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RIEMANN-HILBERT PROBLEMS OF A SIX-COMPONENT MKDV SYSTEM AND ITS SOLITON SOLUTIONS*

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Abstract Based on a 4×4 matrix spectral problem, an AKNS soliton hierarchy with six potentials is generated. Associated with this spectral problem, a kind of Riemann-Hilbert problems is formulated for a six-component system of mKdV equations in the resulting AKNS hierarchy. Soliton solutions to the considered system of coupled mKdV equations are computed, through a reduced Riemann-Hilbert problem where an identity jump matrix is taken.

Key words integrable hierarchy; Riemann-Hilbert problem; soliton solution

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1 Introduction

It is known that the Riemann-Hilbert approach is one of the most powerful techniques to generate integrable equations and their soliton solutions [1]. The approach starts with a kind of matrix spectral problems, which possess bounded eigenfunctions analytically extendable to the upper or lower half-plane. It is closely connected with the inverse scattering method in soliton theory [2]. The normalization conditions at infinity on the real axis in constructing the scattering coefficients is used in solving the corresponding Riemann-Hilbert problems [1]. Upon taking the jump matrix to be the identity matrix, reduced Riemann-Hilbert problems generate

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soliton solutions, whose special limits can lead to rational solutions and periodic solutions. Applications were made for a few integrable equations, including the multiple wave interaction equations [1], the Harry Dym eqution [3], the generalized Sasa-Satsuma equation [4] and the general coupled nonlinear Schrödinger equations [5].

We follow the standard procedure suited for Riemann-Hilbert problems, where the unit imaginary number i is consistently used. We, therefore, start with a pair of matrix spectral problems of the following form:

$$-\mathrm{i}\phi_x = U\phi,\ -\mathrm{i}\phi_t = V\phi,\ U = A(\lambda) + P(u,\lambda),\ V = B(\lambda) + Q(u,\lambda),$$

where λ is a spectral parameter, u is a potential, ϕ is an $n \times n$ matrix eigenfunction, A, B are constant commuting $n \times n$ matrices, and P, Q are trace-less $n \times n$ matrices. Their compatibility condition is the zero curvature equation

$$U_t - V_x + \mathrm{i}[U, V] = 0,$$

where $[\cdot,\cdot]$ is the matrix commutator. To formulate a Riemann-Hilbert problem for this zero curvature equation, we adopt the following pair of equivalent matrix spectral problems

$$\psi_x = \mathrm{i}[A(\lambda),\psi] + \check{P}(u,\lambda)\psi, \psi_t = \mathrm{i}[B(\lambda),\psi] + \check{Q}(u,\lambda)\psi,$$

where ψ is an $n \times n$ matrix eigenfunction, $\check{P} = \mathrm{i}P$ and $\check{Q} = \mathrm{i}Q$. The relation between ϕ and ψ is

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{\mathrm{i}A(\lambda)x + \mathrm{i}B(\lambda)t}.$$

This provides us with a possibility to have two analytical matrix eigenfunctions with the asymptotic conditions

$$\psi^{\pm} \to I_n,\ \text{when } x, t \to \pm\infty,$$

where I_n stands for the identity matrix of size n. Then we try to determine two analytical related matrix functions $P^{\pm}(x,t,\lambda)$, which are analytical in the upper and lower half-planes $C^{\pm} = \{z \in \mathbb{C} |\pm \mathrm{Im}(z) > 0\}$ and continuous in the closed upper and lower half-planes $C_0^{\pm} = \{z \in \mathbb{C} |\pm \mathrm{Im}(z) \geq 0\}$, respectively, to build a Riemann-Hilbert problem

$$G^+(x,t,\lambda) = G^-(x,t,\lambda)G(x,t,\lambda),\ \lambda \in \mathbb{R},$$

where

$$G^+(x,t,\lambda) = P^+(x,t,\lambda),\ \lambda \in \mathbb{C}_0^+,\ G^-(x,t,\lambda) = (P^-)^{-1}(x,t,\lambda),\ \lambda \in \mathbb{C}_0^-.$$

If we take the jump matrix G to be the identity matrix I_n, the corresponding Riemann-Hilbert problem can be normally solved, and soliton solutions can be generated through observing asymptotic behaviors of the matrix functions $P^{\pm}$ at infinity of λ. In this paper, we shall present an application example by considering a six-component system of mKdV equations and generate its soliton solutions by a special Riemann-Hilbert problem.

The rest of the paper is organized as follows. In Section 2, within the zero-curvature formulation, we rederive the AKNS soliton hierarchy with six potentials and furnish its bi-Hamiltonian structure, based on a new matrix spectral problem suited for the Riemann-Hilbert theory. In Section 3, taking a system of coupled mKdV equations as an example, we analyze analytical properties of matrix eigenfunctions for an equivalent spectral problem, and build a kind of Riemann-Hilbert problems associated with the newly introduced spectral problem.

In Section 4, we compute soliton solutions to the considered six-component system of coupled mKdV equations from a specific Riemann-Hilbert problem on the real axis, in which the jump matrix is taken as the identity matrix. In the last section, we give a summary of the results and some discussions.

2 AKNS Soliton Hierarchy with Six Components

2.1 Zero Curvature Formulation

Let us first recall the zero curvature formulation to construct soliton hierarchies [6]. Let u be a vector potential and λ, a spectral parameter. Choose a square spectral matrix $U=U(u,\lambda)$ from a given matrix loop algebra. Assume that

$$W=W(u,\lambda)=\sum_{k=0}^{\infty} W_k\lambda^{-k}=\sum_{k=0}^{\infty} W_k(u)\lambda^{-k} \tag{2.1}$$

solves the corresponding stationary zero curvature equation

$$W_x=\mathrm{i}[U,W]. \tag{2.2}$$

Based on this solution W, we introduce a series of Lax matrices

$$V^{[r]}=V^{[r]}(u,\lambda)=(\lambda^r W)_+ +\Delta_r,\ r\geq 0, \tag{2.3}$$

where the subscript $+$ denotes the operation of taking a polynomial part in λ, and Δ_r, $r\geq 0$, are appropriate modification terms, and then generate a soliton hierarchy

$$u_t=K_r(u)=K_r(x,t,u,u_x,\cdots),\ r\geq 0, \tag{2.4}$$

from a series of zero curvature equations

$$U_t-V_x^{[r]}+\mathrm{i}[U,V^{[r]}]=0,\ r\geq 0. \tag{2.5}$$

The two matrices U and $V^{[r]}$ are called a Lax pair [7] of the r-th soliton equation in the hierarchy (2.4). Obviously, the zero curvature equations in (2.5) are the compatibility conditions of the spatial and temporal matrix spectral problems

$$-\mathrm{i}\phi_x=U\phi=U(u,\lambda)\phi,\ -\mathrm{i}\phi_t=V^{[r]}\phi=V^{[r]}(u,\lambda)\phi,\ r\geq 0, \tag{2.6}$$

where ϕ is the matrix eigenfunction.

To show the Liouville integrability of the soliton hierarchy (2.4), we normally furnish a bi-Hamiltonian structure [8]

$$u_t=K_r=J\frac{\delta\tilde{H}_{r+1}}{\delta u}=M\frac{\delta\tilde{H}_r}{\delta u},\ r\geq 1, \tag{2.7}$$

where J and M form a Hamiltonian pair and $\frac{\delta}{\delta u}$ denotes the variational derivative (see e.g., [9]). The Hamiltonian structures can be often achieved through the trace identity [6]

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial\lambda})\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\Big[\lambda^{\gamma}\mathrm{tr}(W\frac{\partial U}{\partial u})\Big],\ \gamma=-\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\mathrm{tr}(W^2)|, \tag{2.8}$$

or more generally, the variational identity [10]

$$\frac{\delta}{\delta u}\int\langle W,\frac{\partial U}{\partial\lambda}\rangle\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\Big[\lambda^{\gamma}\langle W,\frac{\partial U}{\partial u}\rangle\Big],\ \gamma=-\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle W,W\rangle|, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra [11]. The bi-Hamiltonian structure guarantees that there exist infinitely many commuting Lie symmetries $\{K_n\}_{n=0}^{\infty}$ and conserved quantities $\{\tilde{H}_n\}_{n=0}^{\infty}$:

$$[K_{n_1},K_{n_2}]=K'_{n_1}[K_{n_2}]-K'_{n_2}[K_{n_1}]=0,$$
$$\{\tilde{\mathcal{H}}_{n_1},\tilde{\mathcal{H}}_{n_2}\}_N=\int\Big(\frac{\delta\tilde{\mathcal{H}}_{n_1}}{\delta u}\Big)^T N\frac{\delta\tilde{\mathcal{H}}_{n_2}}{\delta u}\,\mathrm{d}x=0,$$

where $n_1,n_2\geq 0$, $N=J$ or M, and K' stands for the Gateaux derivative of K with respect to u,

$$K'(u)[S]=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}K(u+\varepsilon S,u_x+\varepsilon S_x,\cdots).$$

It is known that for an evolution equation with a vector potential u, $\tilde{H}=\int H\,\mathrm{d}x$ is a conserved functional iff $\frac{\delta\tilde{H}}{\delta u}$ is an adjoint symmetry [12], and thus, the Hamiltonian structures links conserved functionals to adjoint symmetries and further symmetries.

When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations engender classical soliton hierarchies [13]; when semisimple, the associated zero curvature equations generate a collection of different soliton hierarchies; and when non-semisimple, we get hierarchies of integrable couplings [14], which require extra care in presenting soliton solutions.

2.2 AKNS Hierarchy with Six Components

Let us start with a 4×4 matrix spectral problem

$$-\mathrm{i}\phi_x=U\phi=U(u,\lambda)\phi,\ U=(U_{kl})_{4\times 4}=\begin{bmatrix}\alpha_1\lambda & p_1 & p_2 & p_3\\ q_1 & \alpha_2\lambda & 0 & 0\\ q_2 & 0 & \alpha_2\lambda & 0\\ q_3 & 0 & 0 & \alpha_2\lambda\end{bmatrix}, \tag{2.10}$$

where α_1 and α_2 are real constants, λ is a spectral parameter and u is a six-dimensional potential

$$u=(p,q^T)^T,\ p=(p_1,p_2,p_3),\ q=(q_1,q_2,q_3)^T. \tag{2.11}$$

A special case of $p_2=p_3=q_2=q_3=0$ transforms (2.10) into the AKNS spectral problem [15], and therefore it is called a six-component AKNS spectral problem. Since $\Lambda=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_2,\alpha_2)$ has a multiple eigenvalue, the spectral problem (2.10) is degenerate.

To derive the associated AKNS soliton hierarchy, we first solve the stationary zero curvature equation (2.2) corresponding to (2.10). We suppose that a solution W is given by

$$W=\begin{bmatrix}a & b\\ c & d\end{bmatrix}, \tag{2.12}$$

where a is a scalar, b^T and c are three-dimensional columns, and d is a 3×3 matrix. It is easy to see that the stationary zero curvature equation (2.2) becomes

$$a_x=\mathrm{i}(pc-bq),\ b_x=\mathrm{i}(\alpha\lambda b+pd-ap),\ c_x=\mathrm{i}(-\alpha\lambda c+qa-dq),\ d_x=\mathrm{i}(qb-cp), \tag{2.13}$$

where $\alpha=\alpha_1-\alpha_2$. We look for a formal series solution as

$$W=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\sum_{m=0}^{\infty}W_m\lambda^{-m},\ W_m=W_m(u)=\begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix},\ m\geq 0 \tag{2.14}$$

with $b^{[m]}, c^{[m]}$ and $d^{[m]}$ being assumed to be

$$b^{[m]}=(b_1^{[m]},b_2^{[m]},b_3^{[m]}),\ c^{[m]}=(c_1^{[m]},c_2^{[m]},c_3^{[m]})^T,\ d^{[m]}=(d_{kl}^{[m]})_{3\times 3},\ m\geq 0. \tag{2.15}$$

Then system (2.13) is equivalent to the following recursion relations

$$b^{[0]}=0,\ c^{[0]}=0,\ a_x^{[0]}=0,\ d_x^{[0]}=0, \tag{2.16a}$$

$$b^{[m+1]}=\frac{1}{\alpha}(-\mathrm{i}b_x^{[m]}-pd^{[m]}+a^{[m]}p),\ m\geq 0, \tag{2.16b}$$

$$c^{[m+1]}=\frac{1}{\alpha}(\mathrm{i}c_x^{[m]}+qa^{[m]}-d^{[m]}q),\ m\geq 0, \tag{2.16c}$$

$$a_x^{[m]}=\mathrm{i}(pc^{[m]}-b^{[m]}q),\ d_x^{[m]}=\mathrm{i}(qb^{[m]}-c^{[m]}p),\ m\geq 1. \tag{2.16d}$$

Let us now choose the initial values as follows

$$a^{[0]}=\beta_1,\ d^{[0]}=\beta_2 I_3, \tag{2.17}$$

where β_1,β_2 are arbitrary real constants and I_3 is the identity matrix of size 3, and take constants of integration in (2.16d) to be zero, that is, require

$$W_m|_{u=0}=0,\ m\geq 1. \tag{2.18}$$

Thus, with $a^{[0]}$ and $d^{[0]}$ given by (2.17), all matrices $W_m,\ m\geq 1$, will be uniquely determined. For example, a direct computation, based on (2.16), generates that

$$b_k^{[1]}=\frac{\beta}{\alpha}p_k,\ c_k^{[1]}=\frac{\beta}{\alpha}q_k,\ a^{[1]}=0,\ d_{kl}^{[1]}=0, \tag{2.19a}$$

$$b_k^{[2]}=-\frac{\beta}{\alpha^2}\mathrm{i}p_{k,x},\ c_k^{[2]}=\frac{\beta}{\alpha^2}\mathrm{i}q_{k,x},\ a^{[2]}=-\frac{\beta}{\alpha^2}\sum_{l=1}^{3}p_lq_l,\ d_{kl}^{[2]}=\frac{\beta}{\alpha^2}p_lq_k, \tag{2.19b}$$

$$b_k^{[3]}=-\frac{\beta}{\alpha^3}\left[p_{k,xx}+2\left(\sum_{l=1}^{3}p_lq_l\right)p_k\right],\ c_k^{[3]}=-\frac{\beta}{\alpha^3}\left[q_{k,xx}+2\left(\sum_{l=1}^{3}p_lq_l\right)q_k\right], \tag{2.19c}$$

$$a^{[3]}=-\frac{\beta}{\alpha^3}\mathrm{i}\sum_{l=1}^{3}(p_lq_{l,x}-p_{l,x}q_l),\ d_{kl}^{[3]}=-\frac{\beta}{\alpha^3}\mathrm{i}(p_{l,x}q_k-p_lq_{k,x}); \tag{2.19d}$$

$$b_k^{[4]}=\frac{\beta}{\alpha^4}\mathrm{i}\left[p_{k,xxx}+3\left(\sum_{l=1}^{3}p_lq_l\right)p_{k,x}+3\left(\sum_{l=1}^{3}p_{l,x}q_l\right)p_k\right], \tag{2.19e}$$

$$c_k^{[4]}=-\frac{\beta}{\alpha^4}\mathrm{i}\left[q_{k,xxx}+3\left(\sum_{l=1}^{3}p_lq_l\right)q_{k,x}+3\left(\sum_{l=1}^{3}p_lq_{l,x}\right)q_k\right], \tag{2.19f}$$

$$a^{[4]}=\frac{\beta}{\alpha^4}\left[3\left(\sum_{l=1}^{3}p_lq_l\right)^2+\sum_{l=1}^{3}(p_lq_{l,xx}-p_{l,x}q_{l,x}+p_{l,xx}q_l)\right], \tag{2.19g}$$

$$d_{kl}^{[4]}=-\frac{\beta}{\alpha^4}\left[3p_l\left(\sum_{l=1}^{3}p_lq_l\right)q_k+p_{l,xx}q_k-p_{l,x}q_{k,x}+p_lq_{k,xx}\right], \tag{2.19h}$$

where $\beta=\beta_1-\beta_2$ and $1\le k,l\le 3$. Based on (2.16d), we can obtain, from (2.16b) and (2.16c), a recursion relation for $b^{[m]}$ and $c^{[m]}$,

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix},\ m\ge 1, \tag{2.20}$$

where Ψ is a 6×6 matrix operator

$$\Psi=\frac{\mathrm{i}}{\alpha}\begin{bmatrix} \Big(\partial+\sum_{k=1}^{3} q_k\partial^{-1}p_k\Big)I_3+q\partial^{-1}p & -q\partial^{-1}q^T-(q\partial^{-1}q^T)^T \\ p^T\partial^{-1}p+(p^T\partial^{-1}p)^T & -\Big(\partial+\sum_{k=1}^{3} p_k\partial^{-1}q_k\Big)I_3-p^T\partial^{-1}q^T \end{bmatrix}. \tag{2.21}$$

To generate the AKNS soliton hierarchy with six components, we introduce, for all integers $r\ge 0$, the following Lax matrices

$$V^{[r]}=V^{[r]}(u,\lambda)=(V^{[r]}_{kl})_{4\times 4}=(\lambda^r W)_+=\sum_{k=0}^{r} W_k\lambda^{r-k},\ r\ge 0, \tag{2.22}$$

where the modification terms are taken as zero. The compatibility conditions of (2.6), i.e., the zero curvature equations (2.5), lead to the AKNS soliton hierarchy with six components

$$u_t=\begin{bmatrix} p^T \\ q \end{bmatrix}_t = K_r=\mathrm{i}\begin{bmatrix} \alpha b^{[r+1]T} \\ -\alpha c^{[r+1]} \end{bmatrix},\ r\ge 0. \tag{2.23}$$

The first two nonlinear systems in the above soliton hierarchy (2.23) read

$$p_{k,t}=-\frac{\beta}{\alpha^2}\mathrm{i}\Big[p_{k,xx}+2\Big(\sum_{l=1}^{3}p_lq_l\Big)p_k\Big],\ 1\le k\le 3, \tag{2.24a}$$

$$q_{k,t}=\frac{\beta}{\alpha^2}\mathrm{i}\Big[q_{k,xx}+2\Big(\sum_{l=1}^{3}p_lq_l\Big)q_k\Big],\ 1\le k\le 3, \tag{2.24b}$$

and

$$p_{k,t}=-\frac{\beta}{\alpha^3}\Big[p_{k,xxx}+3\Big(\sum_{l=1}^{3}p_lq_l\Big)p_{k,x}+3\Big(\sum_{l=1}^{3}p_{l,x}q_l\Big)p_k\Big],\ 1\le k\le 3, \tag{2.25a}$$

$$q_{k,t}=-\frac{\beta}{\alpha^3}\Big[q_{k,xxx}+3\Big(\sum_{l=1}^{3}p_lq_l\Big)q_{k,x}+3\Big(\sum_{l=1}^{3}p_lq_{l,x}\Big)q_k\Big],\ 1\le k\le 3, \tag{2.25b}$$

which are the six-component versions of the AKNS systems of coupled nonlinear Schrödinger equations and coupled mKdV equations, respectively. Under a symmetric reduction, the six-component AKNS systems (2.24) can be reduced to the Manokov system [16], for which a decomposition into finite-dimensional integrable Hamiltonian systems was made in [17], whileas the six-component AKNS systems (2.25) contain various systems of mKdV equations, for which there exist different kinds of integrable decompositions under symmetry constraints (see e.g., [18, 19]).

The AKNS soliton hierarchy (2.23) with six components possesses a Hamiltonian structure [12], which can be generated through the trace identity [6], or more generally, the variational identity [10]. Precisely, we have

$$-\mathrm{i}\,\mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)=\alpha_1 a+\alpha_2\mathrm{tr}(d)=\sum_{m=0}^{\infty}(\alpha_1 a^{[m]}+\alpha_2 d_{11}^{[m]}+\alpha_2 d_{22}^{[m]})\lambda^{-m}$$

and

$$-\mathrm{i}\,\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big)=\begin{bmatrix} c \\ b^T\end{bmatrix}=\sum_{m\ge 0}G_{m-1}\lambda^{-m}.$$

Inserting these into the trace identity and considering the case of $m=2$ tell $\gamma=0$, and thus

$$\frac{\delta \tilde{H}_m}{\delta u}=\mathrm{i}G_{m-1},\ \tilde{H}_m=-\frac{\mathrm{i}}{m}\int(\alpha_1 a^{[m+1]}+\alpha_2 d_{11}^{[m+1]}+\alpha_2 d_{22}^{[m+1]})\,\mathrm{d}x,\ G_{m-1}=\begin{bmatrix} c^{[m]} \\ b^{[m]T}\end{bmatrix},\ m\ge 1. \tag{2.26}$$

A bi-Hamiltonian structure of the six-component AKNS systems (2.23) then follows

$$u_t=K_r=JG_r=J\frac{\delta \tilde{H}_{r+1}}{\delta u}=M\frac{\delta \tilde{H}_r}{\delta u},\ r\ge 1, \tag{2.27}$$

where the Hamiltonian pair $(J, M=J\Psi)$ is defined by

$$J=\begin{bmatrix} 0 & \alpha I_3 \\ -\alpha I_3 & 0\end{bmatrix}, \tag{2.28a}$$

$$M=\mathrm{i}\begin{bmatrix} p^T\partial^{-1}p+(p^T\partial^{-1}p)^T & -\Big(\partial+\sum_{k=1}^{3}p_k\partial^{-1}q_k\Big)I_3-p^T\partial^{-1}q^T \\ -\Big(\partial+\sum_{k=1}^{3}p_k\partial^{-1}q_k\Big)I_3-q\partial^{-1}p & q\partial^{-1}q^T+(q\partial^{-1}q^T)^T\end{bmatrix}. \tag{2.28b}$$

Adjoint symmetry constraints (or equivalently symmetry constraints) decompose the six-component AKNS systems into two commuting finite-dimensional Liouville integrable Hamiltonian systems [12]. In the next section, we'll concentrate on the six-component system of coupled mKdV equations (2.25).

3 Riemann-Hilbert Problems

The spectral problems of the six-component system of mKdV equations (2.25) are

$$-\mathrm{i}\phi_x=U\phi=U(u,\lambda)\phi,\ -\mathrm{i}\phi_t=V^{[3]}\phi=V^{[3]}(u,\lambda)\phi \tag{3.1}$$

with

$$U=\lambda\Lambda+P,\ V^{[3]}=\lambda^3\Omega+Q, \tag{3.2}$$

where $\Lambda=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_2,\alpha_2)$, $\Omega=\mathrm{diag}(\beta_1,\beta_2,\beta_2,\beta_2)$, and

$$P=\begin{bmatrix} 0 & p \\ q & 0\end{bmatrix},\ Q=\begin{bmatrix} a^{[1]}\lambda^2+a^{[2]}\lambda+a^{[3]} & b^{[1]}\lambda^2+b^{[2]}\lambda+b^{[3]} \\ c^{[1]}\lambda^2+c^{[2]}\lambda+c^{[3]} & d^{[1]}\lambda^2+d^{[2]}\lambda+d^{[3]}\end{bmatrix}, \tag{3.3}$$

u,p,q being defined by (2.11), and $a^{[m]},b^{[m]},c^{[m]},d^{[m]}$, $1\le m\le 3$, being defined in (2.19).

In this section, we discuss the scattering and inverse scattering for the six-component mKdV system (2.25) using the Riemann-Hilbert formulation [1] (see also [20, 21]). The resulting results lay the groundwork for soliton solutions in the following section. Assume that all the six potentials rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$ and satisfy the integrable conditions

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^{m_1}|t|^{m_2}\sum_{k=1}^{3}(|p_k|+|q_k|)\,\mathrm{d}x\mathrm{d}t < \infty,\ \ m_1, m_2 = 0, 1. \tag{3.4}$$

For the sake of presentation, we also assume that

$$\alpha = \alpha_1 - \alpha_2 < 0,\ \ \beta = \beta_1 - \beta_2 < 0. \tag{3.5}$$

From the spectral problems in (3.1), we note, under (3.4), that when $x, t \to \pm\infty$, we have the asymptotic behavior: $\phi \sim \mathrm{e}^{\mathrm{i}\lambda\Lambda x + \mathrm{i}\lambda^3\Omega t}$. Therefore, upon making the variable transformation

$$\phi = \psi E_g,\ \ E_g = \mathrm{e}^{\mathrm{i}\lambda\Lambda x + \mathrm{i}\lambda^3\Omega t}, \tag{3.6}$$

we have the canonical normalization

$$\psi \to I_4,\ \text{ when } x, t \to \pm\infty, \tag{3.7}$$

where I_4 is the identity matrix of size 4. The equivalent pair of spectral problems to (3.1) reads

$$\psi_x = \mathrm{i}\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.8}$$

$$\psi_t = \mathrm{i}\lambda^3[\Omega, \psi] + \check{Q}\psi, \tag{3.9}$$

where $\check{P} = \mathrm{i}P$ and $\check{Q} = \mathrm{i}Q$. Noting $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$, we have

$$\det\psi = 1 \tag{3.10}$$

by a generalized Liouville's formula [22].

Let us now formulate an associated Riemann-Hilbert problem with the variable x. In the scattering problem, we first introduce the matrix solutions $\psi^{\pm}(x, \lambda)$ of (3.8) with the asymptotic conditions

$$\psi^{\pm} \to I_4,\ \text{ when } x \to \pm\infty, \tag{3.11}$$

respectively. The above superscripts refers to which end of the x-axis the boundary conditions are required. Then, by (3.10), we see $\det\psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since $\phi^{\pm} = \psi^{\pm}E, E = \mathrm{e}^{\mathrm{i}\lambda\Lambda x}$ are both solutions of (3.1), they are linearly dependent, and therefore, one can have

$$\psi^- E = \psi^+ E S(\lambda),\ \ \lambda \in \mathbb{R}, \tag{3.12}$$

where

$$S(\lambda) = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix},\ \ \lambda \in \mathbb{R} \tag{3.13}$$

is the scattering matrix. Note that $\det S(\lambda) = 1$ due to $\det\psi^{\pm} = 1$.

Applying the method of variation in parameters and using the boundary condition (3.11), we can turn the x-part of (3.1) into the following Volterra integral equations for $\psi^{\pm}$ [1]:

$$\psi^-(\lambda, x) = I_4 + \int_{-\infty}^{x} \mathrm{e}^{\mathrm{i}\lambda\Lambda(x-y)}\check{P}(y)\psi^-(\lambda, y)\mathrm{e}^{\mathrm{i}\lambda\Lambda(y-x)}\,\mathrm{d}y, \tag{3.14}$$

$$\psi^+(\lambda,x) = I_4 - \int_x^\infty \mathrm{e}^{\mathrm{i}\lambda\Lambda(x-y)}\check{P}(y)\psi^+(\lambda,y)\mathrm{e}^{\mathrm{i}\lambda\Lambda(y-x)}\,\mathrm{d}y. \tag{3.15}$$

Therefore, $\psi^\pm$ allows analytical continuations off the real axis $\lambda\in\mathbb{R}$ provided that the integrals on their right hand sides converge. Based on the diagonal form of Λ, we can directly see that the integral equation for the first column of ψ^- contains only the exponential factor $\mathrm{e}^{-\mathrm{i}\alpha\lambda(x-y)}$, which decays because of $y<x$ in the integral, when λ is in the closed upper half-plane, and the integral equation for the last three columns of ψ^+ contains only the exponential factor $\mathrm{e}^{\mathrm{i}\alpha\lambda(x-y)}$, which also decays because of $y>x$ in the integral, when λ is in the closed upper half-plane. Thus, these four columns can be analytically continued to the closed upper half-plane. In a similar manner, we can find that the last three columns of ψ^- and the first column of ψ^+ can be analytically continued to the closed lower half-plane. Upon expressing

$$\psi^\pm = (\psi_1^\pm, \psi_2^\pm, \psi_3^\pm, \psi_4^\pm), \tag{3.16}$$

that is, $\psi_k^\pm$ stands for the kth column of $\phi^\pm$ $(1\le k\le 4)$, the matrix solution

$$P^+ = P^+(x,\lambda) = (\psi_1^-, \psi_2^+, \psi_3^+, \psi_4^+) = \psi^- H_1 + \psi^+ H_2 \tag{3.17}$$

is analytic in $\lambda\in\mathbb{C}^+$ and continuous in $\lambda\in\mathbb{C}_0^+$, and the matrix solution

$$(\psi_1^+, \psi_2^-, \psi_3^-, \psi_4^-) = \psi^+ H_1 + \psi^- H_2 \tag{3.18}$$

is analytic in $\lambda\in\mathbb{C}^-$ and continuous in $\lambda\in\mathbb{C}_0^-$, where

$$H_1 = \mathrm{diag}(1,0,0,0),\ H_2 = \mathrm{diag}(0,1,1,1). \tag{3.19}$$

In addition, from the Volterra integral equation (3.14), we see that

$$P^+(x,\lambda)\to I_4,\ \text{when } \lambda\in\mathbb{C}_0^+\to\infty \tag{3.20}$$

and

$$(\psi_1^+, \psi_2^-, \psi_3^-, \psi_4^-)\to I_4,\ \text{when } \lambda\in\mathbb{C}_0^-\to\infty. \tag{3.21}$$

Next we construct the analytic counterpart of P^+ in the lower half-plane $\mathbb{C}^-$. Note that the adjoint equation of the x-part of (3.1) and the adjoint equation of (3.8) read as

$$\mathrm{i}\tilde{\phi}_x = \tilde{\phi}U \tag{3.22}$$

and

$$\mathrm{i}\tilde{\psi}_x = \lambda[\tilde{\psi},\Lambda] + \tilde{\psi}P. \tag{3.23}$$

It is easy to see that the inverse matrices $\tilde{\phi}^\pm = (\phi^\pm)^{-1}$ and $\tilde{\psi}^\pm = (\psi^\pm)^{-1}$ solve these adjoint equations, respectively. If we express $\tilde{\psi}^\pm$ as follows

$$\tilde{\psi}^\pm = \begin{bmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \tilde{\psi}^{\pm,3} \\ \tilde{\psi}^{\pm,4} \end{bmatrix}, \tag{3.24}$$

that is, $\tilde{\psi}^{\pm,k}$ stands for the kth row of $\tilde{\psi}^{\pm}$ $(1 \le k \le 4)$, then we can verify by similar arguments that the adjoint matrix solution

$$P^- = \begin{bmatrix} \tilde{\psi}^{-,1} \\ \tilde{\psi}^{+,2} \\ \tilde{\psi}^{+,3} \\ \tilde{\psi}^{+,4} \end{bmatrix} = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1} \tag{3.25}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}_0^-$, and the other matrix solution

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{-,3} \\ \tilde{\psi}^{-,4} \end{bmatrix} = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1} \tag{3.26}$$

is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}_0^+$. In the same way, we see that

$$P^-(x,\lambda) \to I_4, \ \text{when } \lambda \in \mathbb{C}_0^- \to \infty \tag{3.27}$$

and

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{-,3} \\ \tilde{\psi}^{-,4} \end{bmatrix} \to I_4, \ \text{when } \lambda \in \mathbb{C}_0^+ \to \infty. \tag{3.28}$$

Now we have built the two matrix functions P^+ and P^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$, respectively. We can directly see that on the real line, the two matrix functions P^+ and P^- are related by

$$P^-(x,\lambda)P^+(x,\lambda) = G(x,\lambda), \ \lambda \in \mathbb{R}, \tag{3.29}$$

where

$$G(x,\lambda) = E(H_1 + H_2S)(H_1 + S^{-1}H_2)E^{-1} = E\begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \hat{s}_{14} \\ s_{21} & 1 & 0 & 0 \\ s_{31} & 0 & 1 & 0 \\ s_{41} & 0 & 0 & 1 \end{bmatrix}E^{-1}, \ \lambda \in \mathbb{R} \tag{3.30}$$

in which $S^{-1} = (\hat{s}_{ij})_{4\times 4}$. Therefore, the associated matrix Riemann-Hilbert problem we wanted to build reads

$$G^+(x,\lambda) = G^-(x,\lambda)G(x,\lambda), \ \lambda \in \mathbb{R}, \tag{3.31}$$

where G is defined by (3.30) and

$$G^+(x,\lambda) = P^+(x,\lambda), \ \lambda \in \mathbb{C}_0^+, \ G^-(x,\lambda) = (P^-)^{-1}(x,\lambda), \ \lambda \in \mathbb{C}_0^-. \tag{3.32}$$

The asymptotic properties

$$G^{\pm}(x,\lambda) \to I_4, \ \text{when } \lambda \in \mathbb{C}_0^{\pm} \to \infty, \tag{3.33}$$

provide the canonical normalization conditions for the presented Riemann-Hilbert problem.

To complete the direct scattering transform, let us now take the derivative of (3.12) with time t and use the vanishing conditions of the potentials. This way, we can show that S satisfies

$$S_t = \mathrm{i}\lambda^3[\Omega, S], \tag{3.34}$$

which gives the time evolution of the scattering coefficients

$$\begin{cases} s_{11,t} = s_{22,t} = s_{33,t} = s_{44,t} = s_{23,t} = s_{24,t} = s_{32,t} = s_{34,t} = s_{42,t} = s_{43,t} = 0, \\ s_{12} = s_{12}(0,\lambda)\mathrm{e}^{\mathrm{i}\beta\lambda^3 t},\ s_{13} = s_{13}(0,\lambda)\mathrm{e}^{\mathrm{i}\beta\lambda^3 t},\ s_{14} = s_{14}(0,\lambda)\mathrm{e}^{\mathrm{i}\beta\lambda^3 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-\mathrm{i}\beta\lambda^3 t},\ s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-\mathrm{i}\beta\lambda^3 t},\ s_{41} = s_{41}(0,\lambda)\mathrm{e}^{-\mathrm{i}\beta\lambda^3 t}. \end{cases} \tag{3.35}$$

4 Soliton Solutions

The Riemann-Hilbert problems with zeros generate soliton solutions and can be solved by transforming into the ones without zeros [1]. The uniqueness of the associated Riemann-Hilbert problem (3.31) does not hold unless the zeros of $\det P^+$ and $\det P^-$ in the upper and lower half-planes are specified and the kernel structures of $P^\pm$ at these zeros are determined [23, 24]. From the definitions of $P^\pm$ and the scattering relation between ψ^+ and ψ^-, we find, using $\det \psi^\pm = 1$, that

$$\det P^+(x,\lambda) = s_{11}(\lambda),\ \det P^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{4.1}$$

where, based on $\det S = 1$, we have

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} \\ s_{32} & s_{33} & s_{34} \\ s_{42} & s_{43} & s_{44} \end{vmatrix}. \tag{4.2}$$

Assume that s_{11} has zeros $\{\lambda_k \in \mathbb{C}^+,\ 1 \le k \le N\}$, and $\hat{s}_{11}$ has zeros $\{\hat{\lambda}_k \in \mathbb{C}^-,\ 1 \le k \le N\}$. To get soliton solutions, we also assume that these zeros, λ_k and $\hat{\lambda}_k,\ 1 \le k \le N$, are simple. Then, each of $\ker P^+(\lambda_k)$, $1 \le k \le N$, contains only a single column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker P^-(\hat{\lambda}_k)$, $1 \le k \le N$, a row vector, denoted by $\hat{v}_k$, $1 \le k \le N$,

$$P^+(\lambda_k)v_k = 0,\ \hat{v}_k P^-(\hat{\lambda}_k) = 0,\ 1 \le k \le N. \tag{4.3}$$

The Riemann-Hilbert problem (3.31) with the canonical normalization conditions in (3.33) and the zero structures in (4.3) can be solved explicitly [1, 25], and thus one can readily work out the matrix P determining the potentials as follows. Note that P^+ is a solution to the spectral problem (3.8). Therefore, as long as we expand P^+ at large λ as

$$P^+(x,\lambda) = I_4 + \frac{1}{\lambda}P_1^+(x) + O(\frac{1}{\lambda^2}),\ \lambda \to \infty, \tag{4.4}$$

inserting this series expansion into (3.8) and balancing $O(1)$ terms generate

$$\check{P} = -\mathrm{i}[\Lambda, P_1^+], \tag{4.5}$$

which tells that

$$P=-[\Lambda,P_1^+]=\begin{bmatrix}0&-\alpha(P_1^+)_{12}&-\alpha(P_1^+)_{13}&-\alpha(P_1^+)_{14}\\ \alpha(P_1^+)_{21}&0&0&0\\ \alpha(P_1^+)_{31}&0&0&0\\ \alpha(P_1^+)_{41}&0&0&0\end{bmatrix}, \tag{4.6}$$

where $P_1^+=((P_1^+)_{kl})_{1\le k,l\le 4}$. Furthermore, the six potentials p_i and q_i, $1\le i\le 3$, can be computed as follows

$$\begin{cases}p_1=-\alpha(P_1^+)_{12},\ p_2=-\alpha(P_1^+)_{13},\ p_3=-\alpha(P_1^+)_{14},\\ q_1=\alpha(P_1^+)_{21},\ q_2=\alpha(P_1^+)_{31},\ q_3=\alpha(P_1^+)_{41}.\end{cases} \tag{4.7}$$

To compute soliton solutions, we set $G=I_4$ in the above Riemann-Hilbert problem (3.31). This can be achieved if we assume $s_{12}=s_{13}=s_{14}=s_{21}=s_{31}=s_{41}=0$, which means that no reflection exists in the scattering problem. The solutions to this specific Riemann-Hilbert problem can be obtained by (see e.g., [1, 25])

$$P^+(x,\lambda)=I_4-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat v_l}{\lambda-\hat\lambda_l},\ P^-(x,\lambda)=I_4+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat v_l}{\lambda-\lambda_l}, \tag{4.8}$$

where $M=(m_{kl})_{N\times N}$ is a square matrix whose entries are defined by

$$m_{kl}=\frac{\hat v_k v_l}{\lambda_l-\hat\lambda_k},\ 1\le k,l\le N. \tag{4.9}$$

Note that the zeros λ_k and $\hat\lambda_k$ are constants, i.e., space and time independent, and thus, we can easily determine the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat v_k(x,t)$, $1\le k\le N$. For instance, let us compute the x-derivative of both sides of the equations

$$P^+(\lambda_k)v_k=0,\ 1\le k\le N. \tag{4.10}$$

By using (3.8) first and then (4.10), we obtain

$$P^+(\lambda_k)\Big(\frac{\mathrm{d}v_k}{\mathrm{d}x}-\mathrm{i}\lambda_k\Lambda v_k\Big)=0,\ 1\le k\le N,$$

which implies that $\frac{\mathrm{d}v_k}{\mathrm{d}x}-\mathrm{i}\lambda_k\Lambda v_k$ belongs to $\ker P^+(\lambda_k)$. Without loss of generality, we assume that

$$\frac{\mathrm{d}v_k}{\mathrm{d}x}=\mathrm{i}\lambda_k\Lambda v_k,\ 1\le k\le N. \tag{4.11}$$

The time dependence of v_k,

$$\frac{\mathrm{d}v_k}{\mathrm{d}t}=\mathrm{i}\lambda_k^3\Omega v_k,\ 1\le k\le N \tag{4.12}$$

can be determined similarly through the t-part of the matrix spectral problem in (3.9). To conclude, we have

$$v_k(x,t)=\mathrm{e}^{\mathrm{i}\lambda_k\Lambda x+\mathrm{i}\lambda_k^3\Omega t}w_k,\ 1\le k\le N, \tag{4.13}$$

$$\hat v_k(x,t)=\hat w_k\mathrm{e}^{-\mathrm{i}\hat\lambda_k\Lambda x-\mathrm{i}\hat\lambda_k^3\Omega t},\ 1\le k\le N, \tag{4.14}$$

where w_k and $\hat w_k$, $1\le k\le N$, are arbitrary constant column and row vectors, respectively.

Finally, from the solutions in (4.8), we get

$$P_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.15}$$

and thus further through the presentations in (4.7), the N-soliton solution to the six-component system of coupled mKdV equations (2.25)

$$\begin{cases} p_1 = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,2}, \ p_2 = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,3}, \ p_3 = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,4}, \\ q_1 = -\alpha \sum_{k,l=1}^{N} v_{k,2} (M^{-1})_{kl} \hat{v}_{l,1}, \ q_2 = -\alpha \sum_{k,l=1}^{N} v_{k,3} (M^{-1})_{kl} \hat{v}_{l,1}, \ q_3 = -\alpha \sum_{k,l=1}^{N} v_{k,4} (M^{-1})_{kl} \hat{v}_{l,1}, \end{cases} \tag{4.16}$$

where $v_k = (v_{k,1}, v_{k,2}, v_{k,3}, v_{k,4})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3}, \hat{v}_{k,4})$, $1 \le k \le N$, are defined by (4.13) and (4.14), respectively.

5 Concluding Remarks

The paper is dedicated to formulation of Riemann-Hilbert problems and generation of associated soliton solutions to integrable equations. A crucial step is to take a kind of equivalent spectral problems, which guarantee the existence of analytical eigenfunctions in the upper or lower half-plane. We considered a 4×4 degenerate AKNS matrix spatial spectral problem and generated its soliton hierarchy possessing a bi-Hamiltonian structure. Taking the system of coupled mKdV equations as an example, we built its associated Riemann-Hilbert problems and presented an explicit formula for jump matrices. Upon taking the identity jump matrix in the presented Riemann-Hilbert problems, we computed soliton solutions to the considered six-component system of coupled mKdV equations.

The Riemann-Hilbert approach is very effective in generating soliton solutions (see also, e.g. [3–5]). Moreover, it was generalized to solve initial-boundary value problems of integrable equations on the half-line [26]. There are many other approaches to soliton solutions in the field of integrable equations, which include the Hirota direct method [27], the generalized bilinear technique [28], the Wronskian technique [29, 30] and the Darboux transformation [31]. Connections between different approaches would be interesting. About coupled mKdV equations, there were many other studies such as integrable couplings [32, 33], super hierarchies [34] and fractional analogous equations [35], and an important topic for further study is a Riemann-Hilbert formulation for solving those generalized integrable counterparts.

It would be also particularly interesting to study other kinds of exact solutions to integrable equations, including positon and complexiton solutions [36, 37], lump solutions [38–42], involutive solutions [43–46], and algebro-geometric solutions [47, 48], using Riemann-Hilbert techniques. It is hoped that our results could be helpful in recognizing those exact solutions from the perspective of Riemann-Hilbert problems.

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References

[1] Novikov S P, Manakov S V, Pitaevskii L P, Zakharov V E. Theory of Solitons: the Inverse Scattering Method. New York: Consultants Bureau, 1984

[2] Ablowitz M J, Clarkson P A. Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge: Cambridge University Press, 1991

[3] Xiao Y, Fan E G. A Riemann-Hilbert approach to the Harry-Dym equation on the line. Chin Ann Math Ser B, 2016, **37**(3): 373–384

[4] Geng X G, Wu J P. Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation. Wave Motion, 2016, 60:62–72

[5] Wang D S, Zhang D J, Yang J. Integrable properties of the general coupled nonlinear Schrödinger equations. J Math Phys, 2010, **51**(2): 023510

[6] Tu G Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. J Phys A: Math Gen, 1989, **22**(13): 2375–2392

[7] Lax P D. Integrals of nonlinear equations of evolution and solitary waves. Comm Pure Appl Math, 1968, **21**(5): 467–490

[8] Magri F. A simple model of the integrable Hamiltonian equation. J Math Phys, 1978, **19**(5): 1156–1162

[9] Ma W X, Fuchssteiner B. Integrable theory of the perturbation equations. Chaos Solitons Fractals, 1996, **7**(8): 1227–1250

[10] Ma W X, Chen M. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras. J Phys A: Math Gen, 2006, **39**(34): 10787–10801

[11] Ma W X. Variational identities and applications to Hamiltonian structures of soliton equations. Nonlinear Anal, 2009, **71**(12): e1716–e1726

[12] Ma W X, Zhou R G. Adjoint symmetry constraints leading to binary nonlinearization. J Nonlinear Math Phys, 2002, **9**(Suppl 1): 106–126

[13] Drinfeld V G, Sokolov V V. Equations of Korteweg-de Vries type, and simple Lie algebras. Soviet Math Dokl, 1982, **23**(3): 457–462

[14] Ma W X, Xu X X, Zhang Y F. Semi-direct sums of Lie algebras and continuous integrable couplings. Phys Lett A, 2006, **351**(3): 125–130

[15] Ablowitz M J, Kaup D J, Newell A C, Segur H. The inverse scattering transform-Fourier analysis for nonlinear problems. Stud Appl Math, 1974, **53**(4): 249–315

[16] Manakov S V. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. Sov Phys JETP, 1972, **38**(2): 248–253

[17] Chen S T, Zhou R G. An integrable decomposition of the Manakov equation. Comput Appl Math, 2012, **31**(1): 1–18

[18] Ma W X. Symmetry constraint of MKdV equations by binary nonlinearization. Phys A, 1995, **219**(3/4): 467–481

[19] Yu J, Zhou R G. Two kinds of new integrable decompositions of the mKdV equation. Phys Lett A, 2006, **349**(6): 452–461

[20] Gerdjikov V S. Basic aspects of soliton theory//Mladenov I M, Hirshfeld A C, eds. Proceedings of the 6th International Conference on Geometry, Integrability and Quantization (Varna, June 3-10, 2004), Sofia: Softex, 2005: 78–125

[21] Doktorov E V, Leble S B. A Dressing Method in Mathematical Physics, Mathematical Physics Studies 28. Dordrecht: Springer, 2007

[22] Ma W X, Yong X L, Qin Z Y, Gu X, Zhou Y. A generalized Liouville's formula. preprint, 2016

[23] Shchesnovich V S. Perturbation theory for nearly integrable multicomponent nonlinear PDEs. J Math Phys, 2002, **43**(3): 1460–1486

[24] Shchesnovich V S, Yang J. General soliton matrices in the Riemann-Hilbert problem for integrable nonlinear equations. J Math Phys, 2003, **44**(10): 4604–4639

[25] Kawata T. Riemann spectral method for the nonlinear evolution equation//Advances in Nonlinear Waves Vol I, Res Notes in Math 95. Boston, MA: Pitman, 1984: 210–225

[26] Fokas A S, Lenells J. The unified method: I. Nonlinearizable problems on the half-line. J Phys A: Math Theor, 2012, **45**(19): 195201

[27] Hirota R. The Direct Method in Soliton Theory. New York: Cambridge University Press, 2004

[28] Ma W X. Generalized bilinear differential equations. Stud Nonlinear Sci, 2011, **2**(4): 140–144

[29] Freeman N C, Nimmo J J C. Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique. Phys Lett A, 1983, **95**(1): 1–3

[30] Ma W X, You Y. Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. Trans Amer Math Soc, 2015, **357**(5): 1753–1778

[31] Matveev V B, Salle M A. Darboux Transformations and Solitons. Berlin: Springer, 1991

[32] Xu X X. An integrable coupling hierarchy of the MKdV$_-$ integrable systems, its Hamiltonian structure and corresponding nonisospectral integrable hierarchy. Appl Math Comput, 2010, **216**(1): 344–353

[33] Wang X R, Zhang X E, Zhao P Y. Binary nonlinearization for AKNS-KN coupling system. Abstr Appl Anal, 2014, **2014**: Article ID 253102

[34] Dong H H, Zhao K, Yang H W, Li Y Q. Generalised (2+1)-dimensional super MKdV hierarchy for integrable systems in soliton theory. East Asian J Appl Math, 2015, **5**(3): 256–272

[35] Dong H H, Guo B Y, Yin B S. Generalized fractional supertrace identity for Hamiltonian structure of NLS-MKdV hierarchy with self-consistent sources. Anal Math Phys, 2016, **6**(2): 199–209

[36] Matveev V B. Generalized Wronskian formula for solutions of the KdV equations: first applications. Phys Lett A, 1992, **166**(3/4): 205–208

[37] Ma W X. Complexiton solutions to the Korteweg-de Vries equation. Phys Lett A, 2002, **301**(12): 35–44

[38] Satsuma J, Ablowitz M J. Two-dimensional lumps in nonlinear dispersive systems. J Math Phys, 1979, **20**(7): 1496–1503

[39] Ma W X, Zhou Y, Dougherty R. Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations. Int J Modern Phys B, 2016, **30**(28/29): 1640018

[40] Zhang Y, Dong H H, Zhang X E, Yang H W. Rational solutions and lump solutions to the generalized (3+1)-dimensional shallow water-like equation. Comput Math Appl, 2017, **73**(2): 246–252

[41] Zhang Y, Sun S L, Dong H H. Hybrid solutions of (3+1)-dimensional Jimbo-Miwa equation. Math Probl Eng, 2017, **2017**: Article ID 5453941

[42] Ma W X, Zhou Y. Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. J Differential Equations, 2018, **264**(4): 2633–2659

[43] Li X Y, Zhao Q L, Li Y X, Dong H H. Binary Bargmann symmetry constraint associated with 3×3 discrete matrix spectral problem. J Nonlinear Sci Appl, 2015, **8**(5): 496–506

[44] Zhao Q L, Li X Y. A Bargmann system and the involutive solutions associated with a new 4-order lattice hierarchy. Anal Math Phys, 2016, **6**(3): 237–254

[45] Dong H H, Zhang Y, Zhang X E. The new integrable symplectic map and the symmetry of integrable nonlinear lattice equation. Commun Nonlinear Sci Numer Simulat, 2016, **36**: 354–365

[46] Li X Y, Zhao Q L. A new integrable symplectic map by the binary nonlinearization to the super AKNS system. J Geom Phys, 2017, **121**: 123–137

[47] Belokolos E D, Bobenko A I, Enol'skii V Z, Its A R, Matveev V B. Algebro-geometric approach to nonlinear integrable equations. Berlin: Springer, 1994

[48] Gesztesy F, Holden H. Soliton Equations and Their Algebro-Geometric Solutions: (1+1)-Dimensional Continuous Models. Cambridge: Cambridge University Press, 2003