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AKNS type reduced integrable bi-Hamiltonian hierarchies with four potentials

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ABSTRACT

The aim of this paper is to derive a kind of integrable hierarchies with four potentials, which possess bi-Hamiltonian structures, from reduced AKNS matrix spectral problems. The associated recursion operators are worked out explicitly. The Lax pair formulation and the trace identity are basic tools in the analysis. Two nonlinear examples in the resulting integrable hierarchies are integrable nonlinear Schrödinger type equations and integrable modified Korteweg–de Vries type equations.

1. Introduction

The Lax pair formulation is an important technique for constructing integrable equations [1]. It involves finding a pair of linear partial differential equations, known as the Lax pair, which are compatible with a given nonlinear equation. The Lax pair provides a bridge between linear and nonlinear equations, and the integrability of the nonlinear equation is related to the spectral properties of the Lax pair of matrix spectral problems. Many famous integrable equations, such as the Korteweg–de Vries equation and the nonlinear Schrödinger equation, have been derived using the Lax pair formulation.

The general procedure of the Lax pair formulation to construct integrable equations is as follows. We begin with a matrix spatial spectral problem with an appropriately chosen spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = u_1 e_1(\lambda) + \cdots + u_q e_q(\lambda) + e_0(\lambda), \tag{1.1}$$

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where λ is the spectral parameter, $u = (u_1, \dots, u_q)^T$ is the dependent variable, and $e_1, \dots, e_q$ are linear independent elements and e_0 is a pseudo-regular element in a given loop algebra $\tilde{g}$. The pseudo-regular conditions, $\mathrm{Ker}\,\mathrm{ad}_{e_0} \oplus \mathrm{Im}\,\mathrm{ad}_{e_0} = \tilde{g}$ and $\mathrm{Ker}\,\mathrm{ad}_{e_0}$ is commutative, guarantee that there exists a Laurent series solution $\mathcal{Y} = \sum_{s\geq 0} \lambda^{-s}\mathcal{Y}^{[s]}$ to the stationary zero curvature equation:

$$\mathcal{Y}_x = i[\mathcal{M}, \mathcal{Y}]. \tag{1.2}$$

Based on this solution $\mathcal{Y}$, an integrable hierarchy can then be presented through zero curvature equations:

$$\mathcal{M}_t - \mathcal{N}^{[r]}_x + i[\mathcal{M}, \mathcal{N}^{[r]}] = 0,\ r \geq 0, \tag{1.3}$$

where $\mathcal{N}^{[r]} = (\lambda^r \mathcal{Y})_+ + \Delta^{[r]} = \sum_{s=0}^{r} \lambda^{r-s}\mathcal{Y}^{[s]} + \Delta^{[r]},\ \Delta^{[r]} \in \tilde{g},\ r \geq 0$. Those zero curvature equations are the compatibility conditions between the spatial and temporal matrix spectral problems:

$$-i\phi_x = \mathcal{M}\phi,\ -i\phi_t = \mathcal{N}^{[r]}\phi,\ r \geq 0. \tag{1.4}$$

A basic tool to obtain Hamiltonian structures of the associated zero curvature equations is the trace identity [2]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial \lambda}\big)\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial u}\big), \tag{1.5}$$

where γ is a constant, independent of λ. Bi-Hamiltonian structures can be often furnished with a recursion operator, thereby exhibiting the Liouville integrability of the associated zero curvature equations [2,3].

Various integrable hierarchies are generated in this way, with the underlying matrix algebra being the special linear algebras (see, e.g., [4–8]), or the special orthogonal algebras (see, e.g., [9,10]). The well-known integrable hierarchies with two potentials include the Ablowitz–Kaup–Newell–Segur hierarchy, the Heisengerg hierarchy, the Kaup–Newell hierarchy and the Wadati–Konno–Ichikawa hierarchy.

This paper aims to present integrable hierarchies of bi-Hamiltonian equations with four potentials. The Lax pair formulation and the trace identity are basic tools to present integrable hierarchies. Two illustrative examples are a sort of integrable nonlinear Schrödinger type equations and integrable modified Korteweg–de Vries type equations. The last section is devoted to concluding remarks.

2. Lax pairs and integrable hierarchies

Let m and n be two natural integers. We consider a matrix spectral problem of the form:

$$-i\phi_x = \mathcal{M}\phi = \mathcal{M}(u,\lambda)\phi,\ \mathcal{M} = \begin{bmatrix} \alpha_1\lambda & \mathbf{p}_1 & \mathbf{p}_2 \\ \mathbf{q}_1 & \alpha_2\lambda I_m & 0 \\ \mathbf{q}_2 & 0 & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.1}$$

where λ is the spectral parameter and u is the four-dimensional potential

$$u = u(x,t) = (p_1, p_2, q_1, q_2)^T. \tag{2.2}$$

In the above spectral matrix $\mathcal{M}$, $\alpha_1, \alpha_2 \in \mathbb{C}$ are two distinct constants, I_k is the kth-order identity matrix, and

$$\mathbf{p}_1 = (\underbrace{p_1, \dots, p_1}_{m}),\ \mathbf{q}_1 = (\underbrace{q_1, \dots, q_1}_{m})^T,\ \mathbf{p}_2 = (\underbrace{p_2, \dots, p_2}_{n}),\ \mathbf{q}_2 = (\underbrace{q_2, \dots, q_2}_{n})^T. \tag{2.3}$$

Those matrix spectral problems are specific reductions of the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem with vector potentials (see, e.g., [4,11,12] for details). We would like to show that all those reduced spectral problems will still yield integrable hierarchies.

To derive an associated integrable hierarchy for each pair of m and n, let us start to compute a Laurent series solution to the stationary zero curvature Eq. (1.2):

$$\mathcal{Y} = \begin{bmatrix} a & \mathbf{b}_1 & \mathbf{b}_2 \\ \mathbf{c}_1 & d_{1,1}E_{m,m} & d_{1,2}E_{m,n} \\ \mathbf{c}_2 & d_{2,1}E_{n,m} & d_{2,2}E_{n,n} \end{bmatrix} = \sum_{s\geq 0} \lambda^{-s}\mathcal{Y}^{[s]},\ \mathcal{Y}^{[s]} = \begin{bmatrix} a^{[s]} & \mathbf{b}_1^{[s]} & \mathbf{b}_2^{[s]} \\ \mathbf{c}_1^{[s]} & d_{1,1}^{[s]}E_{m,n} & d_{1,2}^{[s]}E_{m,n} \\ \mathbf{c}_2^{[s]} & d_{2,1}^{[s]}E_{n,m} & d_{2,2}^{[s]}E_{n,n} \end{bmatrix}, \tag{2.4}$$

where $E_{k,l}$ is the $k \times l$ matrix of ones, and

$$\mathbf{b}_1 = (\underbrace{b_1,\ldots,b_1}_{m}),\ \mathbf{c}_1 = (\underbrace{c_1,\ldots,c_1}_{m})^T,\ \mathbf{b}_2 = (\underbrace{b_2,\ldots,b_2}_{n}),\ \mathbf{c}_2 = (\underbrace{c_2,\ldots,c_2}_{n})^T,$$

$$\mathbf{b}_1^{[s]} = (\underbrace{b_1^{[s]},\ldots,b_1^{[s]}}_{m}),\ \mathbf{c}_1^{[s]} = (\underbrace{c_1^{[s]},\ldots,c_1^{[s]}}_{m})^T,\ \mathbf{b}_2^{[s]} = (\underbrace{b_2^{[s]},\ldots,b_2^{[s]}}_{n}),\ \mathbf{c}_2^{[s]} = (\underbrace{c_2^{[s]},\ldots,c_2^{[s]}}_{n})^T.$$

It is straightforward to observe that the corresponding stationary zero curvature equation leads to the initial conditions on $\mathcal{Y}^{[0]}$:

$$a_x^{[0]} = 0,\ b_j^{[0]} = c_j^{[0]} = 0,\ (d_{k,l}^{[0]})_x = 0,\ 1 \leq j,k,l \leq 2, \tag{2.5}$$

and the recursion relations for determining $\mathcal{Y}^{[s]},\ s \geq 1$:

$$b_j^{[s+1]} = \frac{1}{\alpha}(-ib_{j,x}^{[s]} + p_j a^{[s]} - mp_1 d_{1,j}^{[s]} - np_2 d_{2,j}^{[s]}),\ 1 \leq j \leq 2, \tag{2.6}$$

$$c_j^{[s+1]} = \frac{1}{\alpha}(ic_{j,x}^{[s]} + q_j a^{[s]} - mq_1 d_{j,1}^{[s]} - nq_2 d_{j,2}^{[s]}),\ 1 \leq j \leq 2, \tag{2.7}$$

$$(d_{k,l}^{[s+1]})_x = i(q_k b_l^{[s+1]} - p_l c_k^{[s+1]}),\ 1 \leq k,l \leq 2, \tag{2.8}$$

$$a_x^{[s+1]} = i(-mq_1 b_1^{[s+1]} - nq_2 b_2^{[s+1]} + mp_1 c_1^{[s+1]} + np_2 c_2^{[s+1]}), \tag{2.9}$$

where $s \geq 0$. To satisfy (2.5), let us take the initial values as follows:

$$a^{[0]} = \beta,\ d_{k,l}^{[0]} = 0,\ 1 \leq k,l \leq 2, \tag{2.10}$$

where $\beta \in \mathbb{C}$ is an arbitrary constant, and choose the constant of integration as zero,

$$a^{[s]}|_{u=0} = 0,\ d_{k,l}^{[s]}|_{u=0} = 0,\ 1 \leq k,l \leq 2,\ s \geq 1, \tag{2.11}$$

so that we can determine all required differential polynomials $a^{[s]}, b_j^{[s]}, c_j^{[s]}, d_{k,l}^{[s]},\ 1 \leq j,k,l \leq 2,\ s \geq 1$, uniquely. Under such conditions, we can obtain that

$$b_j^{[1]} = \frac{\beta}{\alpha}p_j,\ c_j^{[1]} = \frac{\beta}{\alpha}q_j,\ a^{[1]} = 0,\ d_{k,l}^{[1]} = 0,\ 1 \leq j,k,l \leq 2;$$

$$b_j^{[2]} = -\frac{\beta}{\alpha^2}ip_{j,x},\ c_j^{[2]} = \frac{\beta}{\alpha^2}iq_{j,x},\ a^{[2]} = -\frac{\beta}{\alpha^2}(mp_1q_1 + np_2q_2),\ d_{k,l}^{[2]} = \frac{\beta}{\alpha^2}p_l q_k,\ 1 \leq j,k,l \leq 2;$$

$$\begin{cases} b_1^{[3]} = -\frac{\beta}{\alpha^3}(p_{1,xx} + 2np_1p_2q_2 + 2mp_1^2q_1),\ b_2^{[3]} = -\frac{\beta}{\alpha^3}(p_{2,xx} + 2np_2^2q_2 + 2mp_1p_2q_1), \\ c_1^{[3]} = -\frac{\beta}{\alpha^3}(q_{1,xx} + 2np_2q_1q_2 + 2mp_1q_1^2),\ c_2^{[3]} = -\frac{\beta}{\alpha^3}(q_{2,xx} + 2np_2q_2^2 + 2mp_1q_1q_2), \\ a^{[3]} = \frac{\beta}{\alpha^3}i(mp_{1,x}q_1 + np_{2,x}q_2 - mp_1q_{1,x} - np_2q_{2,x}),\ d_{k,l}^{[3]} = \frac{\beta}{\alpha^3}i(p_lq_{k,x} - q_kp_{l,x}),\ 1 \leq k,l \leq 2; \end{cases}$$

and

$$\begin{cases} b_1^{[4]} = \frac{\beta}{\alpha^4}i(p_{1,xxx} + 3np_1p_{2,x}q_2 + 6mp_1p_{1,x}q_1 + 3np_{1,x}p_2q_2), \\ b_2^{[4]} = \frac{\beta}{\alpha^4}i(p_{2,xxx} + 3mp_1p_{2,x}q_1 + 3\ mp_{1,x}p_2q_1 + 6np_2p_{2,x}q_2), \end{cases}$$

$$\left\{\begin{array}{l} c_1^{[4]} = -\frac{\beta}{\alpha^4} i(q_{1,xxx} + 6mp_1q_1q_{1,x} + 3np_2q_1q_{2,x} + 3np_2q_{1,x}q_2), \\ c_2^{[4]} = -\frac{\beta}{\alpha^4} i(q_{2,xxx} + 3mp_1q_{1,x}q_2 + 3mp_1q_1q_{2,x} + 6np_2q_2q_{2,x}), \end{array}\right.$$

$$\left\{\begin{array}{l} a^{[4]} = \frac{\beta}{\alpha^4}\, \big[\, mp_{1,xx}q_1 + mp_1q_{1,xx} + np_{2,xx}q_2 + np_2q_{2,xx} \\ \qquad -mp_{1,x}q_{1,x} - np_{2,x}q_{2,x} + 3(mp_1q_1 + np_2q_2)^2 \,\big], \\ d_{k,l}^{[4]} = -\frac{\beta}{\alpha^4}(3mp_1q_1p_lq_k + 3np_2q_2p_lq_k + p_{l,xx}q_k + p_lq_{k,xx} - p_{l,x}q_{k,x}),\ 1 \le k,l \le 2; \end{array}\right.$$

which will be used to present examples of integrable Hamiltonian equations later.

Following the Lax pair formulation and based on the previous computations, we can take the following temporal matrix spectral problems

$$-i\phi_t = \mathcal{N}^{[r]}\phi = \mathcal{N}^{[r]}(u,\lambda)\phi,\ \mathcal{N}^{[r]} = (\lambda^r\mathcal{Y})_+ = \sum_{s=0}^{r} \lambda^s \mathcal{Y}^{[r-s]},\ r \ge 0, \tag{2.12}$$

to form appropriate other parts of Lax pairs. Now, the compatibility conditions (1.3) of the resulting Lax pairs yield a four-component soliton hierarchy:

$$u_{t_r} = K^{[r]} = (\alpha i b_1^{[r+1]}, \alpha i b_2^{[r+1]}, -\alpha i c_1^{[r+1]}, -\alpha i c_2^{[r+1]})^T,\ r \ge 0. \tag{2.13}$$

Making use of the previous expressions of $b_j^{[s]}$ and $c_j^{[s]}$, we immediately work out the first two nonlinear examples. The first one is the nonlinear Schrödinger type equations:

$$\left\{\begin{array}{l} ip_{j,t_2} = \frac{\beta}{\alpha^2}[p_{j,xx} + 2(mp_1q_1 + np_2q_2)p_j], \\ iq_{j,t_2} = -\frac{\beta}{\alpha^2}[q_{j,xx} + 2(mp_1q_1 + np_2q_2)q_j], \end{array}\right. \tag{2.14}$$

where $1 \le j \le 2$, and the second is the modified Korteweg–de Vries type equations:

$$\left\{\begin{array}{l} p_{j,t_3} = -\frac{\beta}{\alpha^3}[p_{j,xxx} + 3m(p_1p_j)_xq_1 + 3n(p_2p_j)_xq_2], \\ q_{j,t_3} = -\frac{\beta}{\alpha^3}[q_{j,xxx} + 3mp_1(q_1q_j)_x + 3np_2(q_2q_j)_x], \end{array}\right. \tag{2.15}$$

where $1 \le j \le 2$. Those examples enrich the category of integrable coupled nonlinear Schrödinger equations and modified Korteweg–de Vries equations (see, e.g., [13–15]).

3. Bi-Hamiltonian structures

To furnish bi-Hamiltonian structures for the presented soliton hierarchy (2.13), we apply the trace identity (1.5) associated with the matrix spectral problem (2.1). By virtue of the solution $\mathcal{Y}$ determined by (2.4), we can derive

$$\mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial \lambda}\big) = \alpha_1 a + \alpha_2(md_{1,1} + nd_{2,2}),\ \mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial u}\big) = (mc_1, nc_2, mb_1, nb_2)^T,$$

and consequently, we obtain

$$\frac{\delta}{\delta u}\int [\alpha_1 a^{[s+1]} + \alpha_2(md_{1,1}^{[s]} + nd_{2,2}^{[s]})]\lambda^{-s-1}\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma-s}(mc_1^{[s]}, nc_2^{[s]}, mb_1^{[s]}, nb_2^{[s]})^T,\ s \ge 0.$$

Checking the case with $s = 2$, we see $\gamma = 0$. Therefore, we have

$$\frac{\delta}{\delta u}\mathcal{H}^{[s]} = (mc_1^{[s+1]}, nc_2^{[s+1]}, mb_1^{[s+1]}, nb_2^{[s+1]})^T,\ \mathcal{H}^{[s]} = -\int \frac{\alpha_1 a^{[s+2]} + \alpha_2(md_{1,1}^{[s+2]} + nd_{2,2}^{[s+2]})}{s+1}dx, \tag{3.1}$$

where $s \ge 0$.

All these identities enable us to furnish the Hamiltonian structures for the obtained integrable hierarchy (2.13):

$$u_{t_r} = K^{[r]} = J\frac{\delta \mathcal{H}^{[r]}}{\delta u},\ J = \begin{bmatrix} 0 & J_0 \\ -J_0 & 0 \end{bmatrix},\ J_0 = \begin{bmatrix} \frac{1}{m}\alpha i & 0 \\ 0 & \frac{1}{n}\alpha i \end{bmatrix},\ r \ge 0, \tag{3.2}$$

where the Hamiltonian functionals $\mathcal{H}^{[r]}$, $r \geq 0$, are given by (3.1). It is known that the Hamiltonian structures establishes a connection from a conserved functional $\mathcal{H}$ to a symmetry S by $S = J\frac{\delta \mathcal{H}}{\delta u}$.

A straightforward calculation tells that we can obtain an isospectral Lax operator algebra (see [16] for details):

$$[\![\mathcal{N}^{[s_1]}, \mathcal{N}^{[s_2]}]\!] = \mathcal{N}^{[s_1]\prime}(u)[K^{[s_2]}] - \mathcal{N}^{[s_2]\prime}(u)[K^{[s_1]}] + [\mathcal{N}^{[s_1]}, \mathcal{N}^{[s_2]}] = 0,\ s_1, s_2 \geq 0, \tag{3.3}$$

which comes from the algebraic structure of isospectral zero curvature equations [16]. It follows from this Lax operator algebra that we have the Abelian algebra of infinitely many symmetries $\{K^{[s]}\}_{s=0}^{\infty}$:

$$[\![K^{[s_1]}, K^{[s_2]}]\!] = K^{[s_1]\prime}(u)[K^{[s_2]}] - K^{[s_2]\prime}(u)[K^{[s_1]}] = 0,\ s_1, s_2 \geq 0. \tag{3.4}$$

Further, the Hamiltonian structures tell that the conserved functionals $\{\mathcal{H}^{[s]}\}_{s=0}^{\infty}$ form an Abelian algebra, too:

$$\{\mathcal{H}^{[s_1]}, \mathcal{H}^{[s_2]}\}_J = \int (\frac{\delta \mathcal{H}^{[s_1]}}{\delta u})^T J \frac{\delta \mathcal{H}^{[s_2]}}{\delta u}\, dx = 0,\ s_1, s_2 \geq 0. \tag{3.5}$$

Moreover, from $K^{[s+1]} = \Phi K^{[s]}$, we can work out the recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$:

$$\begin{cases} \Phi_{11} = -\frac{i}{\alpha}(\partial_x + 2mp_1\partial_x^{-1}q_1 + np_2\partial_x^{-1}q_2),\ \Phi_{12} = -\frac{i}{\alpha}np_1\partial_x^{-1}q_2, \\ \Phi_{13} = -\frac{i}{\alpha}(2mp_1\partial_x^{-1}p_1),\ \Phi_{14} = -\frac{i}{\alpha}n(p_1\partial_x^{-1}p_2 + p_2\partial_x^{-1}p_1), \end{cases} \tag{3.6}$$

$$\begin{cases} \Phi_{21} = -\frac{i}{\alpha}mp_2\partial_x^{-1}q_1,\ \Phi_{22} = -\frac{i}{\alpha}(\partial_x + mp_1\partial_x^{-1}q_1 + 2np_2\partial_x^{-1}q_2), \\ \Phi_{23} = -\frac{i}{\alpha}m(p_2\partial_x^{-1}p_1 + p_1\partial_x^{-1}p_2),\ \Phi_{24} = -\frac{i}{\alpha}(2np_2\partial_x^{-1}p_2), \end{cases} \tag{3.7}$$

$$\begin{cases} \Phi_{31} = \frac{i}{\alpha}(2mq_1\partial_x^{-1}q_1),\ \Phi_{32} = \frac{i}{\alpha}n(q_1\partial_x^{-1}q_2 + q_2\partial_x^{-1}q_1), \\ \Phi_{33} = \frac{i}{\alpha}(\partial_x + 2mq_1\partial_x^{-1}p_1 + nq_2\partial_x^{-1}p_2),\ \Phi_{34} = \frac{i}{\alpha}nq_1\partial_x^{-1}p_2, \end{cases} \tag{3.8}$$

$$\begin{cases} \Phi_{41} = \frac{i}{\alpha}m(q_2\partial_x^{-1}q_1 + q_1\partial_x^{-1}q_2),\ \Phi_{42} = \frac{i}{\alpha}(2nq_2\partial_x^{-1}q_2), \\ \Phi_{43} = \frac{i}{\alpha}mq_2\partial_x^{-1}p_1,\ \Phi_{44} = \frac{i}{\alpha}(\partial_x + mq_1\partial_x^{-1}p_1 + 2nq_2\partial_x^{-1}p_2). \end{cases} \tag{3.9}$$

It is direct to see that a combination of J with the recursion operator Φ [17], leads to bi-Hamiltonian structures [3] for the hierarchy:

$$u_{t_r} = K^{[r]} = J\frac{\delta \mathcal{H}^{[r]}}{\delta u} = M\frac{\delta \mathcal{H}^{[r-1]}}{\delta u},\ r \geq 1, \tag{3.10}$$

where the second Hamiltonian operator is $M = \Phi J$.

All this implies that each equation in the resulting hierarchy (2.13) is Liouville integrable, or more precisely, each possesses infinitely many commuting conserved densities $\{\mathcal{H}^{[s]}\}_{s=0}^{\infty}$ and symmetries $\{K^{[s]}\}_{s=0}^{\infty}$.

4. Concluding remarks

A set of integrable hierarchies with four potentials has been presented, from a class of AKNS reduced special matrix spectral problems, within the Lax pair formulation. Bi-Hamiltonian structures have been furnished for the resulting integrable equations through applying the trace identity and recursion operators.

Other generalizations could be generated by taking more potentials in matrix spatial spectral problems to generate integrable Hamiltonian equations with six or more potentials. Nonlocal integrable counterparts could also be formulated under similarity transformations of spectral matrices (see, e.g., [18–20] for novel nonlocal nonlinear Schrödinger equations). The field of integrable equations is vast and continues to evolve, with new approaches and techniques being developed. It requires a deep understanding of mathematical methods and theories, as well as creativity and insight to construct new integrable equations.

Data availability

No data was used for the research described in the article.

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