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LUMP AND INTERACTION SOLUTIONS TO LINEAR (4+1)-DIMENSIONAL PDES*

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Abstract Taking a class of linear (4+1)-dimensional partial differential equations as examples, we would like to show that there exist lump solutions and interaction solutions in (4+1)-dimensions. We will compute abundant lump solutions and interaction solutions to the considered linear (4+1)-dimensional partial differential equations via symbolic computations, and plot three specific solutions with Maple plot tools, which supplements the existing literature on lump, rogue wave and breather solutions and their interaction solutions in soliton theory.

Key words symbolic computation; lump solution; interaction solution

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1 Introduction

Differential equations played a prominent role in a bunch of disciplines including engineering, physics, chemistry, economics and biology; and they were studied from different perspectives, mostly concerned with their solutions – functions that satisfy the differential equations [1, 2]. One of the fundamental problems in the theory of differential equations, called the Cauchy problem, is to find a solution of a differential equation satisfying what are known as initial data. Laplace's method and the Fourier transform method are established for solving Cauchy problems for linear ordinary and partial differential equations, respectively. Soliton

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scientists had thought about nonlinearity innovatively and developed novel solution techniques – the isomonodromic transform method and the inverse scattering transform method – for dealing with Cauchy problems for nonlinear ordinary and partial differential equations, respectively [3, 4].

Only the simplest differential equations, often linear, are solvable by explicit formulas. However, soliton theory does bring many different approaches for finding explicit solutions to nonlinear differential equations. Recently, some systematical studies were made on a kind of interesting explicit solutions called lumps, originated from the study on formulation of solitons [5–7]. Mathematically, lumps are a kind of rational function solutions that are localized in all directions in space, historically found for nonlinear integrable equations, and solitons are analytic solutions exponentially localized in all directions in space and time. Particular lumps can be generated from solitons by taking long wave limits [8]. There also exist positons and complexitons to nonlinear integrable equations, enriching the diversity of solitons [9, 10]. Furthermore, interaction solutions between two different kinds of solutions are found to exist in soliton theory [11], and they can explain various nonlinear phenomena in sciences.

Within the Hirota bilinear formulation, solitons can be usually generated as follows

$$u = 2(\ln f)_{xx}, \ f = \sum_{\mu=0,1} \exp(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}), \tag{1.1}$$

where

$$\begin{cases} \xi_i = k_i x - \omega_i t + \xi_{i,0}, & 1 \le i \le N, \\ \mathrm{e}^{a_{ij}} = -\dfrac{P(k_i - k_j, \omega_j - \omega_i)}{P(k_i + k_j, \omega_j + \omega_i)}, & 1 \le i < j \le N \end{cases} \tag{1.2}$$

with k_i and ω_i satisfying the so-called dispersion relation and $\xi_{i,0}$ being arbitrary phase shifts. The polynomial P determines a Hirota bilinear form

$$P(D_x, D_t) f \cdot f = 0, \tag{1.3}$$

where D_x and D_t are Hirota's bilinear derivatives, for a partial differential equation with the dependent variable u. As an example of lumps, we point out that the KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.4}$$

possesses a class of lump solutions [12]

$$u = 2(\ln f)_{xx}, \ f = \big(a_1 x + a_2 y + a_3 t + a_4\big)^2 + \big(a_5 x + a_6 y + a_7 t + a_8\big)^2 + a_9, \tag{1.5}$$

where two wave frequencies and a positive position shift are given by

$$a_3 = \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2\, a_2 a_5 a_6}{{a_1}^2 + {a_5}^2}, \ a_7 = \frac{2 a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2}, \ a_9 = \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \tag{1.6}$$

and four wave numbers and two translation shifts are arbitrary but need to satisfy $a_1 a_6 - a_2 a_5 \ne 0$, which guarantees rational localization in all directions in the (x, y)-plane. Other integrable equations that possess lump solutions contain the three-dimensional three-wave resonant interaction [13], the BKP equation [14, 15], the Davey-Stewartson equation II [8], the Ishimori-I equation [16] and many others (see e.g., [5, 7, 17]).

It is recognized through symbolic computations that many nonintegrable equations possess lump solutions as well, which include (2+1)-dimensional generalized KP, BKP, KP-Boussinesq and Sawada-Kotera equations [18–21]. Moreover, many recent works exhibited interaction solutions between lumps and other kinds of exact solutions to nonlinear integrable equations in (2+1)-dimensions, including lump-kink interaction solutions (see e.g., [22–25]) and lump-soliton interaction solutions (see e.g., [26–29]). In the (3+1)-dimensional case, lump-type solutions, which are rationally localized in almost all directions in space, were worked out for the integrable Jimbo-Miwa equations. Abundant such solutions were generated for the (3+1)-dimensional Jimbo-Miwa equation [30–32] and the (3+1)-dimensional Jimbo-Miwa like equation [33]. It is definitely interesting to search for lump and interaction solutions to partial differential equations in (4+1)-dimensions or higher dimensions.

This paper aims at exploring lump solutions and their interaction solutions to a class of linear partial differential equations in (4+1)-dimensions. Concrete examples of (4+1)-dimensional linear equations will be presented to show lump solution phenomena. Both lump solutions and interaction solutions, including lump-periodic, lump-kink and lump-soliton solutions, will be computed explicitly through Maple symbolic computations. Sufficient conditions which guarantee the existence of lump and interaction solutions will be acquired, and three-dimensional plots and contour plots of specific examples of the presented solutions will be made via Maple plot tools. A few concluding remarks will be presented in the last section.

2 Abundant Lump and Interaction Solutions

Let $u = u(x_1, x_2, x_3, x_4, t)$ be a real function of the variables $x_1, x_2, x_3, x_4, t \in \mathbb{R}$. We consider a class of linear (4+1)-dimensional partial differential equations (PDEs)

$$\begin{aligned}&\alpha_1 u_{x_1x_2} + \alpha_2 u_{x_1x_3} + \alpha_3 u_{x_1x_4} + \alpha_4 u_{tx_1} + \alpha_5 u_{x_2x_3}\\&+\alpha_6 u_{x_2x_4} + \alpha_7 u_{tx_2} + \alpha_8 u_{x_3x_4} + \alpha_9 u_{tx_3} + \alpha_{10} u_{tx_4} = 0,\end{aligned} \tag{2.1}$$

where the subscripts denote partial differentiation and $\alpha_i,\ 1 \le i \le 10$, are real constants.

We look for a kind of exact solutions

$$u = v(\xi_1, \xi_2, \xi_3, \xi_4.\xi_5), \tag{2.2}$$

where v is an arbitrary real function, and $\xi_i,\ 1 \le i \le 5$, are five linear functions of the dependent variables

$$\xi_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i5}t + a_{i6},\ 1 \le i \le 5 \tag{2.3}$$

in which a_{ij}, $1 \le i \le 5$ and $1 \le j \le 6$, are real constants to be determined. Then, the above class of linear PDEs (2.1) becomes

$$\sum_{i=1}^{5}\sum_{j=i}^{5} w_{ij} v_{\xi_i\xi_j} = 0, \tag{2.4}$$

where w_{ij}, $1 \le i \le j \le 5$, are quadratic functions of the parameters a_{ij}, $1 \le i, j \le 5$. By equating all coefficients of the second partial derivatives of v to zero, we get a system of fifteen equations on the parameters a_{ij}, $1 \le i, j \le 5$, and the coefficients $\alpha_i,\ 1 \le i \le 10$,

$$\begin{cases} \alpha_1 a_{i1}a_{i2} + \alpha_2 a_{i1}a_{i3} + \alpha_3 a_{i1}a_{i4} + \alpha_4 a_{i1}a_{i5} + \alpha_5 a_{i2}a_{i3} \\ \quad + \alpha_6 a_{i2}a_{i4} + \alpha_7 a_{i2}a_{i5} + \alpha_8 a_{i3}a_{i4} + \alpha_9 a_{i3}a_{i5} + \alpha_{10} a_{i4}a_{i5} = 0,\ 1 \le i \le 5, \\ \alpha_1(a_{i1}a_{j2} + a_{j1}a_{i2}) + \alpha_2(a_{i1}a_{j3} + a_{j1}a_{i3}) + \alpha_3(a_{i1}a_{j4} + a_{j1}a_{i4}) + \alpha_4(a_{i1}a_{j5} + a_{j1}a_{i5}) \\ \quad + \alpha_5(a_{i2}a_{j3} + a_{j2}a_{i3}) + \alpha_6(a_{i2}a_{j4} + a_{j2}a_{i4}) + \alpha_7(a_{i2}a_{j5} + a_{j2}a_{i5}) \\ \quad + \alpha_8(a_{i3}a_{j4} + a_{j3}a_{j4}) + \alpha_9(a_{i3}a_{j5} + a_{j3}a_{i5}) + \alpha_{10}(a_{i4}a_{j5} + a_{j4}a_{i5}) = 0,\ 1 \le i < j \le 5. \end{cases} \tag{2.5}$$

The arbitrariness of the parameters $a_{i6},\ 1 \le i \le 5$, is due to the translation invariance of the equations in (2.1).

Through Maple direct symbolic computations, we can determine many solutions to this system of cubic equations. We classify the solutions we obtain into the following three categories

$$\begin{aligned} &\Big\{a_{13} = a_{14} = a_{15} = 0,\ a_{21} = a_{23} = a_{24} = a_{25} = 0,\ a_{34} = a_{35} = 0,\ a_{55} = \frac{a_{45}a_{54}}{a_{44}}, \\ &\quad \alpha_1 = \alpha_2 = 0,\ \alpha_3 = -\frac{a_{45}\alpha_4}{a_{44}},\ \alpha_5 = 0,\ \alpha_6 = -\frac{a_{45}\alpha_7}{a_{44}},\ \alpha_8 = -\frac{a_{45}\alpha_9}{a_{44}},\ \alpha_{10} = 0\Big\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\Big\{a_{13} = a_{14} = a_{15} = 0,\ a_{21} = a_{23} = a_{24} = a_{25} = 0,\ a_{34} = 0,\ a_{53} = \frac{p}{a_{35}a_{44}}, \\ &\quad \alpha_1 = 0,\ \alpha_2 = -\frac{a_{35}\alpha_4}{a_{33}},\ \alpha_3 = -\frac{(a_{33}a_{45} - a_{35}a_{43})\alpha_4}{a_{33}a_{44}},\ \alpha_5 = -\frac{a_{35}\alpha_7}{a_{33}}, \\ &\quad \alpha_6 = -\frac{(a_{33}a_{45} - a_{35}a_{43})\alpha_7}{a_{33}a_{44}},\ \alpha_8 = \alpha_9 = \alpha_{10} = 0\Big\}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\Big\{a_{13} = a_{14} = 0,\ a_{21} = a_{23} = a_{24} = a_{25} = 0,\ \ a_{34} = 0,\ a_{51} = \frac{q}{a_{15}a_{33}a_{44}}, \\ &\quad \alpha_1 = -\frac{a_{15}\alpha_7}{a_{11}},\ \alpha_2 = \alpha_3 = \alpha_4 = 0,\ \alpha_5 = -\frac{(a_{11}a_{35} - a_{15}a_{31})\alpha_7}{a_{11}a_{33}}, \\ &\quad \alpha_6 = -\frac{(a_{11}a_{33}a_{45} - a_{11}a_{35}a_{43} + a_{15}a_{31}a_{43} - a_{15}a_{33}a_{41})\alpha_7}{a_{11}a_{33}a_{44}},\ \alpha_8 = \alpha_9 = \alpha_{10} = 0\Big\}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} p &= a_{33}a_{44}a_{55} - a_{33}a_{45}a_{54} + a_{35}a_{43}a_{54}, \\ q &= a_{11}a_{33}a_{44}a_{55} - a_{11}a_{33}a_{45}a_{54} + a_{11}a_{35}a_{43}a_{54} - a_{11}a_{35}a_{44}a_{53} \\ &\quad - a_{15}a_{31}a_{43}a_{54} + a_{15}a_{31}a_{44}a_{53} + a_{15}a_{33}a_{41}a_{54}. \end{aligned}$$

In each set of the three solutions above, the constants not determined in the set are arbitrary provided that all expressions in the set will make sense. Those three categories of the constants will present lumps and their interaction solutions, since a sufficient condition for u to be a lump

$$\det(a_{ij})_{4\times 4} \ne 0 \tag{2.9}$$

can be achieved, though all the three categories of the constants satisfy a determinant equation

$$\det(a_{ij})_{5\times 5} = 0. \tag{2.10}$$

Specifically, we can generate the corresponding three examples on lump and interaction solutions as follows.

Example 1 Upon taking $a_{45}=-a_{44}$ and $a_{55}=-a_{54}$, the first solution (2.6) shows that the following linear (4+1)-dimensional PDE

$$u_{x_1x_4}+u_{tx_1}+u_{x_2x_4}+u_{tx_2}+u_{x_3x_4}+u_{tx_3}=0 \tag{2.11}$$

possesses a kind of exact and explicit solutions

$$u=2(\ln f)_{xx},\ f=\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2+g(\xi_5), \tag{2.12}$$

where $\xi_i,\ 1\le i\le 5$, are defined by

$$\begin{cases}\xi_1=a_{11}x_1+a_{12}x_2+a_{16},\\ \xi_2=a_{22}x_2+a_{26},\\ \xi_3=a_{31}x_1+a_{32}x_2+a_{33}x_3+a_{36},\\ \xi_4=a_{41}x_1+a_{42}x_2+a_{43}x_3+a_{44}x_4-a_{44}t+a_{46},\\ \xi_5=a_{51}x_1+a_{52}x_2+a_{53}x_3+a_{54}x_4-a_{54}t+a_{56},\end{cases} \tag{2.13}$$

and the function g is arbitrary.

Example 2 Upon taking $a_{35}=-a_{33}$ and $a_{45}=-(a_{43}+a_{44})$, solution (2.7) tells that the following linear (4+1)-dimensional PDE

$$u_{x_1x_3}+u_{x_1x_4}+u_{tx_1}+u_{x_2x_3}+u_{x_2x_4}+u_{tx_2}=0 \tag{2.14}$$

possesses a kind of exact and explicit solutions

$$u=2(\ln f)_{xx},\ f=\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2+g(\xi_5), \tag{2.15}$$

where $\xi_i,\ 1\le i\le 5$, are defined by

$$\begin{cases}\xi_1=a_{11}x_1+a_{12}x_2+a_{16},\\ \xi_2=a_{22}x_2+a_{26},\\ \xi_3=a_{31}x_1+a_{32}x_2+a_{33}x_3-a_{33}t+a_{36},\\ \xi_4=a_{41}x_1+a_{42}x_2+a_{43}x_3+a_{44}x_4-(a_{43}+a_{44})t+a_{46},\\ \xi_5=a_{51}x_1+a_{52}x_2-(a_{54}+a_{55)}x_3+a_{54}x_4+a_{55}t+a_{56},\end{cases} \tag{2.16}$$

and the function g is again arbitrary.

Example 3 Upon taking $a_{15}=-a_{11},\ a_{35}=-(a_{31}+a_{33})$ and $a_{45}=-(a_{41}+a_{43}+a_{44})$, solution (2.8) implies that the following linear (4+1)-dimensional PDE

$$u_{x_1x_2}+u_{x_2x_3}+u_{x_2x_4}+u_{tx_2}=0 \tag{2.17}$$

possesses a kind of exact and explicit solutions

$$u=2(\ln f)_{xx},\ f=\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2+g(\xi_5), \tag{2.18}$$

where $\xi_i,\ 1 \le i \le 5$, are defined by

$$\begin{cases} \xi_1 = a_{11}x_1 + a_{12}x_2 - a_{11}t + a_{16}, \\ \xi_2 = a_{22}x_2 + a_{26}, \\ \xi_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - (a_{31}+a_{33})t + a_{36}, \\ \xi_4 = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 - (a_{41}+a_{43}+a_{44})t + a_{46}, \\ \xi_5 = -(a_{53}+a_{54}+a_{55})x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 + a_{55}t + a_{56}, \end{cases} \tag{2.19}$$

and the function g is once again arbitrary.

Now, further taking

$$g(\xi_5) = \beta_1,\ \beta_2 + \beta_3\cos\xi_5,\ \beta_4 e^{\xi_5},\ \text{or}\ \beta_5\cosh\xi_5, \tag{2.20}$$

where $\beta_i,\ 1 \le i \le 5$, are proper constants which need to ensure the positivity of the generating function f, we can obtain lump solutions, and interaction solutions: lump-periodic, lump-kink and lump-soliton solutions to the above three linear (4+1)-dimensional PDEs, (2.11), (2.14) and (2.17), as follows

$$\begin{aligned} u = \frac{2(f_{xx}f - f_x^2)}{f^2} &= \frac{2[2a_{11}^2 + 2a_{22}^2 + 2a_{33}^2 + 2a_{44}^2 + a_{55}^2 g''(\xi_5)]}{f} \\ &- \frac{2[2a_{11}\xi_1 + 2a_{22}\xi_2 + 2a_{33}\xi_3 + 2a_{44}\xi_4 + a_{55}g'(\xi_5)]^2}{f^2}. \end{aligned} \tag{2.21}$$

All solutions obtained above provide supplements to the theories available on soliton solutions and dromion-type solutions, formulated through basic approaches such as the Hirota perturbation technique and symmetry constraints (see e.g., [34–39]).

Particularly taking

$$\begin{cases} a_{11}=1,\ a_{12}=-2,\ a_{16}=5,\ a_{22}=-2,\ a_{26}=-3, \\ a_{31}=-1,\ a_{32}=2,\ a_{33}=-5,\ a_{36}=2, \\ a_{41}=1,\ a_{42}=-3,\ a_{43}=-1,\ a_{44}=6,\ a_{45}=-1,\ a_{46}=-2, \\ a_{51}=1,\ a_{52}=2,\ a_{53}=1,\ a_{54}=-3,\ a_{55}=6,\ a_{56}=1, \\ \beta_1=1,\ \beta_2=2,\ \beta_3=\frac{1}{15},\ \beta_5=50, \end{cases} \tag{2.22}$$

we get the three specific solutions to (2.17)

$$\begin{cases} u_1 = \dfrac{12f_1 - 8(3x_1 - 7x_2 + 4x_3 + 6x_4 - 13t + 1)^2}{f_1^2}, \\ f_1 = (x_1 - 2x_2 - t + 5)^2 + (2x_2+3)^2 + (-x_1 + 2x_2 - 5x_3 + 6t + 2)^2 \\ \qquad +(x_1 - 3x_2 - x_3 + 6x_4 - 6t - 2)^2 + 1, \end{cases} \tag{2.23}$$

$$\begin{cases} u_2 = \dfrac{(12 - \frac{32}{15}\cos\xi_5)f_2 - 8(3x_1 - 7x_2 + 4x_3 + 6x_4 - 13t + 1 + \frac{2}{15}\sin\xi_5)^2}{f_2^2}, \\ f_2 = (x_1 - 2x_2 - t + 5)^2 + (2x_2 + 3)^2 + (-x_1 + 2x_2 - 5x_3 + 6t + 2)^2 \\ \qquad + (x_1 - 3x_2 - x_3 + 6x_4 - 6t - 2)^2 + \frac{1}{15}\cos\xi_5 + 2, \end{cases} \tag{2.24}$$

and

$$\begin{cases} u_3 = \dfrac{(12 + 1600\cosh\xi_5)f_3 - 8(3x_1 - 7x_2 + 4x_3 + 6x_4 - 13t + 1 - 100\sinh\xi_5)^2}{f_3^2}, \\ f_3 = (x_1 - 2x_2 - t + 5)^2 + (2x_2 + 3)^2 + (-x_1 + 2x_2 - 5x_3 + 6t + 2)^2 \\ \qquad + (x_1 - 3x_2 - x_3 + 6x_4 - 6t - 2)^2 + 50\cosh\xi_5, \end{cases} \tag{2.25}$$

where $\xi_5 = -4x_1 + 2x_2 + x_3 - 3x_4 + 6t + 1$. The first solution is a lump, and the second and third ones are lump-periodic and lump-soliton solutions, respectively. Three three-dimensional plots and contour plots of those three solutions are made, to shed light on the characteristics of lump and interaction solutions, in Figure 1, Figure 2, and Figure 3.

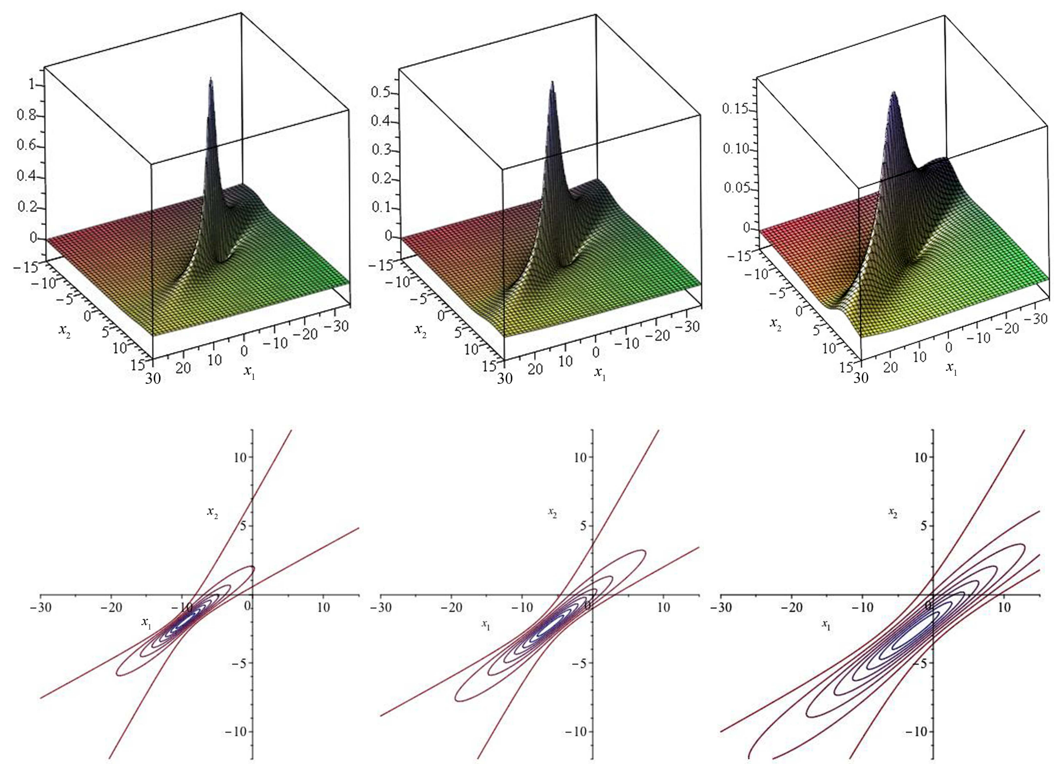

Figure 1 Profiles of u_1 when $t = 0, 1, 2$ and $x_3 = 2, x_4 = 1$: 3d plots (top) and contour plots (bottom)

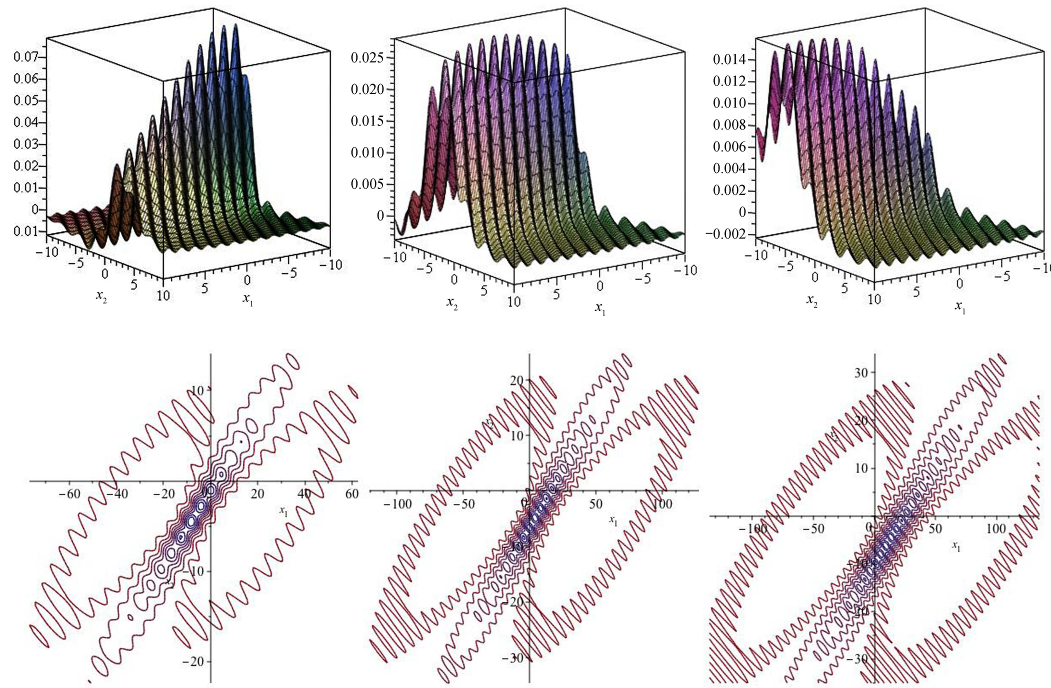

Figure 2 Profiles of u_2 when $t = 0, 3, 5$ and $x_3 = 1, x_4 = -2$: 3d plots (top) and contour plots (bottom)

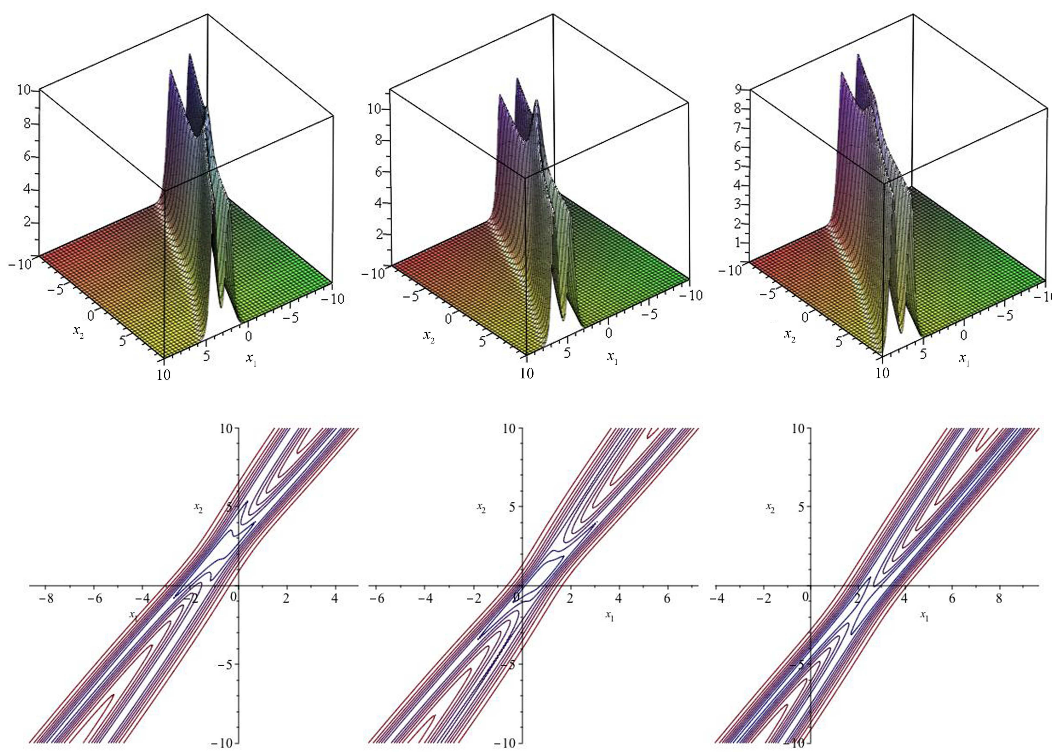

Figure 3 Profiles of u_3 when $t = 0, 1.5, 3$ and $x_3 = 1, x_4 = 3$: 3d plots (top) and contour plots (bottom)

3 Concluding Remarks

We studied a class of linear (4+1)-dimensional partial differential equations to exhibit abundant lump and interaction solutions, including lump-periodic, lump-kink and lump-soliton solutions, via Maple symbolic computations. The results amend the existing soliton theory on nonlinear integrable equations and the recent studies on lumps and interaction solutions to linear partial differential equations in (2+1)- and (3+1)-dimensions (see e.g., [40]). Three concrete examples which possess lump and interaction solutions were explicitly presented, and three-dimensional plots and contour plots of three specially chosen solutions were made via Maple plot tools.

We remark that the obtained lump and interaction solutions also provide supplements to exact solutions generated from different kinds of combinations [41–43]. Moreover, it will be interesting to look for lump and interaction solutions to other generalized bilinear and tri-linear differential equations involving generalized bilinear derivatives [44]. The corresponding interaction solutions will generally not be resonant solutions generated through the linear superposition principle [41, 43]. Though lump solutions generated from quadratic functions remain the same as in the Hirota case, integrable equations determined by generalized bilinear derivatives [44] can possess different interaction solutions (see [6] for a detailed discussion).

It is direct to formulate models in $(n+1)$-dimensions and their lump and interaction solutions following the pattern in the examples presented above. Diverse interaction solutions also imply that there exist the corresponding Lie-Bäcklund symmetries, thereby supplementing symmetry theories on partial differential equations. It is known that the Wronskian technique can solve nonlinear integrable equations, and therefore, our study brings up a new question: how can we formulate novel Wronskian solutions by adopting matrix entries of new type? It is also absolutely important to establish a basic theory of lumps and interaction solutions to difference-differential equations, and to see if such solutions can be constructed via Riemann-Hilbert problems [45]. All those problems deserve further studies.

References

[1] Ince E L. Ordinary Differential Equations. Mineola, NY: Dover, 1956

[2] Evans L C. Partial Differential Equations. Providence, RI: American Mathematical Society, 1998

[3] Novikov S, Manakov S V, Pitaevskii, L P, Zakharov V E. Theory of Solitons – the Inverse Scattering Method. New York: Consultants Bureau, 1984

[4] Ablowitz M J, Clarkson P A. Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge: Cambridge University Press, 1991

[5] Ma W X, Zhou Y. Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. J Diff Eqn, 2018, **264**(4): 2633–2659

[6] Ma W X, Zhou Y, Dougherty R. Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations. Int J Mod Phys B, 2016, **30**(28/29): 1640018

[7] Tan W, Dai H P, Dai Z D, Zhong W Y. Emergence and space-time structure of lump solution to the (2+1)-dimensional generalized KP equation. Pramana J Phys, 2017, **89**(5): 77

[8] Satsuma J, Ablowitz M J. Two-dimensional lumps in nonlinear dispersive systems. J Math Phys, 1979, **20**(7): 1496–1503

[9] Ma W X. Wronskian solutions to integrable equations. Discrete Contin Dynam Syst, 2009, (Suppl): 506–515

[10] Wazwaz A-M, El-Tantawy S A. New (3+1)-dimensional equations of Burgers type and Sharma-Tasso-Olver type: multiple-soliton solutions. Nonlinear Dynam, 2017, **87**(4): 2457–2461

[11] Ma W X, You Y. Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. Trans Amer Math Soc, 2005, **357**(5): 1753–1778

[12] Ma W X. Lump solutions to the Kadomtsev-Petviashvili equation. Phys Lett A, 2015, **379**(36): 1975–1978

[13] Kaup D J. The lump solutions and the Bäcklund transformation for the three-dimensional three-wave resonant interaction. J Math Phys, 1981, **22**(6): 1176–1181

[14] Gilson C R, Nimmo J J C. Lump solutions of the BKP equation. Phys Lett A, 1990, **147**(8/9): 472–476

[15] Yang J Y, Ma W X. Lump solutions of the BKP equation by symbolic computation. Int J Mod Phys B, 2016, **30**(28/29): 1640028

[16] Imai K. Dromion and lump solutions of the Ishimori-I equation. Prog Theor Phys, 1997, **98**(5): 1013–1023

[17] Zhang Y, Dong H H, Zhang X E, Yang H W. Rational solutions and lump solutions to the generalized (3+1)-dimensional shallow water-like equation. Comput Math Appl, 2017, **73**(2): 246–252

[18] Ma W X, Qin Z Y, Lü X. Lump solutions to dimensionally reduced p-gKP and p-gBKP equations. Nonlinear Dynam, 2016, **84**(2): 923–931

[19] Lü X, Chen S T, Ma W X. Constructing lump solutions to a generalized Kadomtsev-Petviashvili-Boussinesq equation. Nonlinear Dynam, 2016, **86**(1): 523–534

[20] Zhang H Q, Ma W X. Lump solutions to the (2+1)-dimensional Sawada-Kotera equation. Nonlinear Dynam, 2017, **87**(4): 2305–2310

[21] Yu J P, Sun Y L. Study of lump solutions to dimensionally reduced generalized KP equations. Nonlinear Dynam, 2017, **87**(4): 2755–2763

[22] Tang Y N, Tao S Q, Qing G. Lump solitons and the interaction phenomena of them for two classes of nonlinear evolution equations. Comput Math Appl, 2016, **72**(9): 2334–2342

[23] Zhao H Q, Ma W X. Mixed lump-kink solutions to the KP equation. Comput Math Appl, 2017, **74**(6): 1399–1405

[24] Zhang J B, Ma W X. Mixed lump-kink solutions to the BKP equation. Comput Math Appl, 2017, **74**(3): 591–596

[25] Kofane T C, Fokou M, Mohamadou A, Yomba E. Lump solutions and interaction phenomenon to the third-order nonlinear evolution equation. Eur Phys J Plus, 2017, **132**(11): 465

[26] Yang J Y, Ma W X. Abundant interaction solutions of the KP equation. Nonlinear Dynam, 2017, **89**(2): 1539–1544

[27] Yang J Y, Ma W X, Qin Z Y. Lump and lump-soliton solutions to the (2+1)-dimensional Ito equation. Anal Math Phys, 2018, **8**(3): 427–436

[28] Ma W X, Yong X L, Zhang H Q. Diversity of interaction solutions to the (2+1)-dimensional Ito equation. Comput Math Appl, 2018, **75**(1): 289–295

[29] Yang Y J, Ma W X, Qin Z Y. Abundant mixed lump-soliton solutions of the BKP equation. East Asian J Appl Math, 2018, **8**(2): 224–232

[30] Ma W X. Lump-type solutions to the (3+1)-dimensional Jimbo-Miwa equation. Int J Nonlinear Sci Numer Simulat, 2016, **17**(7/8): 355–359

[31] Yang J Y, Ma W X. Abundant lump-type solutions of the Jimbo-Miwa equation in (3+1)-dimensions. Comput Math Appl, 2017, **73**(2): 220–225

[32] Zhang Y, Sun S L, Dong H H. Hybrid solutions of (3+1)-dimensional Jimbo-Miwa equation. Math Probl Eng, 2017, **2017**: Article ID 5453941

[33] Roshid-Harun-Or, Ali M Z. Lump solutions to a Jimbo-Miwa like equation. arXiv:1611.04478, 2016

[34] Dorizzi B, Grammaticos B, Ramani A, Winternitz P. Are all the equations of the Kadomtsev-Petviashvili hierarchy integrable? J Math Phys, 1986, **27**(12): 2848–2852

[35] Konopelchenko B, Strampp W. The AKNS hierarchy as symmetry constraint of the KP hierarchy. Inverse Probl, 1991, **7**(2): L17–L24

[36] Li X Y, Zhao Q L, Li Y X, Dong H H. Binary Bargmann symmetry constraint associated with 3×3 discrete matrix spectral problem. J Nonlinear Sci Appl, 2015, **8**(5): 496–506

[37] Zhao Q L, Li X Y. A Bargmann system and the involutive solutions associated with a new 4-order lattice hierarchy. Anal Math Phys, 2016, **6**(3): 237–254

[38] Dong H H, Zhang Y, Zhang X E. The new integrable symplectic map and the symmetry of integrable nonlinear lattice equation. Commun Nonlinear Sci Numer Simulat, 2016, **36**: 354–365

[39] Li X Y, Zhao Q L. A new integrable symplectic map by the binary nonlinearization to the super AKNS system. J Geom Phys, 2017, **121**: 123–137

[40] Ma W X. Abundant lumps and their interaction solutions of (3+1)-dimensional linear PDEs. J Geom Phys, 2018, **133**: 10–16

[41] Ma W X, Fan E G. Linear superposition principle applying to Hirota bilinear equations. Comput Math Appl, 2011, **61**(4): 950–959

[42] Xu Z H, Chen H L, Dai Z D. Rogue wave for the (2+1)-dimensional Kadomtsev-Petviashvili equation. Appl Math Lett, 2014, **37**: 34–38

[43] Ünsal Ö, Ma W X. Linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations. Comput Math Appl, 2016, **71**(6): 1242–1247

[44] Ma W X. Generalized bilinear differential equations. Stud Nonlinear Sci, 2011, **2**(4): 140–144

[45] Ablowitz M J, Fokas A S. Complex Variables: Introduction and Applications. New York: Cambridge University Press, 2003