -\lambda \end{bmatrix}, \tag{2.1}$$

where λ is the spectral parameter as usual, and u is the six-dimensional vector of potentials:

$$u = u(x,t) = (p_1, p_2, p_3, q_1, q_2, q_3)^T. \tag{2.2}$$

This spectral problem can not be reduced from the matrix Ablowitz-Kaup-Newell-Segur spectral problem (see, e.g., [21]). It is determined by a Maple symbolic computation process.

In order to compute an associated integrable hierarchy, we first solve the stationary zero curvature equation (1.3) by assuming

$$W = \begin{bmatrix} a & b_1 & b_2 & b_3 & b_1 & 0 \\ c_1 & 0 & d_1 & d_2 & 0 & b_1 \\ c_2 & -d_1 & 0 & d_3 & -d_1 & b_2 \\ c_3 & -d_2 & -d_3 & 0 & -d_2 & b_3 \\ c_1 & 0 & d_1 & d_2 & 0 & b_1 \\ 0 & c_1 & c_2 & c_3 & c_1 & -a \end{bmatrix} = \sum_{s\ge 0} \lambda^{-s} W^{[s]}, \tag{2.3}$$

with

$$W^{[s]} = \begin{bmatrix} a^{[s]} & b_1^{[s]} & b_2^{[s]} & b_3^{[s]} & b_1^{[s]} & 0 \\ c_1^{[s]} & 0 & d_1^{[s]} & d_2^{[s]} & 0 & b_1^{[s]} \\ c_2^{[s]} & -d_1^{[s]} & 0 & d_3^{[s]} & -d_1^{[s]} & b_2^{[s]} \\ c_3^{[s]} & -d_2^{[s]} & -d_3^{[s]} & 0 & -d_2^{[s]} & b_3^{[s]} \\ c_1^{[s]} & 0 & d_1^{[s]} & d_2^{[s]} & 0 & b_1^{[s]} \\ 0 & c_1^{[s]} & c_2^{[s]} & c_3^{[s]} & c_1^{[s]} & -a^{[s]} \end{bmatrix},\ s \ge 0, \tag{2.4}$$

which is determined, again based on symbolic computations. It is direct to see that the corresponding stationary zero curvature equation leads to the initial conditions:

$$b_1^{[0]} = b_2^{[0]} = b_3^{[0]} = c_1^{[0]} = c_2^{[0]} = c_3^{[0]} = 0,\ d_{1,x}^{[0]} = d_{2,x}^{[0]} = d_{3,x}^{[0]} = 0,\ a_x^{[0]} = 0, \tag{2.5}$$

and the recursion relations:

$$\begin{cases} b_1^{[s+1]} = -\mathrm{i} b_{1,x}^{[s]} + p_1 a^{[s]} + p_2 d_1^{[s]} + p_3 d_2^{[s]}, \\ b_2^{[s+1]} = -\mathrm{i} b_{2,x}^{[s]} + p_2 a^{[s]} - 2p_1 d_1^{[s]} + p_3 d_3^{[s]}, \\ b_3^{[s+1]} = -\mathrm{i} b_{3,x}^{[s]} + p_3 a^{[s]} - 2p_1 d_2^{[s]} - p_2 d_3^{[s]}, \end{cases} \tag{2.6}$$

$$\begin{cases} c_1^{[s+1]} = \mathrm{i}c_{1,x}^{[s]} + q_1 a^{[s]} - q_2 d_1^{[s]} - q_3 d_2^{[s]}, \\ c_2^{[s+1]} = \mathrm{i}c_{2,x}^{[s]} + q_2 a^{[s]} + 2q_1 d_1^{[s]} - q_3 d_3^{[s]}, \\ c_3^{[s+1]} = \mathrm{i}c_{3,x}^{[s]} + q_3 a^{[s]} + 2q_1 d_2^{[s]} + q_2 d_3^{[s]}, \end{cases} \tag{2.7}$$

$$\begin{cases} d_{1,x}^{[s+1]} = \mathrm{i}(q_1 b_2^{[s+1]} - q_2 b_1^{[s+1]} + p_1 c_2^{[s+1]} - p_2 c_1^{[s+1]}), \\ d_{2,x}^{[s+1]} = \mathrm{i}(q_1 b_3^{[s+1]} - q_3 b_1^{[s+1]} + p_1 c_3^{[s+1]} - p_3 c_1^{[s+1]}), \\ d_{3,x}^{[s+1]} = \mathrm{i}(q_2 b_3^{[s+1]} - q_3 b_2^{[s+1]} + p_2 c_3^{[s+1]} - p_3 c_2^{[s+1]}), \end{cases} \tag{2.8}$$

and

$$\begin{aligned} a_x^{[s+1]} &= \mathrm{i}(-2q_1 b_1^{[s+1]} - q_2 b_2^{[s+1]} - q_3 b_3^{[s+1]} + 2p_1 c_1^{[s+1]} + p_2 c_2^{[s+1]} + p_3 c_3^{[s+1]}) \\ &= -(2q_1 b_{1,x}^{[s]} + q_2 b_{2,x}^{[s]} + q_3 b_{3,x}^{[s]} + 2p_1 c_{1,x}^{[s]} + p_2 c_{2,x}^{[s]} + p_3 c_{3,x}^{[s]}), \end{aligned} \tag{2.9}$$

where $s \geq 0$. To get the uniqueness of Laurent series solutions, we set the initial values,

$$a^{[0]} = 1,\ d_1^{[0]} = d_2^{[0]} = d_3^{[0]} = 0, \tag{2.10}$$

and take the constant of integration as zero,

$$a^{[s]}|_{u=0} = 0,\ d_1^{[s]}|_{u=0} = d_2^{[s]}|_{u=0} = d_3^{[s]}|_{u=0} = 0,\ s \geq 1. \tag{2.11}$$

Consequently, we can uniquely determine that

$$\begin{cases} b_1^{[1]} = p_1,\ b_2^{[1]} = p_2,\ b_3^{[1]} = p_3, \\ c_1^{[1]} = q_1,\ c_2^{[1]} = q_2,\ c_3^{[1]} = q_3, \\ d_1^{[1]} = d_2^{[1]} = d_3^{[1]} = 0,\ a^{[1]} = 0; \end{cases}$$

$$\begin{cases} b_1^{[2]} = -\mathrm{i}p_{1,x},\ b_2^{[2]} = -\mathrm{i}p_{2,x},\ b_3^{[2]} = -\mathrm{i}p_{3,x}, \\ c_1^{[2]} = \mathrm{i}q_{1,x},\ c_2^{[2]} = \mathrm{i}q_{2,x},\ c_3^{[2]} = \mathrm{i}q_{3,x}, \\ d_1^{[2]} = -p_1 q_2 + p_2 q_1,\ d_2^{[2]} = -p_1 q_3 + p_3 q_1,\ d_3^{[2]} = -p_2 q_3 + p_3 q_2, \\ a^{[2]} = -2p_1 q_1 - p_2 q_2 - p_3 q_3; \end{cases}$$

$$\begin{cases} b_1^{[3]} = -p_{1,xx} - 2p_1^2 q_1 - 2p_1(p_2 q_2 + p_3 q_3) + (p_2^2 + p_3^2) q_1, \\ b_2^{[3]} = -p_{2,xx} + 2p_1^2 q_2 - 4p_1 p_2 q_1 - 2p_2 p_3 q_3 - (p_2^2 - p_3^2) q_2, \\ b_3^{[3]} = -p_{3,xx} + 2p_1^2 q_3 - 4p_1 p_3 q_1 - 2p_2 p_3 q_2 + (p_2^2 - p_3^2) q_3, \end{cases}$$

$$\begin{cases} c_1^{[3]} = -q_{1,xx} - 2p_1 q_1^2 - 2p_2 q_1 q_2 - 2p_3 q_1 q_3 + p_1(q_2^2 + q_3^2), \\ c_2^{[3]} = -q_{2,xx} + 2p_2 q_1^2 - 4p_1 q_1 q_2 - 2p_3 q_2 q_3 - p_2(q_2^2 - q_3^2), \\ c_3^{[3]} = -q_{3,xx} + 2p_3 q_1^2 - 4p_1 q_1 q_3 - 2p_2 q_2 q_3 + p_3(q_2^2 - q_3^2), \end{cases}$$

$$\begin{cases} d_1^{[3]} = -\mathrm{i}(p_1 q_{2,x} - p_2 q_{1,x} - p_{1,x} q_2 + p_{2,x} q_1), \\ d_2^{[3]} = -\mathrm{i}(p_1 q_{3,x} - p_3 q_{1,x} - p_{1,x} q_3 + p_{3,x} q_1), \\ d_3^{[3]} = -\mathrm{i}(p_2 q_{3,x} - p_3 q_{2,x} - p_{2,x} q_3 + p_{3,x} q_2), \end{cases}$$

$$a^{[3]} = -\mathrm{i}(2p_1 q_{1,x} - 2p_{1,x} q_1 + p_2 q_{2,x} - p_{2,x} q_2 + p_3 q_{3,x} - p_{3,x} q_3);$$

and

$$\begin{cases} b_1^{[4]} = \mathrm{i}(p_{1,xxx} + 6p_1p_{1,x}q_1 + 3p_1p_{2,x}q_2 + 3p_1p_{3,x}q_3 \\ \qquad -3p_2p_{2,x}q_1 - 3p_3p_{3,x}q_1 + 3p_{1,x}p_2q_2 + 3p_{1,x}p_3q_3), \\ b_2^{[4]} = \mathrm{i}(p_{2,xxx} + 6p_1p_{2,x}q_1 - 6p_1p_{1,x}q_2 + 6p_{1,x}p_2q_1 \\ \qquad +3p_2p_{2,x}q_2 + 3p_2p_{3,x}q_3 + 3p_{2,x}p_3q_3 - 3p_3p_{3,x}q_2), \\ b_3^{[4]} = \mathrm{i}(p_{3,xxx} + 6p_1p_{3,x}q_1 - 6p_1p_{1,x}q_3 + 6p_{1,x}p_3q_1 \\ \qquad -3p_2p_{2,x}q_3 + 3p_2p_{3,x}q_2 + 3p_{2,x}p_3q_2 + 3p_3p_{3,x}q_3), \end{cases}$$

$$\begin{cases} c_1^{[4]} = -\mathrm{i}(q_{1,xxx} + 6p_1q_1q_{1,x} - 3p_1q_2q_{2,x} - 3p_1q_3q_{3,x} \\ \qquad +3p_2q_{1,x}q_2 + 3p_2q_1q_{2,x} + 3p_3q_{1,x}q_3 + 3p_3q_1q_{3,x}), \\ c_2^{[4]} = -\mathrm{i}(q_{2,xxx} - 6p_2q_1q_{1,x} + 6p_1q_1q_{2,x} + 6p_1q_{1,x}q_2 \\ \qquad +3p_2q_2q_{2,x} - 3p_2q_3q_{3,x} + 3p_3q_2q_{3,x} + 3p_3q_{2,x}q_3), \\ c_3^{[4]} = -\mathrm{i}(q_{3,xxx} + 6p_1q_1q_{3,x} + 6p_1q_{1,x}q_3 - 6p_3q_1q_{1,x} \\ \qquad +3p_2q_2q_{3,x} + 3p_2q_{2,x}q_3 - 3p_3q_2q_{2,x} + 3p_3q_3q_{3,x}), \end{cases}$$

$$\begin{cases} d_1^{[4]} = 6(p_1q_1 + \frac{1}{2}p_2q_2 + \frac{1}{2}p_3q_3)(p_1q_2 - p_2q_1) + p_{1,xx}q_2 - p_{2,xx}q_1 \\ \qquad -p_2q_{1,xx} + p_1q_{2,xx} - p_{1,x}q_{2,x} + p_{2,x}q_{1,x}, \\ d_2^{[4]} = 6(p_1q_1 + \frac{1}{2}p_2q_2 + \frac{1}{2}p_3q_3)(p_1q_3 - p_3q_1) + p_{1,xx}q_3 - p_{3,xx}q_1 \\ \qquad -p_3q_{1,xx} + p_1q_{3,xx} - p_{1,x}q_{3,x} + p_{3,x}q_{1,x}, \\ d_3^{[4]} = 6(p_1q_1 + \frac{1}{2}p_2q_2 + \frac{1}{2}p_3q_3)(p_2q_3 - p_3q_2) + p_{2,xx}q_3 - p_{3,xx}q_2 \\ \qquad -p_3q_{2,xx} + p_2q_{3,xx} - p_{2,x}q_{3,x} + p_{3,x}q_{2,x}, \end{cases}$$

$$\begin{aligned} a^{[4]} = & 3(2q_1^2 - q_2^2 - q_3^2)p_1^2 - \frac{3}{2}(2q_1^2 - q_2^2 + q_3^2)p_2^2 - \frac{3}{2}(2q_1^2 + q_2^2 - q_3^2)p_3^2 \\ & + 12p_1(p_2q_2 + p_3q_3)q_1 + 6p_2p_3q_2q_3 + 2p_1q_{1,xx} + 2p_{1,x,x}q_1 \\ & + p_2q_{2,xx} + p_{2,xx}q_2 + p_3q_{3,xx} + p_{3,xx}q_3 - 2p_{1,x}q_{1,x} - p_{2,x}q_{2,x} - p_{3,x}q_{3,x}. \end{aligned}$$

Based on the structure of the spatial spectral matrix U, we can determine that the temporal matrix spectral problems can be taken as

$$-\mathrm{i}\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi,\ V^{[r]} = (\lambda^r W)_+ = \sum_{s=0}^{r} \lambda^s W^{[r-s]},\ r \geq 0, \tag{2.12}$$

which are the other parts of Lax pairs of matrix spectral problems. The resulting compatibility conditions of the spatial and temporal matrix spectral problems in (2.1) and (2.12), namely, the zero curvature equations in (1.5), yield a six-component integrable hierarchy:

$$u_{t_r} = K^{[r]} = (\mathrm{i}b_1^{[r+1]}, \mathrm{i}b_2^{[r+1]}, \mathrm{i}b_3^{[r+1]}, -\mathrm{i}c_1^{[r+1]}, -\mathrm{i}c_2^{[r+1]}, -\mathrm{i}c_3^{[r+1]})^T,\ r \geq 0, \tag{2.13}$$

or more concretely,

$$\begin{cases} p_{1,t_r} = \mathrm{i}b_1^{[r+1]},\ p_{2,t_r} = \mathrm{i}b_2^{[r+1]},\ p_{3,t_r} = \mathrm{i}b_3^{[r+1]}, \\ q_{1,t_r} = -\mathrm{i}c_1^{[r+1]},\ q_{2,t_r} = -\mathrm{i}c_2^{[r+1]},\ q_{3,t_r} = -\mathrm{i}c_3^{[r+1]}, \end{cases} r \geq 0. \tag{2.14}$$

To illustrate the hierarchy, we work out the first two nonlinear examples. The first gives a system of integrable coupled nonlinear Schrödinger equations:

$$\begin{cases} \mathrm{i}p_{1,t_2} = p_{1,xx} + 2p_1^2 q_1 + 2p_1(p_2q_2 + p_3q_3) - (p_2^2 + p_3^2)q_1, \\ \mathrm{i}p_{2,t_2} = p_{2,xx} - 2p_1^2 q_2 + 4p_1p_2q_1 + 2p_2p_3q_3 + (p_2^2 - p_3^2)q_2, \\ \mathrm{i}p_{3,t_2} = p_{3,xx} - 2p_1^2 q_3 + 4p_1p_3q_1 + 2p_2p_3q_2 - (p_2^2 - p_3^2)q_3, \end{cases} \tag{2.15}$$

and

$$\begin{cases} \mathrm{i}q_{1,t_2} = -q_{1,xx} - 2p_1q_1^2 - 2p_2q_1q_2 - 2p_3q_1q_3 + p_1(q_2^2 + q_3^2), \\ \mathrm{i}q_{2,t_2} = -q_{2,xx} + 2p_2q_1^2 - 4p_1q_1q_2 - 2p_3q_2q_3 - p_2(q_2^2 - q_3^2), \\ \mathrm{i}q_{3,t_2} = -q_{3,xx} + 2p_3q_1^2 - 4p_1q_1q_3 - 2p_2q_2q_3 + p_3(q_2^2 - q_3^2); \end{cases} \tag{2.16}$$

and the seond presents a system of integrable coupled modified Korteweg-de Vries equations:

$$\begin{cases} p_{1,t_3} = -p_{1,xxx} - 6p_1p_{1,x}q_1 - 3p_1p_{2,x}q_2 - 3p_1p_{3,x}q_3 \\ \qquad\qquad +3p_2p_{2,x}q_1 + 3p_3p_{3,x}q_1 - 3p_{1,x}p_2q_2 - 3p_{1,x}p_3q_3, \\ p_{2,t_3} = -p_{2,xxx} - 6p_1p_{2,x}q_1 + 6p_1p_{1,x}q_2 - 6p_{1,x}p_2q_1 \\ \qquad\qquad -3p_2p_{2,x}q_2 - 3p_2p_{3,x}q_3 - 3p_{2,x}p_3q_3 + 3p_3p_{3,x}q_2, \\ p_{3,t_3} = -p_{3,xxx} - 6p_1p_{3,x}q_1 + 6p_1p_{1,x}q_3 - 6p_{1,x}p_3q_1 \\ \qquad\qquad +3p_2p_{2,x}q_3 - 3p_2p_{3,x}q_2 - 3p_{2,x}p_3q_2 - 3p_3p_{3,x}q_3, \end{cases} \tag{2.17}$$

and

$$\begin{cases} q_{1,t_3} = -q_{1,xxx} - 6p_1q_1q_{1,x} + 3p_1q_2q_{2,x} + 3p_1q_3q_{3,x} \\ \qquad\qquad -3p_2q_{1,x}q_2 - 3p_2q_1q_{2,x} - 3p_3q_{1,x}q_3 - 3p_3q_1q_{3,x}, \\ q_{2,t_3} = -q_{2,xxx} + 6p_2q_1q_{1,x} - 6p_1q_1q_{2,x} - 6p_1q_{1,x}q_2 \\ \qquad\qquad -3p_2q_2q_{2,x} + 3p_2q_3q_{3,x} - 3p_3q_2q_{3,x} - 3p_3q_{2,x}q_3, \\ q_{3,t_3} = -q_{3,xxx} - 6p_1q_1q_{3,x} - 6p_1q_{1,x}q_3 + 6p_3q_1q_{1,x} \\ \qquad\qquad -3p_2q_2q_{3,x} - 3p_2q_{2,x}q_3 + 3p_3q_2q_{2,x} - 3p_3q_3q_{3,x}. \end{cases} \tag{2.18}$$

They are different from the counterparts associated with the AKNS standard matrix spectral problems (see, e.g., [22]), and provide novel representatives of integrable coupled nonlinear Schrödinger equations and integrable coupled modified Korteweg-de Vries equations.

3 Hamiltonian Formulation

To establish a Hamiltonian formulation for the integrable hierarchy (2.14), we apply the trace identity (1.6) to the matrix spatial spectral problem (2.1). By using the solution W defined by (2.3), we can directly compute that

$$\mathrm{tr}\big(W\frac{\partial U}{\partial \lambda}\big) = 2a,\ \mathrm{tr}\big(W\frac{\partial U}{\partial u}\big) = 2(2c_1, c_2, c_3, 2b_1, b_2, b_3)^T,$$

and therefore, we have

$$\frac{\delta}{\delta u}\int a\,\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(2c_1, c_2, c_3, 2b_1, b_2, b_3)^T.$$

A check of the case with $s = 2$ leads to $\gamma = 0$, and consequently, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}^{[s]} = (2c_1^{[s+1]}, c_2^{[s+1]}, c_3^{[s+1]}, 2b_1^{[s+1]}, b_2^{[s+1]}, b_3^{[s+1]})^T,\ s \geq 0, \tag{3.1}$$

where the Hamiltonian functionals are defined by

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+2]}}{s+1}\,\mathrm{d}x,\ \ s \geq 0. \tag{3.2}$$

This allows us to furnish the Hamiltonian formulation for the integrable hierarchy (2.14):

$$u_{t_r} = K^{[r]} = J\frac{\delta \mathcal{H}^{[r]}}{\delta u},\ \ J = \left[\begin{array}{ccc|ccc} & & & \frac{1}{2}\mathrm{i} & 0 & 0 \\ & 0 & & 0 & \mathrm{i} & 0 \\ & & & 0 & 0 & \mathrm{i} \\ \hline -\frac{1}{2}\mathrm{i} & 0 & 0 & & & \\ 0 & -\mathrm{i} & 0 & & 0 & \\ 0 & 0 & -\mathrm{i} & & & \end{array}\right],\ \ r \geq 0, \tag{3.3}$$

where J is a Hamiltonian operator and the Hamiltonian functionals $\mathcal{H}^{[r]}$ are given by (3.2). The resulting Hamiltonian formulation gives a relation $S = J\frac{\delta\mathcal{H}}{\delta u}$ from a conserved functional $\mathcal{H}$ to a symmetry S. The commutativity of these symmetries:

$$[\![K^{[s_1]}, K^{[s_2]}]\!] = K^{[s_1]\prime}(u)[K^{[s_2]}] - K^{[s_2]\prime}(u)[K^{[s_1]}] = 0,\ \ s_1, s_2 \geq 0, \tag{3.4}$$

is a consequence of a Lax operator algebra:

$$[\![V^{[s_1]}, V^{[s_2]}]\!] = V^{[s_1]\prime}(u)[K^{[s_2]}] - V^{[s_2]\prime}(u)[K^{[s_1]}] + [V^{[s_1]}, V^{[s_2]}] = 0,\ \ s_1, s_2 \geq 0, \tag{3.5}$$

which can be proved directly (see [23] for details). It further follows from the Hamiltonian formulation that the conserved functionals also commute:

$$\{\mathcal{H}^{[s_1]}, \mathcal{H}^{[s_2]}\}_J = \int \Big(\frac{\delta \mathcal{H}^{[s_1]}}{\delta u}\Big)^T J \frac{\delta \mathcal{H}^{[s_2]}}{\delta u}\,\mathrm{d}x = 0,\ \ s_1, s_2 \geq 0, \tag{3.6}$$

under the Poisson bracket associated with the Hamiltonian operator J. Based on a combination of the Hamiltonian operator J and a recursion operator Φ [24], generated from $K^{[s+1]} = \Phi K^{[s]}$, a bi-Hamiltonian formulation [4] can also be furnished for the integrable hierarchy (2.14).

4 Integrable Hierarchies with Higher-order Spectral Matrices

Fix an arbitrary natural number n. Let us take a generalization of the matrix spatial spectral problem (2.1):

$$-\mathrm{i}\phi_x = U\phi,\ \ U = \left[\begin{array}{c|c|c} \lambda & p_1\ p_2\ p_3\ p_1\ \cdots\ p_1 & 0 \\ \hline q_1 & & p_1 \\ q_2 & & p_2 \\ q_3 & & p_2 \\ q_1 & 0 & p_1 \\ \vdots & & \vdots \\ q_1 & & p_1 \\ \hline 0 & q_1\ q_2\ q_3\ q_1\ \cdots\ q_1 & -\lambda \end{array}\right]_{(n+5)\times(n+5)}. \tag{4.1}$$

Assume that a Laurent series solution to the stationary zero curvature equation (1.3) reads

$$W=\begin{bmatrix} a & b_1 & b_2 & b_3 & b_1 & \cdots & b_1 & 0\\ c_1 & 0 & d_1 & d_2 & 0 & \cdots & 0 & b_1\\ c_2 & -d_1 & 0 & d_3 & -d_1 & \cdots & -d_1 & b_2\\ c_3 & -d_2 & -d_3 & 0 & -d_2 & \cdots & -d_2 & b_3\\ c_1 & 0 & d_1 & d_2 & 0 & \cdots & 0 & b_1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ c_1 & 0 & d_1 & d_2 & 0 & \cdots & 0 & b_1\\ 0 & c_1 & c_2 & c_3 & c_1 & \cdots & c_1 & -a \end{bmatrix}_{(n+5)\times(n+5)} =\sum_{s\ge 0}\lambda^{-s}W^{[s]},$$

with

$$W^{[s]}=\begin{bmatrix} a^{[s]} & b_1^{[s]} & b_2^{[s]} & b_3^{[s]} & b_1^{[s]} & \cdots & b_1^{[s]} & 0\\ c_1^{[s]} & 0 & d_1^{[s]} & d_2^{[s]} & 0 & \cdots & 0 & b_1^{[s]}\\ c_2^{[s]} & -d_1^{[s]} & 0 & d_3^{[s]} & -d_1^{[s]} & \cdots & -d_1^{[s]} & b_2^{[s]}\\ c_3^{[s]} & -d_2^{[s]} & -d_3^{[s]} & 0 & -d_2^{[s]} & \cdots & -d_2^{[s]} & b_3^{[s]}\\ c_1^{[s]} & 0 & d_1^{[s]} & d_2^{[s]} & 0 & \cdots & 0 & b_1^{[s]}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ c_1^{[s]} & 0 & d_1^{[s]} & d_2^{[s]} & 0 & \cdots & 0 & b_1^{[s]}\\ 0 & c_1^{[s]} & c_2^{[s]} & c_3^{[s]} & c_1^{[s]} & \cdots & c_1^{[s]} & -a^{[s]} \end{bmatrix}_{(n+5)\times(n+5)},$$

where $s\ge 0$. Consequently, we can have

$$\begin{cases} b_{1,x}=-\mathrm{i}(p_1a+p_2d_1+p_3d_2-\lambda b_1),\\ b_{2,x}=-\mathrm{i}[p_2a-(n+1)p_1d_1+p_3d_3-\lambda b_2],\\ b_{3,x}=-\mathrm{i}[p_3a-(n+1)p_1d_2-p_2d_3-\lambda b_3], \end{cases} \tag{4.2}$$

$$\begin{cases} c_{1,x}=\mathrm{i}(q_1a-q_2d_1-q_3d_2-\lambda c_1),\\ c_{2,x}=\mathrm{i}[q_2a+(n+1)q_1d_1-q_3d_3-\lambda c_2],\\ c_{3,x}=\mathrm{i}[q_3a+(n+1)q_1d_2+q_2d_3-\lambda c_3], \end{cases} \tag{4.3}$$

$$\begin{cases} d_{1,x}=\mathrm{i}(q_1b_2-q_2b_1+p_1c_2-p_2c_1),\\ d_{2,x}=\mathrm{i}(q_1b_3-q_3b_1+p_1c_3-p_3c_1),\\ d_{3,x}=\mathrm{i}(q_2b_3-q_3b_2+p_2c_3-p_3c_2), \end{cases} \tag{4.4}$$

and

$$\begin{aligned} a_x&=\mathrm{i}[-(n+1)q_1b_1-q_2b_2-q_3b_3+(n+1)p_1c_1+p_2c_2+p_3c_3)\\ &=-\lambda^{-1}[(n+1)q_1b_{1,x}+q_2b_{2,x}+q_3b_{3,x}+(n+1)p_1c_{1,x}+p_2c_{2,x}+p_3c_{3,x}]. \end{aligned} \tag{4.5}$$

A direct computation shows that

$$\frac{\delta}{\delta u}\int a\,\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}((n+1)c_1,c_2,c_3,(n+1)b_1,b_2,b_3)^T. \tag{4.6}$$

Therefore, we obtain the Hamiltonian formulation for those associated integrable equations:

$$u_{t_r} = K^{[r]} = (\mathrm{i}b_1^{[r+1]}, \mathrm{i}b_2^{[r+1]}, \mathrm{i}b_3^{[r+1]}, -\mathrm{i}c_1^{[r+1]}, -\mathrm{i}c_2^{[r+1]}, -\mathrm{i}c_3^{[r+1]})^T = J\frac{\delta \mathcal{H}^{[r]}}{\delta u},\ r \geq 0, \tag{4.7}$$

where

$$J = \left[\begin{array}{ccc|ccc} & & & \frac{1}{n+1}\mathrm{i} & 0 & 0 \\ & 0 & & 0 & \mathrm{i} & 0 \\ & & & 0 & 0 & \mathrm{i} \\ \hline -\frac{1}{n+1}\mathrm{i} & 0 & 0 & & & \\ 0 & -\mathrm{i} & 0 & & 0 & \\ 0 & 0 & -\mathrm{i} & & & \end{array}\right],\ \mathcal{H}^{[r]} = -\int \frac{a^{[r+2]}}{r+1}\mathrm{d}x,\ r \geq 0. \tag{4.8}$$

If we take the initial values in (2.10) and zero constants of integration, then we can compute the first two nonlinear examples in those generalized hierarchies depending on values of n. The first system is the integrable coupled nonlinear Schrödinger equations:

$$\begin{cases} \mathrm{i}p_{1,t_2} = p_{1,xx} + (n+1)p_1^2 q_1 + 2p_1p_2q_2 + 2p_1p_3q_3 - (p_2^2 + p_3^2)q_1, \\ \mathrm{i}p_{1,t_2} = p_{2,xx} - (n+1)p_1^2 q_2 + 2(n+1)p_1p_2q_1 + 2p_2p_3q_3 + (p_2^2 - p_3^2)q_2, \\ \mathrm{i}p_{3,t_2} = p_{3,xx} - (n+1)p_1^2 q_3 + 2(n+1)p_1p_3q_1 + 2p_2p_3q_2 - (p_2^2 - p_3^2)q_3, \end{cases} \tag{4.9}$$

and

$$\begin{cases} \mathrm{i}q_{1,t_2} = -q_{1,xx} - (n+1)p_1 q_1^2 - 2p_2q_1q_2 - 2p_3q_1q_3 + p_1(q_2^2 + q_3^2), \\ \mathrm{i}q_{2,t_2} = -q_{2,xx} + (n+1)p_2 q_1^2 - 2(n+1)p_1q_1q_2 - 2p_3q_2q_3 - p_2(q_2^2 - q_3^2), \\ \mathrm{i}q_{3,t_2} = -q_{3,xx} + (n+1)p_3 q_1^2 - 2(n+1)p_1q_1q_3 - 2p_2q_2q_3 + p_3(q_2^2 - q_3^2); \end{cases} \tag{4.10}$$

and the second system, the integrable coupled modified Korteweg-de Vries equations:

$$\begin{cases} p_{1,t_3} = -p_{1,xxx} - 3(n+1)p_1p_{1,x}q_1 - 3p_1p_{2,x}q_2 - 3p_1p_{3,x}q_3 \\ \qquad\quad +3p_2p_{2,x}q_1 + 3p_3p_{3,x}q_1 - 3p_{1,x}p_2q_2 - 3p_{1,x}p_3q_3, \\ p_{2,t_3} = -p_{2,xxx} - 3(n+1)p_1p_{2,x}q_1 - 3(n+1)p_1p_{1,x}q_2 - 3(n+1)p_{1,x}p_2q_1 \\ \qquad\quad -3p_2p_{2,x}q_2 - 3p_2p_{3,x}q_3 - 3p_{2,x}p_3q_3 + 3p_3p_{3,x}q_2, \\ p_{3,t_3} = -p_{3,xxx} - 3(n+1)p_1p_{3,x}q_1 + 3(n+1)p_1p_{1,x}q_3 - 3(n+1)p_{1,x}p_3q_1 \\ \qquad\quad +3p_2p_{2,x}q_3 - 3p_2p_{3,x}q_2 - 3p_{2,x}p_3q_2 - 3p_3p_{3,x}q_3, \end{cases} \tag{4.11}$$

and

$$\begin{cases} q_{1,t_3} = -q_{1,xxx} - 3(n+1)p_1q_1q_{1,x} + 3p_1q_2q_{2,x} + 3p_1q_3q_{3,x} \\ \qquad\quad -3p_2q_{1,x}q_2 - 3p_2q_1q_{2,x} - 3p_3q_{1,x}q_3 - 3p_3q_1q_{3,x}, \\ q_{2,t_3} = -q_{2,xxx} + 3(n+1)p_2q_1q_{1,x} - 3(n+1)p_1q_1q_{2,x} - 3(n+1)p_1q_{1,x}q_2 \\ \qquad\quad -3p_2q_2q_{2,x} + 3p_2q_3q_{3,x} - 3p_3q_2q_{3,x} - 3p_3q_{2,x}q_3, \\ q_{3,t_3} = -q_{3,xxx} - 3(n+1)p_1q_1q_{3,x} - 3(n+1)p_1q_{1,x}q_3 + 3(n+1)p_3q_1q_{1,x} \\ \qquad\quad -3p_2q_2q_{3,x} - 3p_2q_{2,x}q_3 + 3p_3q_2q_{2,x} - 3p_3q_3q_{3,x}. \end{cases} \tag{4.12}$$

Again, these provide two novel examples of integrable coupled equations.

5 Concluding Remarks

Certain integrable hierarchies of Hamiltonian equations with six components have been constructed from a kind of special matrix spectral problems, through the zero curvature formulation. For each hierarchy, two important objects are a spatial spectral matrix and a Laurent series solution to the corresponding stationary zero curvature equation. The resulting integrable equations possess Hamiltonian formulations, derived from applications of the trace identity to the underlying matrix spectral problems. This guarantees the existence of infinitely many symmetries and conserved Hamiltonian functionals.

On one hand, generalized matrix spectral problems can further be made by taking more copies of p_2 and p_3 as did for p_1. On the other hand, one can also include more dependent variables in matrix spectral problems (see, e.g., [25, 26]). These generalizations need enormous effort, due to computational complexity.

It is definitely interesting to study soliton structures for the resulting integrable equations, and one can try the Riemann-Hilbert technique [27], the Darboux transformation [28, 29] and the determinant approach [30]. Various other types of interesting solutions (see, e.g., [31–34]) can be computed by taking wave number reductions of soliton solutions. Another avenue that needs further investigation is to make group reductions for the considered matrix spectral problems. This will generate local and nonlocal reduced integrable equations (see, e.g., [22, 35] and [36–38], respectively).

Conflict of Interest The author declares no conflict of interest.

References

[1] Das A. Integrable Models. Teaneck, NJ: World Scientific, 1989

[2] Tu G Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. J Phys A: Math Gen, 1989, **22**(13): 2375–2392

[3] Ma W X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. Chin Ann Math Ser A, 1992, **13**(1): 115–123

[4] Magri F. A simple model of the integrable Hamiltonian equation. J Math Phys, 1978, **19**(5): 1156–1162

[5] Ablowitz M J, Kaup D J, Newell A C, Segur H. The inverse scattering transform-Fourier analysis for nonlinear problems. Stud Appl Math, 1974, **53**(4): 249–315

[6] Wu Y T, Geng X G, Hu, X B, Zhu S M. A generalized Hirota-Satsuma coupled Korteweg-de Vries equation and Miura transformations. Phys Lett A, 1999, **255**: 259–264

[7] Xia T C, Yu F J, Zhang Y. The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions. Phys A, 2004, **343**: 238–246

[8] Manukure S. Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints. Commun Nonlinear Sci Numer Simul, 2018, **57**: 125–135

[9] Liu T S, Xia T C. Multi-component generalized Gerdjikov-Ivanov integrable hierarchy and its Riemann-Hilbert problem. Nonlinear Anal Real World Appl, 2022, **68**: 103667

[10] Wang H F, Zhang Y F. Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations. J Comput Appl Math, 2023, **420**: 114812

[11] Ma W X. A Hamiltonian structure associated with a matrix spectral problem of arbitrary-order. Phys Lett A, 2007, **367**(6): 473–477

[12] Yao Y Q, Shen S F, Ma W X. A soliton hierarchy associated with a spectral problem of 2nd degree in a spectral parameter and its bi-Hamiltonian structure. Adv Math Phys, 2016, **2016**: 3589704

[13] Wang Z B, Wang H F. Integrable couplings of two expanded non-isospectral soliton hierarchies and their bi-Hamiltonian structures. Int J Geom Methods Mod Phys, 2022, **19**(10): 2250160

[14] Ma W X. A combined Liouville integrable hierarchy associated with a fourth-order matrix spectral problem. Commun Theor Phys, 2024, **76**(7): 075001

[15] Yang J Y, Ma W X. Four-component Liouville integrable models and their bi-Hamiltonian formulations. Rom J Phys, 2024, **69**(1/2): 101

[16] Ma W X. A four-component hierarchy of combined integrable equations with bi-Hamiltonian formulations. Appl Math Lett, 2024, **153**: 109025

[17] Ma W X. A generalized hierarchy of combined integrable bi-Hamiltonian equations from a specific fourth-order matrix spectral problem. Mathematics, 2024, **12**(6): 927

[18] Kaup D J, Newell A C. An exact solution for a derivative nonlinear Schrödinger equation. J Math Phys, 1978, **19**(4): 798–801

[19] Wadati M, Konno K, Ichikawa Y H. New integrable nonlinear evolution equations. J Phys Soc Jpn, 1979, **47**(5): 1698–1700

[20] Takhtajan L A. Integration of the continuous Heisenberg spin chain through the inverses scattering method. Phys Lett A, 1977, **64**(2): 235–237

[21] Ma W X. Reduced non-local integrable NLS hierarchies by pairs of local and non-local constraints. Int J Appl Comput Math, 2022, **8**(4): 206

[22] Ma W X. Sasa-Satsuma type matrix integrable hierarchies and their Riemann-Hilbert problems and soliton solutions. Phys D, 2023, **446**: 133672

[23] Ma W X. The algebraic structure of zero curvature representations and application to coupled KdV systems. J Phys A: Math Gen, 1993, **26**(11): 2573–2582

[24] Fuchssteiner B, Fokas A S. Symplectic structures, their Bäcklund transformations and hereditary symmetries. Phys D, 1981/82, **4**(1): 47–66

[25] Zhang Y, Dong H H. Multi-component Gerdjikov-Ivanov system and its Riemann-Hilbert problem under zero boundary conditions. Nonlinear Anal Real World Appl, 2021, **60**: 103279

[26] Ye R S, Zhang Y. Two-component complex modified Korteweg-de Vries equations: New soliton solutions from novel binary Darboux transformation. Theor Math Phys, 2023, **214**(2): 183–193

[27] Novikov S P, Manakov S V, Pitaevskii L P, Zakharov V E. Theory of Solitons: the Inverse Scattering Method. New York: Consultants Bureau, 1984

[28] Matveev V B, Salle M A. Darboux Transformations and Solitons. Berlin: Springer-Verlag, 1991

[29] Geng X G, Li R M, Xue B. A vector general nonlinear Schrödinger equation with $(m+n)$ components. J Nonlinear Sci, 2020, **30**(3): 991–1013

[30] Aktosun T, Busse T, Demontis F, van der Mee C. Symmetries for exact solutions to the nonlinear Schrödinger equation. J Phys A: Math Theoret, 2010, **43**(2): 025202

[31] Cheng L, Zhang Y, Lin M J. Lax pair and lump solutions for the (2+1)-dimensional DJKM equation associated with bilinear Bäcklund transformations. Anal Math Phys, 2019, **9**(4): 1741–1752

[32] Sulaiman T A, Yusuf A, Abdeljabbar A, Alquran M. Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation. J Geom Phys, 2021, **169**: 104347

[33] Manukure S, Chowdhury A, Zhou Y. Complexiton solutions to the asymmetric Nizhnik-Novikov-Veselov equation. Int J Mod Phys B, 2019, **33**(11): 1950098

[34] Zhou Y, Manukure S, McAnally M. Lump and rogue wave solutions to a (2+1)-dimensional Boussinesq type equation. J Geom Phys, 2021, **167**: 104275

[35] Ma W X. Riemann-Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction. Math Methods Appl Sci, 2019, **42**(4): 1099–1113

[36] Ma W X. Reduced nonlocal integrable mKdV equations of type $(-\lambda, \lambda)$ and their exact soliton solutions. Commun Theoret Phys, 2022, **74**(6): 065002

[37] Ma W X. Type (λ^*, λ) reduced nonlocal integrable AKNS equations and their soliton solutions. Appl Numer Math, 2024, **199**: 105–113

[38] Ma W X, Huang Y H, Wang F D, et al. Binary Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations. Int J Geom Methods Mod Phys, 2024, **21**: 2450182