functions. For even s (C. B. Ling, On summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 5 (1974) 551–561) and for all positive integral s (I. J. Zucker, The summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 10 (1979) 192–206), $Z(s)$ can be evaluated in terms of elliptic functions. In particular, for $s = 1$, $0 < k < 1$, and $z = K(\sqrt{1-k^2})/K(k)$, we have $\sum_{n=-\infty}^{\infty} \cosh^{-1}(n\pi z) = (\sum_{n=-\infty}^{\infty} e^{-\pi n^2 z})^2 = (2/\pi)K(k)$, where $K(k)$ is the complete elliptic integral of the first kind (see B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, 1991, p. 102 and p. 138).

Solved also by D. Cantor, R. J. Chapman (U. K.), R. Holzsager, and the proposer.

A Matrix of Inequalities

10599 [1997, 566]. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be nonnegative numbers and let (a_{ij}) be an $m \times n$ matrix of nonnegative numbers with at least one nonzero entry in each row. Suppose that the inequality $\sum_{h=1}^{m} a_{hj} x_h \le \sum_{k=1}^{n} a_{ik} y_k$ holds whenever $a_{ij} > 0$. Show that $\sum_{i=1}^{m} x_i \le \sum_{j=1}^{n} y_j$.

Solution by Frank Jelen and Eberhard Triesch, Der Rheinisch-Westfälischen Technischen Hochschule, Aachen, Germany. Let A be the specified matrix, with columns $c_1, \dots, c_n$. Let $x = (x_1, \dots, x_m)^T$ and $y = (y_1, \dots, y_n)^T$, and let $\mathbf{1}_k$ denote the column vector of length k with entries equal to 1.

Define $b = (b_1, \dots, b_m)^T$ by $b_i = \max\{c_j^T x \colon a_{ij} > 0\}$; this is well-defined since each row contains a positive entry. Consider the linear programs

$$\text{minimize} \quad \mathbf{1}_n^T z \qquad \text{subject to} \quad Az \ge b \text{ and } z \ge 0 \tag{1}$$

and

$$\text{maximize} \quad b^T w \qquad \text{subject to} \quad A^T w \le \mathbf{1}_n \text{ and } w \ge 0. \tag{2}$$

These linear programs are duals of each other, and (1) has the feasible solution $z = y$. It thus suffices to show that there exists a feasible solution u of (2) with $b^T u \ge \mathbf{1}_m^T x$, since the Duality Theorem then yields $\mathbf{1}_n^T y \ge b^T u \ge \mathbf{1}_m^T x$.

Consider the nonnegative vector $u = (u_1, \dots, u_m)^T$ defined by $u_i = x_i/b_i$ if $b_i > 0$ and $u_i = 0$ otherwise. Clearly $b^T u = \mathbf{1}_m^T x$.

For $1 \le j \le n$, define $I_j = \{i : a_{ij} > 0 \text{ and } x_i > 0\}$. For $i \in I_j$, we have $b_i \ge c_j^T x > 0$. Feasibility of u now follows from

$$c_j^T u = \sum_{i=1}^{m} a_{ij} u_i = \sum_{i \in I_j} a_{ij} \frac{x_i}{b_i} \le \frac{1}{c_j^T x} \sum_{i \in I_j} a_{ij} x_i = 1.$$

Solved also by the proposer.

A Complex Determinant

10601 [1997, 566]. *Proposed by Wen-Xiu Ma, Universität-GH Paderborn, Paderborn, Germany.* Let $n > 1$ be an integer and let $a_1, a_2, \dots, a_n$ be complex numbers. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{2n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{2n-1} \\ \vdots & & & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{2n-1} \\ 0 & 1 & 2a_1 & \cdots & (2n-1)a_1^{2n-2} \\ 0 & 1 & 2a_2 & \cdots & (2n-1)a_2^{2n-2} \\ \vdots & & & \ddots & \vdots \\ 0 & 1 & 2a_n & \cdots & (2n-1)a_n^{2n-2} \end{vmatrix} = (-1)^{n(n-1)/2} \prod_{1 \le i < j \le n} (a_i - a_j)^4.$$

Solution I by Robin J. Chapman, University of Exeter, Exeter, UK. Consider the Vandermonde matrix for $2n$ complex numbers $a_1, \ldots, a_{2n}$, in which the i, j-entry is a_i^{j-1}. The determinant is $\prod_{1\le j<k\le 2n}(a_k - a_j)$. Subtracting row j from row $n+j$ turns row $n+j$ into

$$(a_{n+j} - a_j)\left[0, 1, a_{n+j} + a_j, \cdots, \sum_{r=0}^{2n-2} a_{n+j}^{2n-2-r} a_j^r\right].$$

These row operations do not change the determinant. When $a_{n+j} \neq a_j$ for each j, we may cancel $\prod_{j=1}^{n}(a_{n+j} - a_j)$ from the two expressions for the determinant to obtain

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{2n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{2n-1} \\ & \vdots & & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{2n-1} \\ 0 & 1 & a_{n+1} + a_1 & \cdots & \sum_{r=0}^{2n-2} a_{n+1}^{2n-2-r} a_1^r \\ 0 & 1 & a_{n+2} + a_2 & \cdots & \sum_{r=0}^{2n-2} a_{n+2}^{2n-2-r} a_2^r \\ & \vdots & & \ddots & \vdots \\ 0 & 1 & a_{2n} + a_n & \cdots & \sum_{r=0}^{2n-2} a_{2n}^{2n-2-r} a_n^r \end{vmatrix} = \prod_{\substack{1\le j<k\le 2n \\ k\neq n+j}} (a_k - a_j). \qquad (*)$$

By continuity, $(*)$ is also valid when $a_{n+j} = a_j$. Setting $a_j = a_{n+j}$ for each j in $(*)$ yields the desired result, since each difference $a_k - a_j$ for $j < k$ appears four times in the product on the right side of $(*)$, once in reverse order.

Solution II by Joseph J. Rushanan, The MITRE Corporation, Bedford, MA. We use the techniques from J. J. Rushanan, On the Vandermonde matrix, this MONTHLY **96** (1989) 921–924. Let A be the matrix whose determinant is given in the problem statement. Given a complex polynomial f defined by $f(z) = \sum_{i=0}^{2n-1} c_i z^i$, let $\mathbf{f} = [c_0, \ldots, c_{2n-1}]^T$. Then

$$A\mathbf{f} = \left[f(a_1), f(a_2), \ldots, f(a_n), f'(a_1), f'(a_2), \ldots, f'(a_n)\right]^T.$$

Let $f_k(z) = \prod_{i=1}^{k-1}(z - a_i)$ for $1 \le k \le n$, and let $f_k(z) = f_{k-n}(z)\prod_{i=1}^{n}(z - a_i)$ for $n+1 \le k \le 2n$. Since f_k is monic with degree $k-1$, the matrix $U = [\mathbf{f}_1 \cdots \mathbf{f}_{2n}]$ is upper-triangular with 1s on the diagonal. Furthermore, $L = AU$ is lower-triangular, since $f_k(a_j) = 0 = f'_{n+k}(a_j)$ if $1 \le j < k \le n$ and $f_{n+k}(a_j) = 0$ for all j.

Thus $\det A$ is the product of the diagonal terms of L, which are $f_k(a_k)$ and $f'_{n+k}(a_k)$ for $1 \le k \le n$. These terms consist only of factors of the form $(a_r - a_s)$ with $r \neq s$. A typical term $(a_s - a_r)$ with $r < s$ appears in $f_s(a_s)$ once, appears negated in $f'_{n+r}(a_r)$, and appears squared in $f'_{n+s}(a_s)$. This shows that A has the desired determinant.

The technique generalizes to higher derivatives.

Editorial comment. David Callan and Wai Wah Lau observed that generalizations involving higher derivatives have appeared in the literature, such as on page 400 of R. A. Horn and C. R. Johnson, *Topics in Matrix Algebra*, Cambridge Univ. Press, 1991. Several others noted that the formula holds for $a_1, \ldots, a_n$ in an arbitrary commutative ring. Indeed, every polynomial identity in $\mathbb{Z}[a_1, \ldots, a_n]$ holds over arbitrary commutative rings.

Solved also by M. Benedicty, J. C. Binz (Switzerland), G. L. Body (U. K.), D. Callan, L. L. Foster, J.-P. Grivaux (France), R. Holzsager, G. Keselman, N. Komanda, O. Kouba (Syria), W. W Lau, J. H. Lindsey II, G. R. Miller, M. McKee, J. H. Nieto (Venezuela), G. Peng, C. Popescu (Belgium), R. Richberg (Germany), J. H. Smith, P. Szeptycki, A. Tissier (France), J. Van hamme (Belgium), Wyoming Problems Circle, WMC Problems Group, and the proposer.