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Type $(-\lambda, -\lambda^*)$ reduced nonlocal integrable mKdV equations and their soliton solutions

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ABSTRACT

A kind of novel reduced nonlocal integrable mKdV equations of odd order is presented by taking two group reductions of the AKNS matrix spectral problems. One reduction is local, replacing the spectral parameter with its negative and the other is nonlocal, replacing the spectral parameter with its negative complex conjugate. Based on distribution of eigenvalues, soliton solutions are generated from the corresponding reflectionless Riemann–Hilbert problems.

1. Introduction

Nonlocal integrable equations have become a new hot research area in soliton theory [1,2]. Group reductions of matrix spectral problems play an essential role in exploring their integrable structures [3,4]. The traditional methods, including the inverse scattering transform, Darboux transformation and the Hirota bilinear method, can be used to construct their soliton solutions (see, e.g., [2,5–7]).

It is known that the Riemann-Hilbert technique is one of powerful approaches to integrable equations, both local and nonlocal, and particularly to their soliton solutions [8]. Various integrable equations have been studied via their associated Riemann-Hilbert problems (see, e.g., [9–11] in the nonlocal case). In this letter, we would like to construct a kind of novel reduced nonlocal integrable mKdV equations and their soliton solutions.

The rest of this letter is organized as follows. In Section 2, we make two group reductions to generate type $(-\lambda, -\lambda^*)$ reduced nonlocal integrable mKdV equations of odd order. In Section 3, based on distribution of eigenvalues, we formulate solutions to the corresponding special Riemann-Hilbert problems with the identity jump matrix, and construct soliton solutions for the resulting reduced nonlocal integrable mKdV equations.

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2. Reduced nonlocal integrable mKdV equations

The matrix AKNS integrable hierarchies revisited: Let us recall the AKNS hierarchies of matrix integrable equations. As usual, let λ denote the spectral parameter, and assume that $m, n \geq 1$ are two given integers, and p and q are two matrix potentials of sizes $m \times n$ and $n \times m$, respectively. The matrix AKNS spectral problems are defined as follows:

$$-i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda\Lambda + P)\phi,\ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi = (\lambda^r\Omega + Q^{[r]})\phi, \tag{2.1}$$

where $r \geq 0$. The constant matrices Λ and Ω are given by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n),\ \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.2}$$

where α_1, α_2 and β_1, β_2 are two arbitrary pairs of distinct real constants, I_s denotes the identity matrix of size s, and the other two involved square matrices of size $m+n$ are defined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix},\ Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.3}$$

where $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$, $s \geq 0$, are defined recursively as follows:

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a^{[0]} = \beta_1 I_m,\ d^{[0]} = \beta_2 I_n, \tag{2.4a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p),\ s \geq 0, \tag{2.4b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q),\ s \geq 0, \tag{2.4c}$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q),\ d_x^{[s]} = i(qb^{[s]} - c^{[s]}p),\ s \geq 1, \tag{2.4d}$$

with zero constants of integration being taken. Particularly, we can have

$$Q^{[1]} = \frac{\beta}{\alpha}P,\ Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2}I_{m,n}(P^2 + iP_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3),$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. The compatibility conditions of the two matrix spectral problems in (2.1), i.e., the zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\ r \geq 0, \tag{2.5}$$

lead to one so-called matrix AKNS integrable hierarchy:

$$p_t = i\alpha b^{[r+1]},\ q_t = -i\alpha c^{[r+1]},\ r \geq 0. \tag{2.6}$$

The second set of nonlinear integrable equations in the hierarchy gives us the AKNS matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp),\ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \tag{2.7}$$

where p and q are two matrix potentials of sizes $m \times n$ and $n \times m$, respectively.

Reduced nonlocal integrable mKdV equations: Let us now construct a kind of novel reduced nonlocal integrable mKdV equations of odd order by taking two group reductions for the matrix AKNS spectral problems in (2.1).

Let Σ_1, Σ_2 and Δ_1, Δ_2 be a pair of constant invertible symmetric matrices and another pair of constant invertible Hermitian matrices, respectively. We consider two group reductions for the spectral matrix U:

$$U^T(x,t,-\lambda) = (U(x,t,-\lambda))^T = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{2.8}$$

and

$$U^\dagger(-x,-t,-\lambda^*) = (U(-x,-t,-\lambda^*))^\dagger = -\Delta U(x,t,\lambda)\Delta^{-1}, \tag{2.9}$$

where T denotes the matrix transpose, $\dagger$ stands for the Hermitian transpose, and the two constant invertible matrices, Σ and Δ, are defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{2.10}$$

These two group reductions in (2.8) and (2.9) lead precisely to

$$P^T(x,t) = -\Sigma P(x,t)\Sigma^{-1}, \tag{2.11}$$

and

$$P^\dagger(-x,-t) = -\Delta P(x,t)\Delta^{-1}, \tag{2.12}$$

respectively, and enable us to make the reductions for the matrix potentials:

$$q(x,t) = -\Sigma_2^{-1}p^T(x,t)\Sigma_1, \tag{2.13}$$

and

$$q(x,t) = -\Delta_2^{-1}p^\dagger(-x,-t)\Delta_1, \tag{2.14}$$

respectively. Therefore, to satisfy both group reductions, we need to impose an additional constraint:

$$\Sigma_2^{-1}p^T(x,t)\Sigma_1 = \Delta_2^{-1}p^\dagger(-x,-t)\Delta_1. \tag{2.15}$$

Moreover, we notice that the group reductions in (2.8) and (2.9) guarantee

$$\begin{cases} V^{[2s+1]T}(x,t,-\lambda) = (V^{[2s+1]}(x,t,-\lambda))^T = -\Sigma V^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2s+1]\dagger}(-x,-t,-\lambda^*) = (V^{[2s+1]}(-x,-t,-\lambda^*))^\dagger = -\Delta V^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.16}$$

where $s \geq 0$. Consequently, under the potential reductions in (2.13) and (2.14), the integrable matrix AKNS equations in (2.6) with $r = 2s+1,\ s \geq 0$, become a hierarchy of reduced nonlocal integrable matrix mKdV equations of odd order:

$$p_t = i\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1=-\Delta_2^{-1}p^\dagger(-x,-t)\Delta_1},\ s \geq 0, \tag{2.17}$$

where p is an $m \times n$ matrix potential, which satisfies (2.15). Each equation in the hierarchy (2.17) possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in (2.1) with $r = 2s+1,\ s \geq 0$, and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (2.6) with $r = 2s+1,\ s \geq 0$.

Let us now fix $s = 1$, i.e., $r = 3$. The reduced matrix integrable mKdV equations in (2.17) with $s = 1$ give a kind of novel reduced nonlocal integrable matrix mKdV equations:

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Sigma_2^{-1}p^T\Sigma_1 p_x - 3p_x\Sigma_2^{-1}p^T\Sigma_1 p) \\ &= -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Delta_2^{-1}p^\dagger(-x,-t)\Delta_1 p_x - 3p_x\Delta_2^{-1}p^\dagger(-x,-t)\Delta_1 p), \end{aligned} \tag{2.18}$$

where p is an $m \times n$ matrix potential satisfying (2.15). Let us below compute a few examples of these novel reduced nonlocal integrable matrix mKdV equations, by taking different values for m, n and different choices for Σ, Δ.

If we consider $m = 1$ and $n = 2$, and take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{2.19}$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$, then the potential constraint (2.15) tells

$$p_2 = \sigma\delta p_1^*(-x, -t), \tag{2.20}$$

where $p = (p_1, p_2)$, and further, the corresponding potential matrix P reads

$$P = \begin{bmatrix} 0 & p_1 & \sigma\delta p_1^*(-x,-t) \\ -\sigma p_1 & 0 & 0 \\ -\delta p_1^*(-x,-t) & 0 & 0 \end{bmatrix}. \tag{2.21}$$

In this way, the corresponding reduced nonlocal integrable mKdV equations in (2.18) become

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1^2 p_{1,x} - 3\sigma p_1^*(-x,-t)(p_1 p_1^*(-x,-t))_x], \tag{2.22}$$

where $\sigma = \pm 1$. In the same manner, a choice of

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \tag{2.23}$$

where σ and δ are real constants satisfying $\sigma^2 = \delta^2 = 1$, leads to another pair of scalar nonlocal integrable mKdV equations:

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\delta p_1 p_1^*(-x,-t) p_{1,x} - 3\delta p_1 (p_1 p_1^*(-x,-t))_x], \tag{2.24}$$

where $\delta = \pm 1$. All these are a different kind of nonlocal integrable equations from the one studied in [12–14]. We point out that if we take the complex field into consideration, the two equations in (2.22), and the two equations in (2.24), can be changed into each other, under the dependent variable transformation $p_1 \to ip_1$, and the independent variable transformation $(x, t) \to (ix, -it)$, respectively.

Next, if we consider $m = 1$ and $n = 4$, and take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_1 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & \delta_2 & 0 \end{bmatrix}, \tag{2.25}$$

where σ_j and δ_j are real constants and satisfy $\sigma_j^2 = \delta_j^2 = 1,\ j = 1, 2$, then the potential constraint (2.15) generates

$$p_2 = \sigma_1\delta_1 p_1^*(-x,-t),\ p_4 = \sigma_2\delta_2 p_3^*(-x,-t), \tag{2.26}$$

where $p = (p_1, p_2, p_3, p_4)$, and further, the corresponding potential matrix P is given by

$$P = \begin{bmatrix} 0 & p_1 & \sigma_1\delta_1 p_1^*(-x,-t) & p_3 & \sigma_2\delta_2 p_3^*(-x,-t) \\ -\sigma_1 p_1 & 0 & 0 & 0 & 0 \\ -\delta_1 p_1^*(-x,-t) & 0 & 0 & 0 & 0 \\ -\sigma_2 p_3 & 0 & 0 & 0 & 0 \\ -\delta_2 p_3^*(-x,-t) & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.27}$$

This enables us to obtain a set of two-component novel reduced nonlocal integrable mKdV equations:

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma_1 p_1^2 p_{1,x} - 3\sigma_1 p_1^*(-x,-t)(p_1 p_1^*(-x,-t))_x \\ \qquad -3\sigma_2 p_3 (p_1 p_3)_x - 3\sigma_2 p_3^*(-x,-t)(p_1 p_3^*(-x,-t))_x], \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}[p_{3,xxx} - 3\sigma_1 p_1 (p_1 p_3)_x - 3\sigma_1 p_1^*(-x,-t)(p_1^*(-x,-t)p_3)_x \\ \qquad -6\sigma_2 p_3^2 p_{3,x} - 3\sigma_2 p_3^*(-x,-t)(p_3 p_3^*(-x,-t))_x], \end{cases} \tag{2.28}$$

where σ_j are real constants and satisfy $\sigma_j^2 = 1,\ j = 1, 2$.

3. Soliton solutions

Distribution of eigenvalues: Note that based on the group reduction in (2.8) (or (2.9)), we can show that λ is an eigenvalue of the matrix spectral problems in (2.1) if and only if $\hat{\lambda} = -\lambda$ (or $\hat{\lambda} = -\lambda^*$) is an adjoint eigenvalue, i.e., it satisfies the adjoint matrix spectral problems:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u, \hat{\lambda}),\ i\tilde{\phi}_t = \tilde{\phi}V^{[r]} = \tilde{\phi}V^{[r]}(u, \hat{\lambda}), \tag{3.1}$$

where $r = 2s+1,\ s \geq 0$. Accordingly, we can assume to have eigenvalues $\lambda :\ \mu,\ \mu^*,\ \nu$, and adjoint eigenvalues $\hat{\lambda} :\ -\mu,\ -\mu^*,\ -\nu$, where $\mu \notin \mathbb{R}$ and $\nu \in \mathbb{R}$. Moreover, if $\phi(\lambda)$ is an eigenfunction of the matrix spectral problems associated with an eigenvalue λ, then

$$\tilde{\phi}(x,t,\lambda) = \phi^T(x,t,-\lambda)\Sigma,\ \tilde{\phi}(x,t,\lambda) = \phi^\dagger(-x,-t,-\lambda^*)\Delta, \tag{3.2}$$

present two adjoint eigenfunctions associated with the same original eigenvalue λ.

General solutions to reflectionless Riemann-Hilbert problems: Let $N_1, N_2 \geq 0$ be two integers such that $N = 2N_1 + N_2 \geq 1$. To present a general formulation of solutions to reflectionless Riemann-Hilbert problems, we take two different sets of eigenvalues and adjoint eigenvalues as follows:

$$\lambda_k,\ 1 \leq k \leq N :\ \mu_1,\ \cdots,\ \mu_{N_1},\ \mu_1^*,\ \cdots,\ \cdots,\ \mu_{N_1}^*,\ \nu_1,\ \cdots,\ \nu_{N_2}, \tag{3.3}$$

and

$$\hat{\lambda}_k,\ 1 \leq k \leq N :\ -\mu_1,\ \cdots,\ -\mu_{N_1}, -\mu_1^*,\ \cdots,\ -\mu_{N_1}^*,\ -\nu_1,\ \cdots,\ -\nu_{N_2}, \tag{3.4}$$

where $\mu_i \notin \mathbb{R},\ 1 \leq i \leq N_1$, and $\nu_i \in \mathbb{R},\ 1 \leq i \leq N_2$, and their corresponding eigenfunctions and adjoint eigenfunctions: $v_k,\ \hat{v}_k,\ 1 \leq k \leq N$, respectively. Let us further introduce

$$G^+(\lambda) = I_{m+n} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ (G^-)^{-1}(\lambda) = I_{m+n} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{3.5}$$

where M is a square matrix $M = (m_{kl})_{N\times N}$ with its entries determined by

$$m_{kl} = \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k},\ 1 \leq k, l \leq N. \tag{3.6}$$

It is direct to show that these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the corresponding reflectionless Riemann-Hilbert problem:

$$(G^-)^{-1}(\lambda) G^+(\lambda) = I_{m+n},\ \lambda \in \mathbb{R}. \tag{3.7}$$

Solving the matrix spectral problems with zero potentials leads to

$$v_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^{2s+1}\Omega t} w_k,\ 1 \leq k \leq N, \tag{3.8}$$

and

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^{2s+1}\Omega t},\ 1 \le k \le N, \tag{3.9}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are constant column and row vectors, respectively. Also, by making an asymptotic expansion

$$G^+(\lambda) = I_{m+n} + \frac{1}{\lambda} G_1^+ + \mathrm{O}(\frac{1}{\lambda^2}), \tag{3.10}$$

as $\lambda \to \infty$, we obtain

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{3.11}$$

and further, plugging it into the matrix spatial spectral problem, we arrive at

$$P = -[\Lambda, G_1^+] = \lim_{\lambda \to \infty} \lambda [G^+(\lambda), \Lambda]. \tag{3.12}$$

This gives the N-soliton solution to the matrix AKNS equations in (2.6):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2,\ q = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl} \hat{v}_l^1, \tag{3.13}$$

where for each $1 \le k \le N$, we have made the splittings, $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, where v_k^1 and $\hat{v}_k^1$ are column and row vectors of dimension m, respectively, while v_k^2 and $\hat{v}_k^2$ are column and row vectors of dimension n, respectively.

To present soliton solutions for the reduced nonlocal matrix integrable mKdV equations in (2.17), one needs to check if G_1^+ determined by (3.11) possesses the involution properties:

$$(G_1^+)^\dagger = \Sigma G_1^+ \Sigma^{-1},\ (G_1^+)^\dagger(-x,-t) = \Delta G_1^+ \Delta^{-1}. \tag{3.14}$$

These mean that the resulting potential matrix P determined by (3.12) will satisfy the group reduction conditions in (2.11) and (2.12). In this way, the above N-soliton solution for the matrix AKNS equations in (2.6) reduces to the N-soliton solution:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{3.15}$$

for the reduced nonlocal matrix integrable mKdV equations in (2.17).

Realization: Following the preceding analysis, all adjoint eigenfunctions can be determined by

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^T(\lambda_k)\Sigma = v_{N_1+k}^\dagger(-x,-t,\lambda_{N_1+k})\Delta,\ 1 \le k \le N_1, \tag{3.16}$$

$$\hat{v}_{N_1+k} = \hat{v}_{N_1+k}(x,t,\hat{\lambda}_{N_1+k}) = v_{N_1+k}^T(\lambda_{N_1+k})\Sigma = v_k^\dagger(-x,-t,\lambda_k)\Delta,\ 1 \le k \le N_1, \tag{3.17}$$

and

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^T(\lambda_k)\Sigma = v_k^\dagger(-x,-t,\lambda_k^*)\Delta,\ 2N_1+1 \le k \le N, \tag{3.18}$$

respectively. It is direct to see that the choices in (3.16), (3.17) and (3.18) require the selections on $w_k,\ 1 \le k \le N$:

$$\begin{cases} w_k^T(\Sigma\Delta^{-1} - \Delta^*\Sigma^{*-1}) = 0,\ 1 \le k \le N_1, \\ w_k = \Sigma^{*-1}\Delta w_{k-N_1}^*,\ N_1+1 \le k \le 2N_1, \\ w_k^T\Sigma = w_k^\dagger\Delta,\ 2N_1+1 \le k \le N, \end{cases} \tag{3.19}$$

where $*$ denotes the complex conjugate of a matrix. We emphasize that all these selections aim to satisfy the reduction conditions in (2.11) and (2.12), which are consequences of the group reductions in (2.8) and

(2.9). To sum up, when the selections in (3.19) are made, the formula (3.15), together with (3.5), (3.6), (3.8), (3.16), (3.17) and (3.18), gives rise to N-soliton solutions to the reduced nonlocal integrable mKdV equations in (2.17).

When $m = n/2 = s = N = 1$, let us take $\lambda_1 = i\nu,\ \hat{\lambda}_1 = -i\nu,\ \nu \in \mathbb{R}$. Based on the last condition in (3.19), we can choose $w_1 = (w_{1,1}, w_{1,2}, -\sigma w_{1,2})^T$, where $w_{1,1}, w_{1,2}$ are real constants and $\sigma = \pm 1$ as involved in (2.22). In this way, we obtain the following one-soliton solution to the reduced nonlocal integrable mKdV equations in (2.22):

$$p_1 = \frac{2i\sigma\nu(\alpha_1 - \alpha_2)w_{1,1}w_{1,2}}{w_{1,1}^2 \mathrm{e}^{-(\alpha_1-\alpha_2)\nu x + (\beta_1-\beta_2)\nu^3 t} + 2\sigma w_{1,2}^2 \mathrm{e}^{(\alpha_1-\alpha_2)\nu x - (\beta_1-\beta_2)\nu^3 t}}, \tag{3.20}$$

where $\nu \in \mathbb{R}$ is arbitrary, and $w_{1,1}, w_{1,2}$ are arbitrary but satisfy $w_{1,1}^2 = 2w_{1,2}^2$, which is a consequence of the involution properties in (3.14). This traveling wave solution is analytic when $\sigma = 1$ but has a singularity line on the (x,t)-plane when $\sigma = -1$. The speed of the wave is $\alpha^{-1}\beta\nu^2$.

We remark that it would also be interesting to search for other kinds of reduced nonlocal integrable mKdV equations by different kinds of group reductions, both local and nonlocal, and to explore dynamic properties of exact solutions to the resulting reduced nonlocal integrable mKdV equations, including lump solutions [15], solitonless solutions [16] and algebro–geometric solutions [17]. All this will greatly enrich the theory of nonlocal integrable equations.

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References

[1] M.J. Ablowitz, Z.H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, Phys. Rev. Lett. 110 (2013) 064105.
[2] M. Gürses, A. Pekcan, Nonlocal nonlinear Schrödinger equations and their soliton solutions, J. Math. Phys. 59 (2018) 051501.
[3] W.X. Ma, Nonlocal PT-symmetric integrable equations and related Riemann-Hilbert problems, Partial Differ. Equ. Appl. Math. 4 (2021) 100190.
[4] W.X. Ma, Riemann-Hilbert problems and inverse scattering of nonlocal real reverse-spacetime matrix AKNS hierarchies, Physica D 430 (2022) 133078.
[5] M.J. Ablowitz, Z.H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, Nonlinearity 29 (2016) 915–946.
[6] C.Q. Song, D.M. Xiao, Z.N. Zhu, Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation, Commun. Nonlinear Sci. Numer. Simul. 45 (2017) 13–28.
[7] B.F. Feng, X.D. Luo, M.J. Ablowitz, Z.H. Musslimani, General soliton solution to a nonlocal nonlinear Schrödinger equation with zero and nonzero boundary conditions, Nonlinearity 31 (2018) 5385–5409.
[8] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Consultants Bureau, New York, 1984.
[9] J. Yang, General N-solitons and their dynamics in several nonlocal nonlinear Schrödinger equations, Phys. Lett. A 383 (2019) 328–337.
[10] W.X. Ma, Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations, Appl. Math. Lett. 102 (2020) 106161.
[11] W.X. Ma, Riemann-Hilbert problems and soliton solutions of nonlocal reverse-time NLS hierarchies, Acta Math. Sci. 42 (2022) 127–140.
[12] M.J. Ablowitz, Z.H. Musslimani, Integrable nonlocal nonlinear equations, Stud. Appl. Math. 139 (2017) 7–59.
[13] J.L. Ji, Z.N. Zhu, On a nonlocal modified korteweg–de vries equation: Integrability, darboux transformation and soliton solutions, Commun. Nonlinear Sci. Numer. Simul. 42 (2017) 699–708.
[14] M. Gürses, A. Pekcan, Nonlocal modified KdV equations and their soliton solutions by Hirota method, Commun. Nonlinear Sci. Numer. Simul. 67 (2019) 427–448.
[15] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Differential Equations 264 (2018) 2633–2659.
[16] W.X. Ma, Long-time asymptotics of a three-component coupled mKdV system, Mathematics 7 (2019) 573.
[17] F. Gesztesy, H. Holden, Soliton Equations and Their Algebro-geometric Solutions: (1 + 1)-Dimensional Continuous Models, Cambridge University Press, Cambridge, 2003.